Engineering Mathematics – I

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Lecture – 12
Differentiability of Two Variables

Welcome to the lectures on Engineering Mathematics-I, and today we will be talking about the Differentiability of Two Variables functions of two variables and this is lecture number 12.

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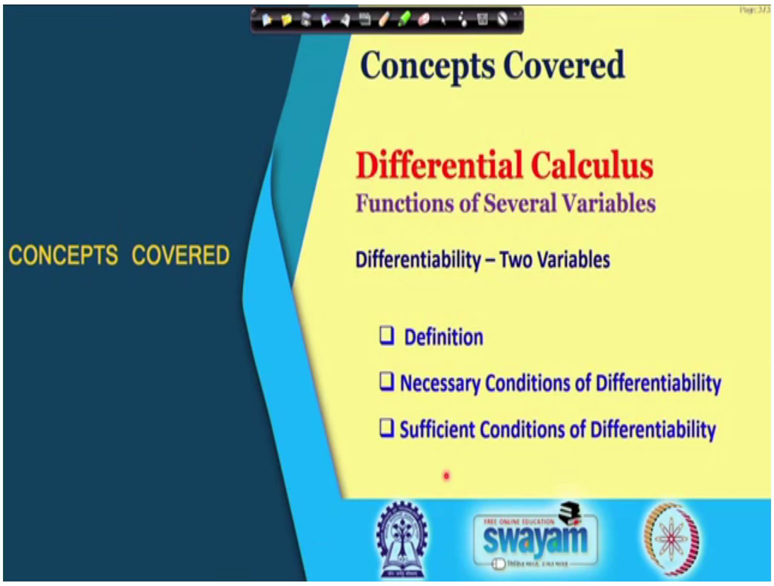

So, we will introduce first the differentiability concept for the functions of two variables, and then we will derive necessary conditions for differentiability, and also sufficient conditions for differentiability.

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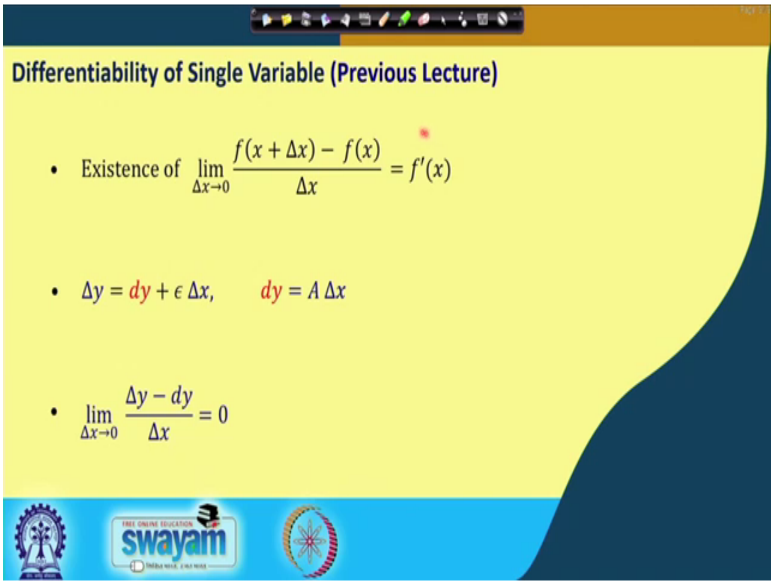

So, just to recall the differentiability of single variable which we have learnt in the previous lecture. So, there were three concepts we have used to define the differentiability of a functions of single variable. So, one was the existence of the derivative. So, if this limit f x plus delta x minus f x over delta x exists, which we call derivative of the function f at x and it was denoted by f prime x. So, if this derivative exists we call the function as differentiable.

The second concept which is more general and we will indeed now use this to derive differentiability of functions of more than one variable. And in this case we have seen that if we can express this delta y which is the change in delta y when we make a change of delta x in x. And if we can express this change in terms of this dy plus epsilon delta x the dy which we call differential was a linear term in delta x and this was A times delta x; here the A was independent of delta x and this epsilon term must go to 0 as delta x goes to 0.

So, the third one was useful for testing the differentiability and what we have seen that if this quotient here delta y minus d y over delta x and when we take this limit delta x goes to 0 is 0 then we call the function differentiable. So, this was just a consequence of this relation when we have taken this dy to the left side and divided by delta x taking this limit since epsilon goes to 0. So, we get this relation out of this relation, but this was useful for testing the differentiability in many cases.

But what was the point here that all these three definitions are equivalent for differentiability. So, the existence of the derivative we have seen that imply implies the differentiability of a function of single variable, but today what we will observe that the existence of derivatives which we call the partial derivatives with respect to x or y that is not sufficient for the differentiability of the functions of two variables for example.

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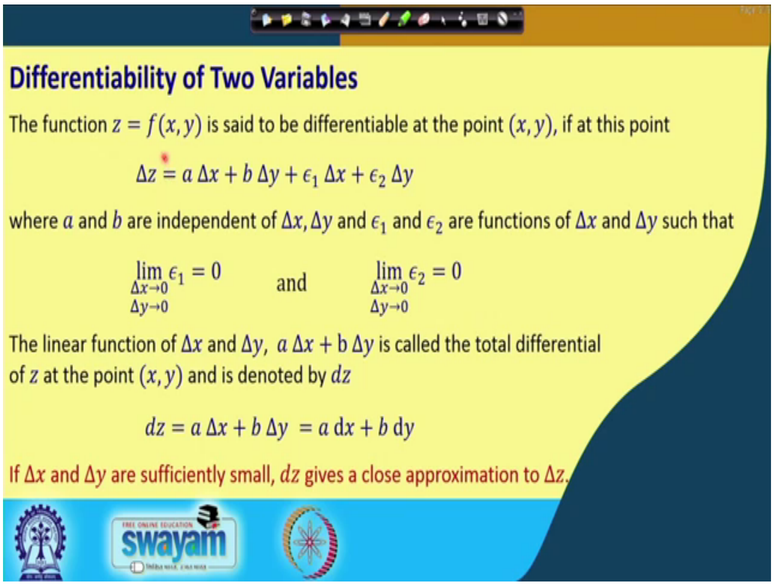

So, here just to define the differentiability of functions of two variables, we take this function z is equal to f x y and this is said to be differentiable at the point x y a general point xy, if at this point we can express this change in z. So now, we have this z as an dependent variable on x and y. So, xy here are independent variables. So, if this change in delta y is equal to a times delta x plus b times delta y plus epsilon 1 times delta x and epsilon 2 times delta y.

So, this is just the extension of the definition which we have used for functions of one variable. And now in this case again this a and b they are independent of delta x and delta y and this epsilon 1 and epsilon 2 they must go to 0 as delta x delta y goes to 0. So, again we have this linear term and then this quadratic or the non-linear rather non-linear term because we do not know; what is the dependency of delta x one epsilon 1 and epsilon 2. So, we have the linear term here and pass this rest.

So, again here the a and b are independent of delta x and delta y and epsilon and epsilon 2 are functions of delta x and delta y such that, this epsilon 1 is equal to 0 when we take this limit delta x goes to 0 and delta y goes to 0 also this epsilon 2 when we take this limit as delta x and delta y both go to 0 epsilon 2 must be 0. So, this linear function again which is a delta x plus b delta y is called the total differential we have also introduced in case of functions of single variables. So, here this is called total differentiable of z at this point general point x y and this is denoted by delta z.

So, now we will denote this by d z which is the linear part of this expression and delta z that is the change in z when we make change in x and y by delta x and delta y. So, this dz which we call the differential term or the total differential this is equal to a delta x and b delta y, that is the linear part of this change in z; which we can also denote because we have observed yesterday that delta x or dx they are the same because they measure the change in x and in this case because we have two variables.

So, the delta x and delta y or dx or dy they make they measure the change along the x and the y axis, while this delta z is the change in the function value. Whereas, this dz is different which is the change in a along the tangent plane in this case in case of one variable it was along the tangent line.

So, if this delta x and delta y are small enough then this dz which is the differential of z at this point will be a close approximation or will be a good approximation of the change in delta change in z that is delta z, because this non-linear term here for small values of delta x and delta y that will go to 0. And we will have a very good approximation by this linear term or at least in the neighborhood of the point x y.

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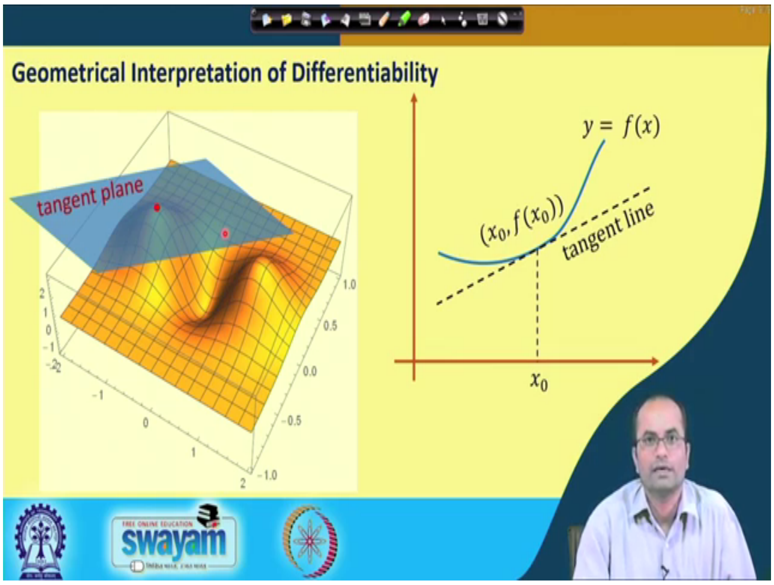

So, the geometrical interpretation of the differentiability: so what we have seen in case of function of single variable that at this point x 0, the value of the function is f x 0. So, this is the tangent line; that means, if the function is differentiable we have observed that we can draw a tangent line or in other words we can approximate this function in the neighborhood of this x 0 point by this tangent line.

So, as we see here in the graph as well in the close neighborhood of this point x naught, we can have a very good approximation by this tangent line at least in the close neighborhood of this point x naught in case of the functions of two variables. So, we have a similar argument or the extension of this concept.

So, if suppose there is a point here x naught y naught, and then in this case if the function is differentiable. Then we can approximate in the close vicinity of this point by the tangent plane; in case of single variable this was tangent line. And now we have the function of two variables in this picture and then we can approximate by this tangent plane which is actually the linear part which we have seen in the definition there.

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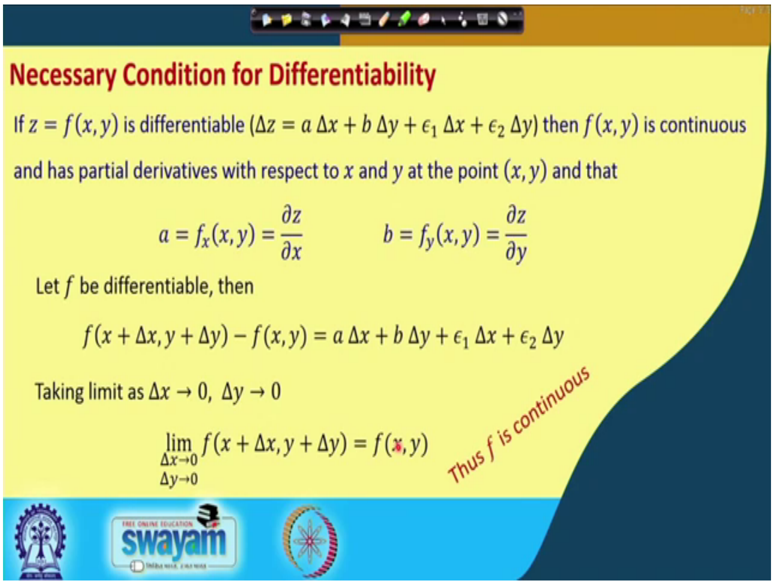

So, what are the necessary conditions for differentiability we will derive now? So, if this z is equal to fx y, this function is differentiable. That means, we can express this delta z is equal to a delta x plus b delta y plus epsilon delta x plus epsilon delta y. Then we will show that the function is continuous and has partial derivatives with respect to x and y, at that point where we are talking about the differentiability. That means, here that discontinuity and the partial derivatives the existence of the partial derivatives at x y point with respect to x and y these two are the necessary conditions for the differentiability, because we will show that the differentiability implies that f is continuous and the partial derivatives exist at the point x y.

Moreover we will also see that this a here which is the constant free from delta x and delta y this is nothing, but the partial derivative of f with respect to x at that point and this b here is nothing, but the partial derivative of f with respect to y at that point. So, let us prove this. So, if we assume that this f is differentiable; that means, we can write down this delta z here which is we have made an increment here delta x and delta y in x and y and this difference is nothing but the delta z. So, delta z we can express as a delta x plus b delta y plus epsilon delta x plus epsilon delta y, the definition of the differentiability.

So, having this now, if we take the limit here as delta x goes to 0 and delta y goes to 0 what will happen? Since delta x goes to 0 and a is a finite number. So, this will disappear.

And here again this will go to 0 here also since the epsilon goes to 0 and also delta x goes to 0. So, this term will go to 0 and also this term will go to 0. So, we will have simply when we take the limit delta x goes to 0 and delta y goes to 0, we will have that the limit of this f x plus delta x y plus delta y as delta x goes to 0 delta y goes to 0 is equal to f x y and this is nothing, but the continuity of f.

So, to show the continuity, we take a point in the neighborhood and then we take the limit and this should approach to the function value x and y independent of this how we approach to this xy point by taking this delta x to 0 delta y to 0 along any path. So, that was the continuity here of f which we have proved. So, what we have assumed that if f is differentiable then we can express it in this form and simply by taking this limit we have seen that the function must be continuous if the function is differentiable.

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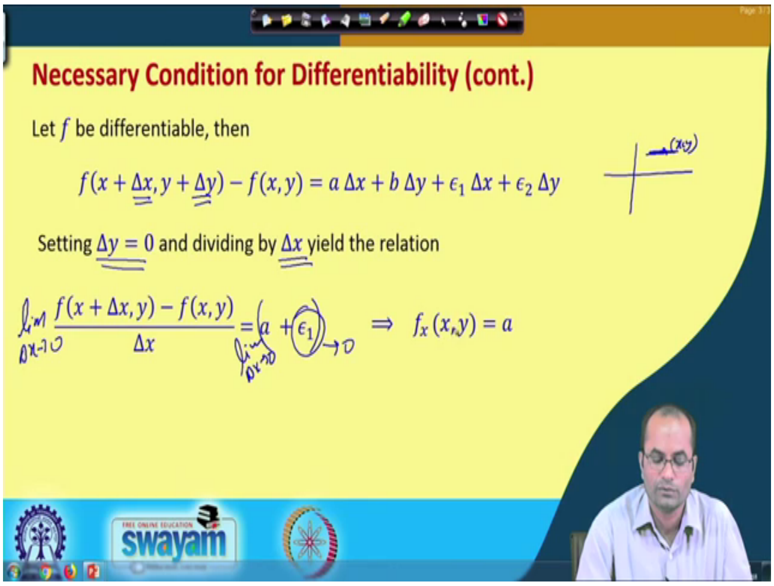

So, next we will continue the necessary condition the second one. So, we again assume that f is differentiable then we can express this delta f or delta z in terms of a delta x b delta y and this epsilon and epsilon 2. And now since delta x and delta y are arbitrary we can choose in the neighborhood anything here. So, what we have set if we say delta y equal to 0 and then divide by delta x then what we will get. So, we will get this here x plus delta x and then delta y is 0 minus this f xy and divided by delta x. So, that will be equal to so here a will remain and then here delta y is set to 0. So, this term is 0 and then epsilon 1 because we are dividing with delta x and this term will go to 0 because we have set delta y to 0, because this is again we are in the neighborhood of this point x y.

So, if we are talking about a point here x y, so we have set just delta y to 0. So, again we are in the neighborhood along this x axis by having this delta x not equal to 0 at this moment here. And so we can do that we can take any delta x and any delta y and this relation must hold. So, what we have done we have taken delta y to 0 and then divided by delta x and now we can we can take the limit of this delta x as it goes to 0. And what we will get this is the partial derivative the definition of the partial derivative of f with respect to x and this will be equal to.

So, we have taken the limit here both the side, so limit delta x goes to 0 and here also then in this case the limit delta x goes to 0. So, a is independent of delta x delta y, so this will remain as a and this epsilon term will go to 0 as delta x go to 0. So, we will get this relation that fx is equal to at x y is equal to a.

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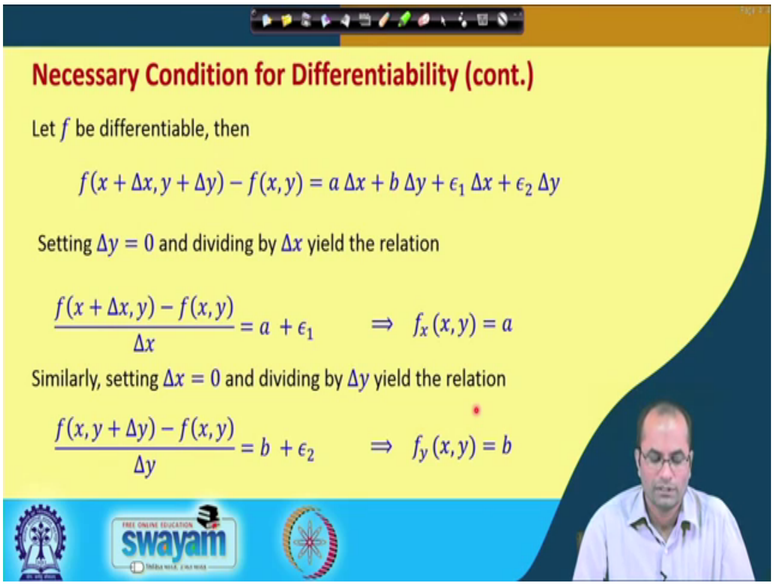

And now similarly what we can do we can set delta x to 0 now and then we can divide by delta y. And again by taking the limit as delta y goes to 0, so we will get that f xy sorry f y xy is equal to b. So, here if we take this limit now delta y goes to 0, so the right hand side will be just b because epsilon 2 will go to 0 as delta y goes to 0. So, in this case you will get that the partial derivative of y at x y is equal to b.

So, what we have observed now that if function is differentiable, in that case the partial derivative of x at this point x y will exist, and that will be nothing but this number a here which is appearing in the definition of this differentiability. And also the partial derivative with respect to y at that point will exist and the value will be b which is appearing here in the definition of the differentiability.

So, what we have seen now that if a function is differentiable then it must be continuous and not only continuous, but the partial derivatives must exist at that point where the function is differentiable. So these are the necessary conditions for differentiability. If one of the conditions like the function is not continuous or one of these two partial derivatives does not exist. In that case we can immediately claim that the function is not differentiable because, these two are the necessary conditions the continuity of the function and the existence of partial derivatives.

So, if one of these conditions is violated then we can immediately conclude that the function is not differentiable, but if the function is continuous and also the partial derivatives exist at the point x y, where we are talking about the differentiability. In that case we have to go for the further test, because these are the necessary conditions based on these 2 conditions we cannot say that the function is differentiable or it is not differentiable because these are the necessary conditions for differentiability.

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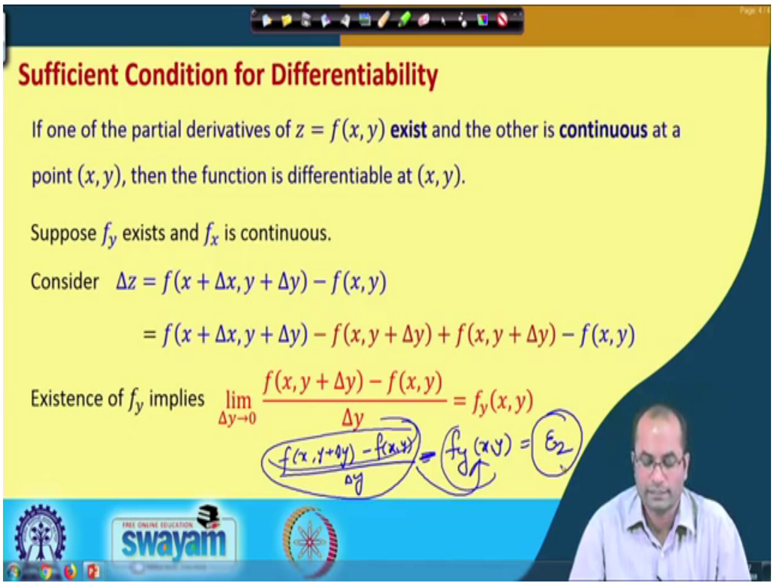

So, now we will move to the sufficient conditions for differentiability and in this case if one of the partial derivatives of z exists and the other is continuous at a point x y then the function is differentiable at x y. So, what we have now for the sufficient conditions that if one partial derivative exists and that is anyway necessary condition for the differentiability. So, we must have the existence of both the partial derivatives for discussing the differencability.

So, this is natural now because this is a necessary condition that both the partial derivatives exist. So, basically for the sufficient condition we have to check that if one of the partial derivative is continuous, x then the function is differentiable at x y. So, we need continuity of one partial derivatives and the existence of both because the existence is necessary condition for the differentiability. So, we suppose here that fy exists and fx is continuous.

So, in this case we will now observe what is this delta z the variation in this z when we vary x by delta x and y by delta y. So, we will consider this difference now and we will see whether we can express this difference in terms of that linear function plus this epsilon delta x plus epsilon to delta y. So, here now moving further we have f x plus this term and then we have subtracted this f x y plus delta y and we have added the same here f x y plus delta y minus f x y which was already there in the difference and then since we have taken here the existence of fy and continuity of f x. So, we will use these two conditions to prove that the function is differentiable.

So, the existence of f y implies that this limit here f x y plus, so here this is the partial derivative with respect to y. So that means, f x y plus delta y we will make an increment in y and minus the function value at that point divided by this increment delta y, and taking this limit delta y that this exist. That is the meaning of the existence of f y that this partial derivative exists.

So, this is equal to f y and now we can use that idea which we have used in the previous lecture, that once the limit is given we can define or we can introduce some epsilon. That means, that f this x and y plus this delta y and minus this f xy over this delta y is equal to or with minus sign here. So, minus this f x y we can set as epsilon in this case we will take this epsilon 2 and then later on we will introduce epsilon 1 as well.

So, if we set this difference to epsilon 2 then we know that when we take the limit as delta x or in this case only this delta y goes to 0 then this epsilon 2 must go to 0, because when delta y goes to 0 this is nothing but exactly the f y at x y. So, this we will introduce now and then we will multiply this whole expression by delta y, and we will get that this difference is equal to f y delta y plus epsilon 2 delta y.

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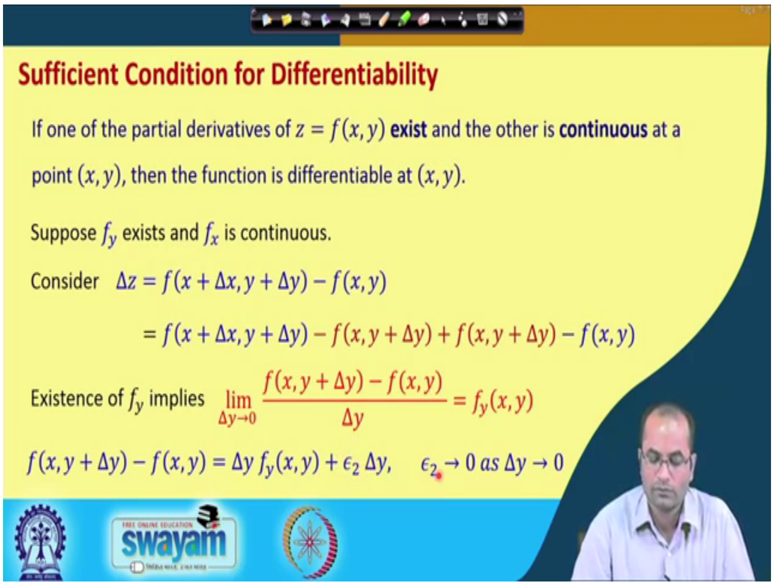

So, having this we will get this f x y plus delta y minus this f x y is equal to this delta y will be multiplied to the right side. So, we will have f y x y and plus this epsilon 2 which we have introduced here and this delta y and note that this epsilon 2 which appeared here now must go to 0 as delta y goes to 0, this is the condition because of this limit we have observed.

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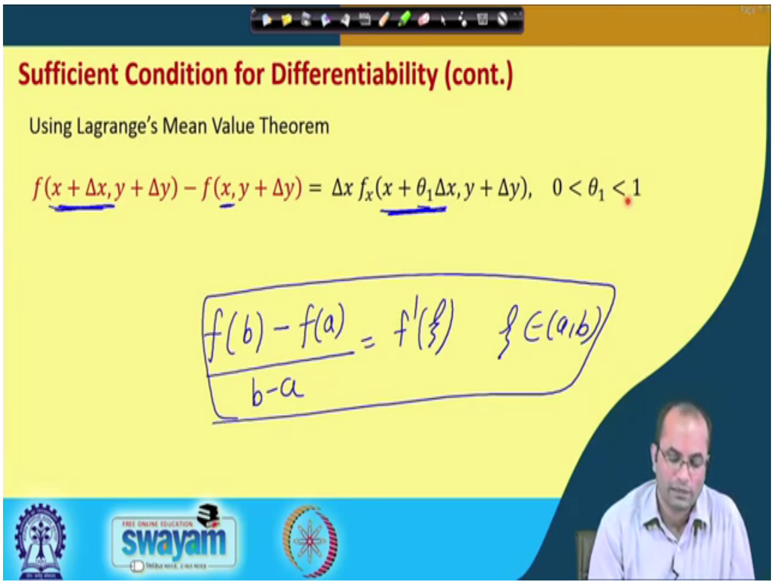

Next now we will use the LaGrange mean value theorem. So, just to recall; what was the LaGrange mean value theorem we have function which is continuous and differentiable in the open interval a b. So, here f b minus f a over b minus a is equal to this quotient will be equal to the derivative at some point in the interval a b, that was the LaGrange mean value theorem we have studied before and now so using this LaGrange mean value theorem to this difference here. So, notice that here f x plus delta x and f x so y is not changed.

So, why we are not changing y plus delta y here also y plus delta y, so this is with respect to x only the change is in x and if we divide. So, this is like f b minus this f a here a is x and this b is x plus delta x and divided by the difference which is b minus a, so in this case this will be delta x but we will multiply to the right side will be delta x and the derivative naturally with respect to x because we are talking about with respect to x the y is unchanged.

So, f x the partial derivative and at which point here the xi was between b and a, so here also this argument will vary between this x and x plus delta x. So, we have introduced this theta 1 so that this will be precisely between x and x plus delta x when theta runs between 0 and 1. So, this theta 1 here is between 0 and 1 when this is close to 0 this argument is close to x and when this is close to 1 this will go to and go to x plus delta x. So, exactly which xi was doing here in the open interval a b, we have this argument x plus theta 1 delta x for this theta 1 0 and 1, so we have this LaGrange mean value theorem now.

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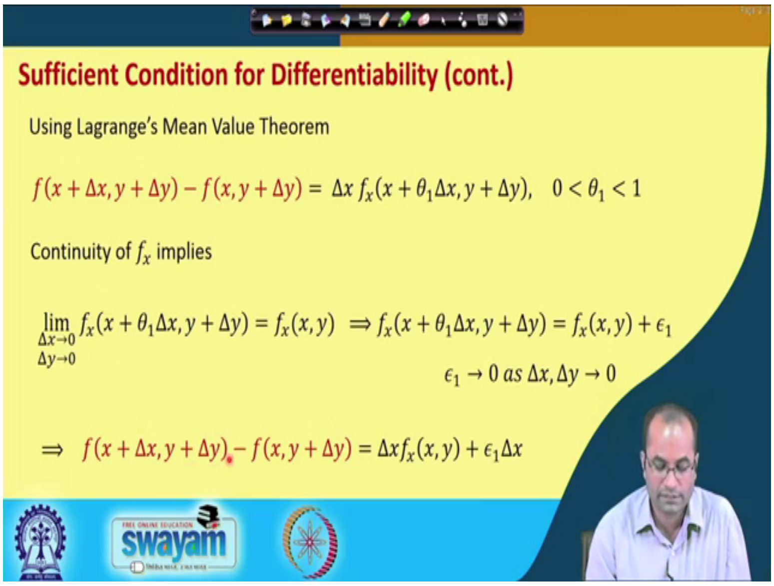

And now the continuity of f x because we have assumed the continuity of f x and the existence of f y. So, this continuity of the partial derivative f x we will apply here. So, if f x is continuous and we take the limit delta x go to 0 and this delta y goes to 0, then this f x x plus theta one delta x y plus delta y will simply go to f x at the point x y this is what we have used now.

So, the continuity of f x will give us that this derivative with respect to x will be f x at x y point and then again we will introduce another epsilon as we have done before, so that is f x and this x plus theta one delta x y plus delta y is equal to f x at x y plus this epsilon and this epsilon will have property that this will go to 0 as delta x and delta y goes to 0. So, now out of this we get this relation now that f x plus delta x and y plus delta y minus f x y plus delta y is equal to delta x is multiplied and f x x y plus epsilon 1 delta x and now so we are here now f x plus delta x and y plus delta y minus f x y is equal to we can write down as delta x times f x at x y plus epsilon times delta x no well.

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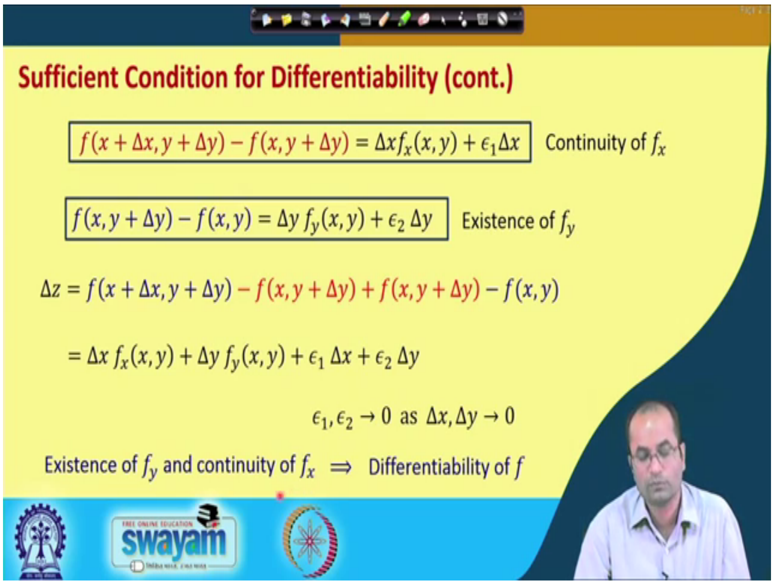

And now what we have we have these two relations we have proved that this difference when we have variation here is delta x and y is unchanged y plus delta y. This was because of the continuity we got this relation, and because of the existence of f y we got this relation and having these 2. So, remember the delta z we had written by subtracting and adding to this difference here. And now we will replace this f x plus delta x y plus delta y minus this by this expression and this f x y plus delta y by this expression here.

So, we will get now by substituting these that delta x f x delta y f y plus epsilon delta x plus epsilon 2 delta y and epsilon and epsilon 2 goes to 0 as delta x delta y go to 0. That means, the existence of f y and the continuity of f x these 2 conditions we have proved that the function is differentiable because, this is the definition of the differentiability. And we have used to arrive to this expression, we have used the fact that the we have used that the function derivative with respect to y exists and the function derivative with respect to x is continuous.

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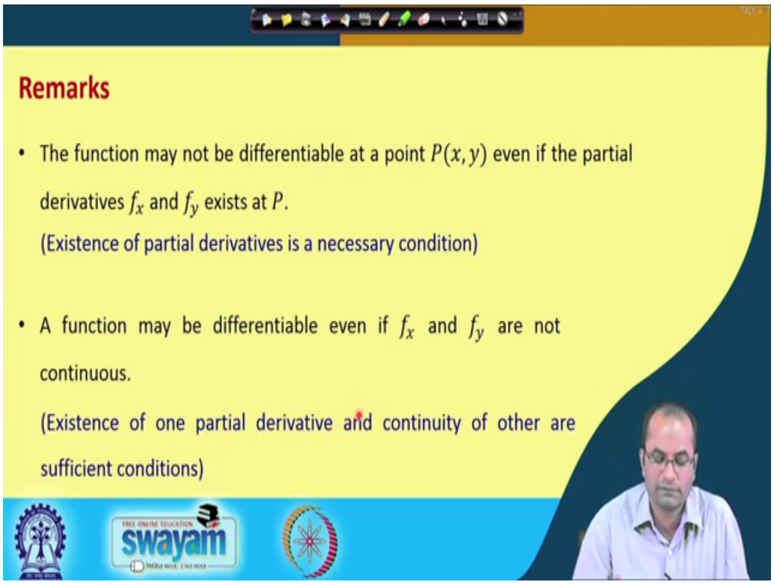

So, just a remark that the function may not be differentiable at a point even if the partial derivatives f x and f y exist at p, why because the existence of partial derivatives at the point p is necessary condition this is not sufficient for differentiability which is written here existence of partial derivatives necessary condition, so they cannot guaranty differentiability.

On the other hand a function may be differentiable if f x and f y are not continuous because, what we have observed that the continuity of f x over f y or both that is the sufficient condition not the necessary condition. So, the function may be differentiable if f x or f y or f x and f y are not continuous because, this existence of partial derivatives and continuity of the other the existence of 1 partial derivative and the continuity of other these 2 are the sufficient conditions.

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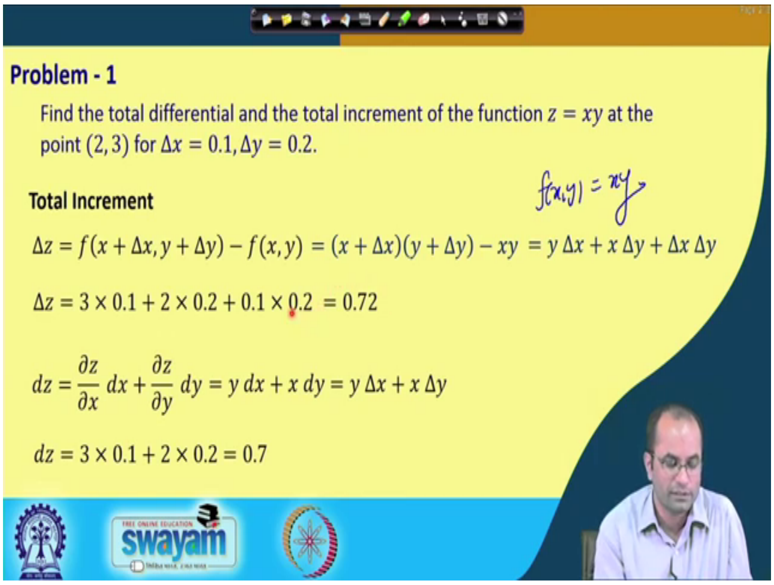

So, let us go through quickly 2 problems, so here find the total differential and total increment of the function z is equal to xy at the point 2 3 for delta x is equal to 0.1 and delta y is equal to 0.2. So, the total increment is defined as f x plus delta xy plus delta y minus f x y which is equal to. So, this is x plus delta x and y plus delta y we can use this z now z is equal to x y, so this is our function f x y is xy f x y is x y. So, in this case we have here the product of this x plus delta x y plus delta y minus this product here x y and then we can open this so x y and then x delta y delta x y delta x delta y so x y will get cancelled and we will have this expression. And then we can compute this delta z at this point two three and the delta x is given point one delta y is given 0.2 and this will come 0.72.

Now, coming to the differential which is dz del z over del x dx plus del z over del y dy and in this case del z over del x is y and del z over del y is x. So, we have this expression now to be computed at this two three point with delta x 0.1 delta y 0.2. So, note that is dx dy is nothing but delta x and delta y because, these are independent variable for independent variables these notations are the same dx is equal to delta x dy is equal to delta y. But for dependent there is a difference which we are going to observe now here and this dz the differential of z now at this point will be 0.7. So, there is a difference naturally on delta z and the dz, but when this delta x and delta y are small numbers, then they will be a good approximation to each other here or dz will approximate well this delta z term.

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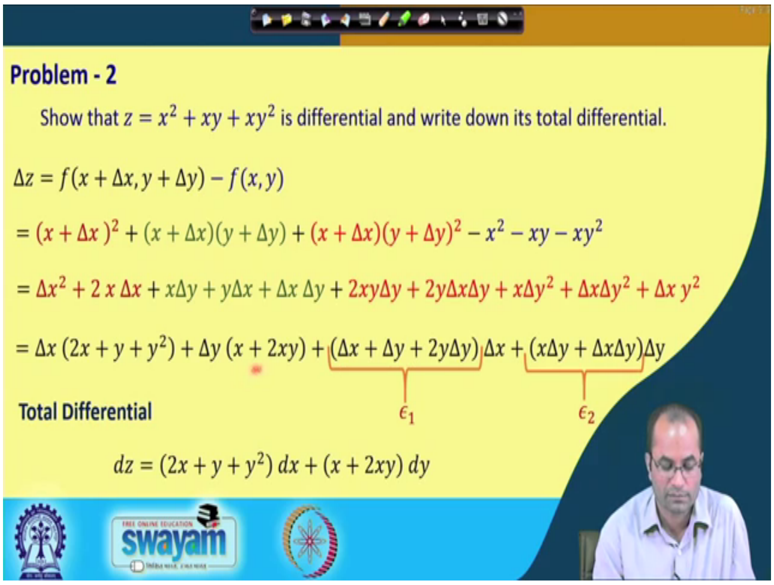

Another problem we will show now that this is differentiable and write down this total differential. So, to show the differentiability we will use this basic definition of expressing delta z in terms of this a b and epsilon-epsilon 2. So, this delta z is nothing but this difference here and then we can compute this, because this is our f x y. So, here we have substituted this x plus delta x and y plus delta y and this is f x y which is given as x square minus x y and minus x y square.

So, we can open now here the square so x squared term will be 3 2 times x delta x term will be there delta x square term here also 1 term will be xy here also there will be a term xy square. So, these terms will be canceled and the other we will collect together. So we will have these many terms this is coming from x plus delta x square other than x square because that was canceled out.

Now, these terms are because of this product here there will be 4 terms, but 1 will be cancelled here and there will be 6 terms. So, we will have 5 terms here because this xy square will be cancelled. And now we will try to get the coefficient of delta x and the delta y and plus in terms of epsilon delta x and epsilon delta 2. So, here if we take common delta x, so this is two x from here and then y from here and then we will get delta x from here also y square. So, this is with the delta x term and from delta y you will get x from here and delta y is appearing here too. So, del 2 x delta 2 x y with delta y, so this is the linear term which we have collected and the rest we have kept in this format that with delta x whatever the coefficients here and we delta y also we got this 1.

So, what we observe that we can introduce now that this is our epsilon 1 and this is our epsilon 2. And they have the property that they will go to 0 as delta x and delta y will go to 0 and this 1 is the total differential term and since this a and this is b in our notation and they are independent of delta x delta y.

So, we have a into delta x b into delta y epsilon 1 delta x epsilon 2 delta y, so we can express this z this function in this form. So, the function is differentiable and its total differential will be given by this linear term which is written here 2 x plus y plus y square dx and x plus 2 x y dy. So, this is the differential.

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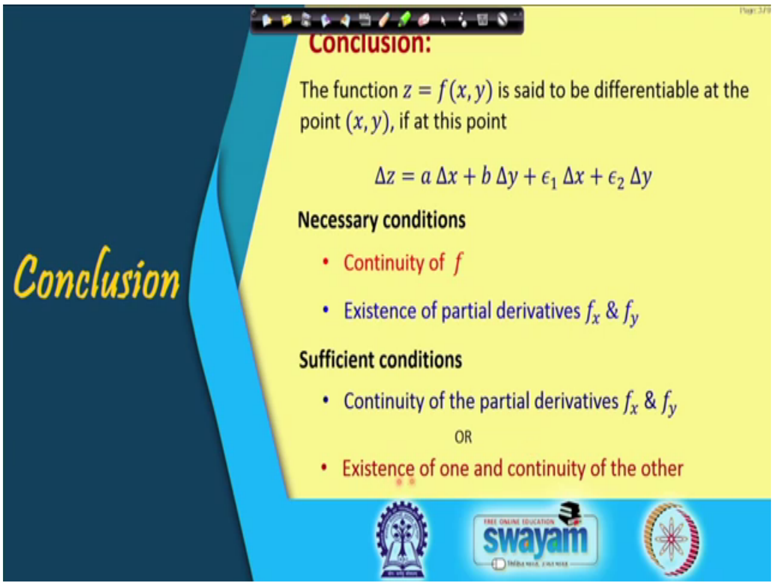

So, coming to the conclusion now: so we have discussed that this function z is said to be differentiable at the point xy; if at this point we can express this delta z as a delta x b delta y plus epsilon delta x epsilon to delta y, where this epsilon 1 and epsilon 2 they go to 0 as delta x delta y go to 0 and also this number a and b they are independent of delta x and delta y. So, the necessary conditions we have a studied they were continuity of f is necessary for differentiability and the existence of partial derivatives of f x and f y at that point where we are talking about the differentiability.

So, these two are the necessary conditions and for the sufficient conditions we have seen that the continuity of the partial derivatives of f x and f y are important or in other words we can say the existence of 1 and the continuity of the other. Meaning the continuity of one is sufficient here, because the existence anyway we have to have to discuss the differentiability because, the existence of partial derivative is important is necessary then only we can talk about the about the differentiability. So, basically the continuity of one partial derivative is sufficient for differentiability.

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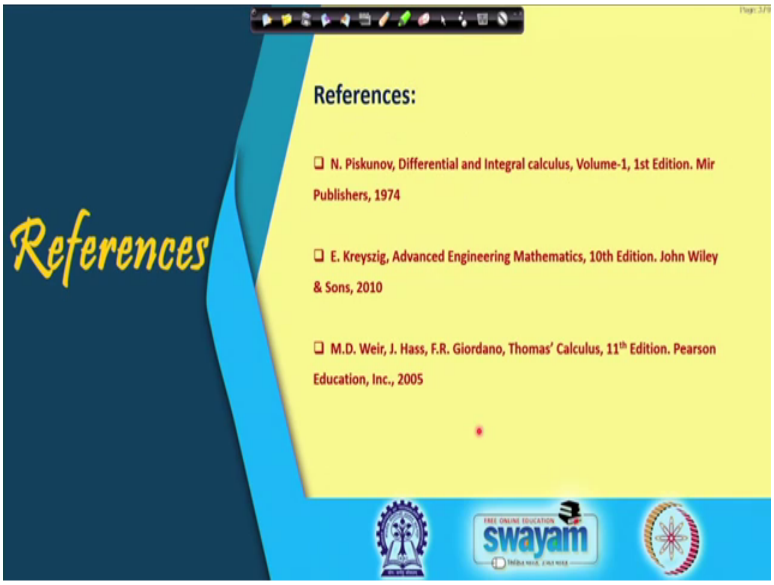

These are the references we have used to prepare these lectures.

Thank you very much.