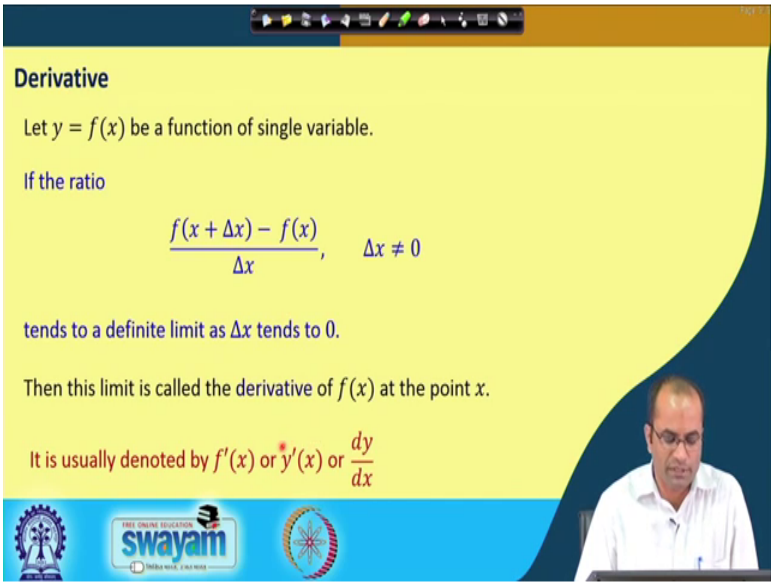
Engineering Mathematics - I
Prof. Jitendra Kumar
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture – 11
Derivative & Differentiability

Welcome to the lectures on Engineering Mathematics - I. And today's we are discussing lecture number 11 on Derivatives and Differentiability of One Variable; actually it is very important to discuss differentiability of one variable before we go for the several variable case in the next lecture.

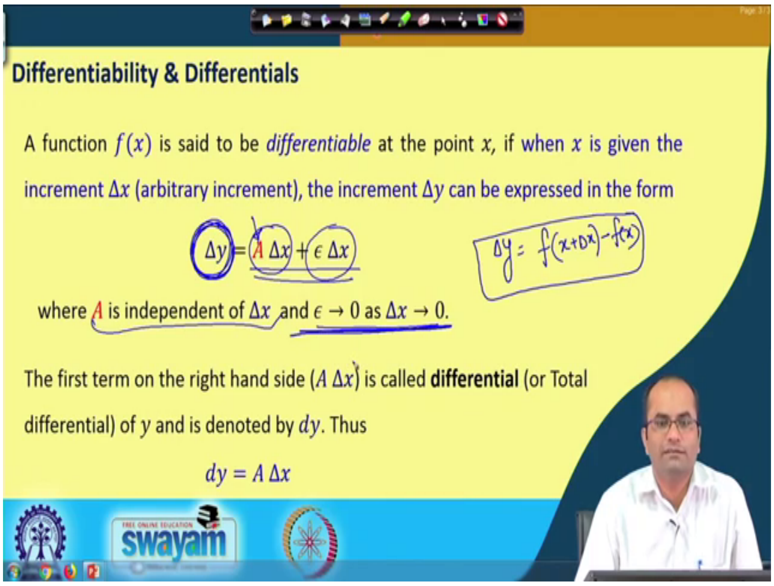
(Refer Slide Time: 00:37)



So, what is derivative for a function of single variable? So, let f x y is equal to f x be a function of single variable and if this ratio f x plus delta x minus f x; so this difference when we make an increment delta x in x and minus the f x the value at x divided by this increment. If this ratio here has a limit a definite limit as delta x tends to 0 then we call that this limit is the derivative of the function f x at the point x. So, that is the definition of the derivative usually we have. So, that limit here when we take the limit delta x goes to 0, if this limit exists we call that this value is the derivative of the function f x.

And, this is usually denoted by this f prime x or sometimes y prime x or t y over d x that is a notation for the derivative which is the limit of this quotient here, when we make the increment in x by delta x and take this difference by the value of the function at x divided by the increment. So, if this limit exists we call the function has the derivative and its value is exactly that limit. And, the notation here we use for the derivative are f prime y prime over d y over d x.

(Refer Slide Time: 02:13)



Now, what is differentiability and differential usually? So, this is more formal definition and the reason behind this we will we can easily extend it to the case of several variable. So, a function f x is said to be differentiable at the point x, if when x is given the increment delta x arbitrary increment and the increment delta y can be expressed in the form of delta y is equal to some constant A into delta x plus epsilon times delta x; where this epsilon goes to 0 as delta x goes to 0 and this number here A is independent of delta x. So, the point is that this f x is said to be differentiable if we can write down the increment delta y when we give the increment delta x in x.

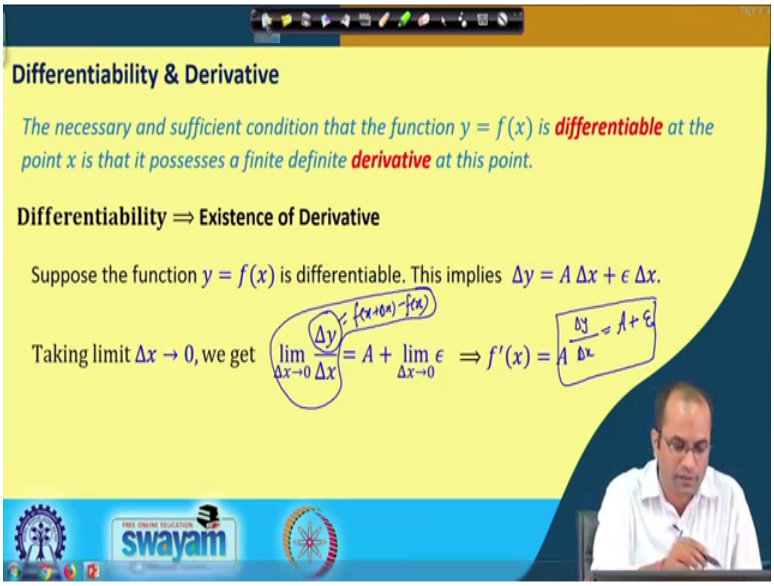
So, there will be an increment in y that is denoted by delta y. So, if we can write down this delta y as A times delta x plus epsilon delta x, this is a linear term here because A is independent of delta x and plus this epsilon times delta x and this epsilon has the property that it goes to 0 as delta x goes to 0. Then we call that this function is differentiable and the first term on this right hand side; that means, this A times delta x.

So, this first term on the right hand side A times delta x, this is called differential or the total differential of y and this is usually denoted by the notation dy. Thus, what we have that dy that is another notation for what we call differential. So, dy is denoted by this A times delta x those are the linear term in this expression of this delta y. So, again this delta y was the increment. So, this delta y is as f because the function is y is equal to f x.

So, delta y is f x plus delta x when we make an increment in x by delta x and the difference so, the f x. So, this delta y is f x plus delta y minus x. So, if this increment here delta y we can write down in this form. So, this linear term A which is independent of delta x times delta x and plus epsilon delta x, then we call that the function is differentiable. And, this epsilon very important that it should go to 0 as delta x goes to 0 and this A should be independent of delta x.

So, if we can do that we call the function is differentiable and now in the next slide we will see that this definition of the differentiability; what we have here that we can express this delta y in terms of this A delta x plus epsilon delta x. And, the earlier definition of the derivative they are actually equivalent.

(Refer Slide Time: 05:33)



So, let us move to the next slide here. So, here differentiability and the derivative what is the relation or basically they are the same what we will observe now here. So, the necessary and sufficient condition that the function y is equal to f x is differentiable at the point x is that it possesses a finite derivative at this point. So, as written here it is a necessary and sufficient; that means if a function is differentiable we have given already the definition of the differentiability. So, if a function is differentiable then it will have derivative at that point or it if it has a derivative at that point then the function must be differentiable at that point.

So, the conclusion is that the both the definition what we have given earlier the derivative and the differentiability they are basically the same in case of the single variable. So, the now we will see the differentiability implies the existence of the derivative at a given point. So, suppose the function y is equal to f x is differentiable. So, we assume that the function is differentiable and what this will imply as per the definition that we can express this delta y increment in delta y when an increment in x is given by delta x.

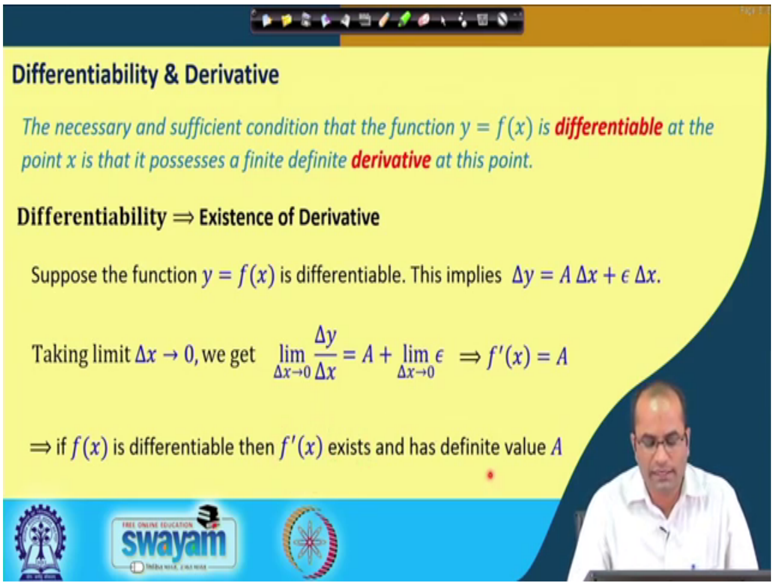
So, if we can write down this delta y as A delta x plus epsilon delta x then this function is differentiable. So, we have assumed that the function is differentiable; that means, we can express delta y is equal to A delta x plus epsilon delta x and now we can take the limit. So, before we take the limit we can divide by this delta x. So, this will be like delta y over delta x is equal to A times epsilon. So, A times epsilon and now we will take the limit as delta x goes to 0.

So, by taking this limit here so, this will be delta y over delta x. So, the limit is equal to here A there is no delta x and A is also independent of delta x as per the definition of the differentiability plus this epsilon and the limit delta x goes to 0. And, we know that this epsilon has the property that it goes to 0 as delta x goes to 0. So, in this case we get now here this limit delta y over delta x.

So, this is the constant in the derivative we have seen f. So, this term here is nothing, but f x plus delta x and minus f x and divided by this delta x. So, this here is f x plus delta x minus f x and divided by this delta x limit delta x goes to 0. So, this term here becomes the derivative which we have discussed in the previous slide so; that means, we get here let me erase this ok.

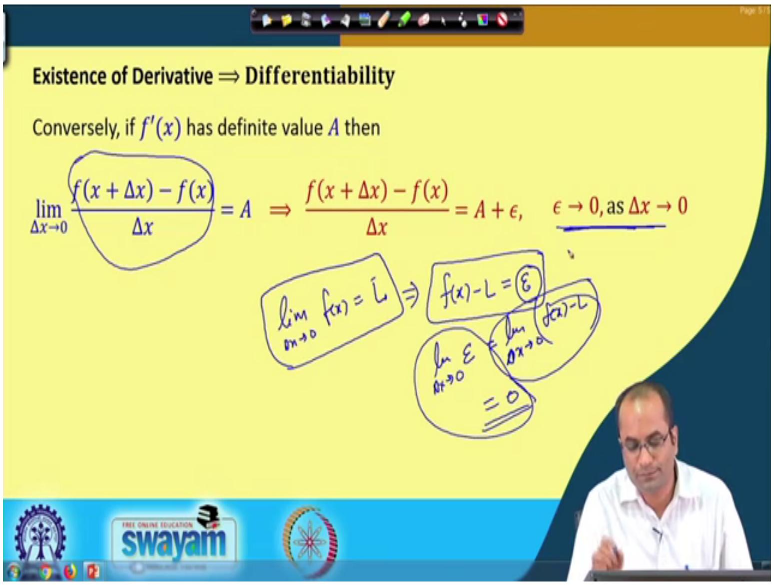
So, we get this here f prime x is equal to A because this will go to 0. So, what we have observed that if the function is differentiable then the derivative f prime x this is the derivative here when take the limit. So, the derivative is nothing, but this A which is written here in the linear term A which is free from delta x. So, this A is nothing, but this f prime x the derivative. So, what we have seen that if the function is differentiable then the derivative exist and that derivative is equal to A, this is given in this expression of delta y.

(Refer Slide Time: 09:18)



So, if f x is differentiable then f prime exist and has definite value A. The next we will see that if the derivative exists that will imply that the function is differentiable.

(Refer Slide Time: 09:27)



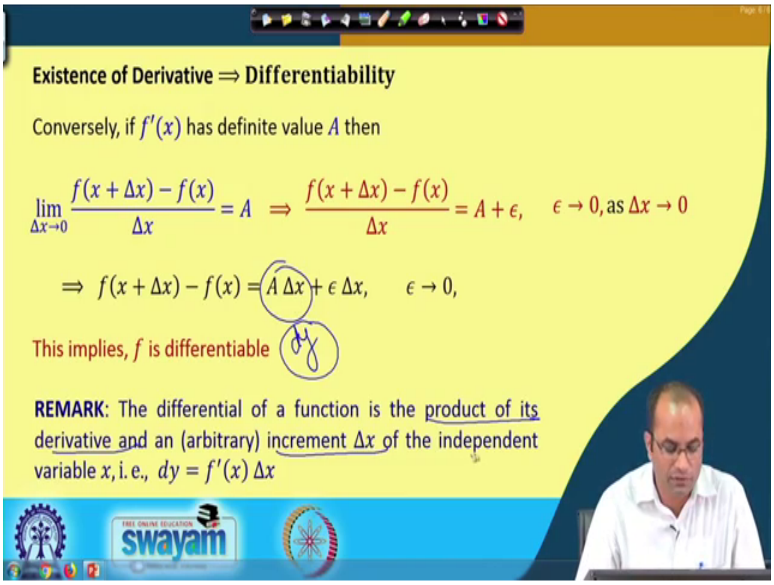
So, we assume that f prime x has definite value A; that means, this f prime exist or the function derivative exist. So, in this case we have that this is given here f x plus delta x minus fx over delta x has the limit which is equal to A because we have assumed that the derivative exist. So; that means, this limit exist which is equal to A this is what we have assumed and now since this limit is equal to A we can define or let me just explain you here. So, if for example, the limit delta x goes to 0 of some function is equal to let us say L then what we can write down out of it that this f x minus L we can define this difference in the neighborhood of point x is equal to some epsilon for example.

So, what this epsilon will have the property that this epsilon will go to 0 when delta x approaches to 0. So, if we take the limit here. So, epsilon and then the limit delta x goes to 0 will be equal to this limit here delta x goes to 0 and f x minus L. So, since the limit f x goes to L so, this will go to 0. So, if we write down this expression here for epsilon as f x minus L then this epsilon must have this property that epsilon will go to 0 as delta x goes to 0.

So, we have a similar situation here this limit delta x goes to 0 of this expression of this quotient is given as A. So, what we have written down that this quotient f x plus delta x minus f x over delta x we can write down as A plus so, the value of this limit which was L there. So, A plus epsilon plus epsilon and this epsilon will have the property that epsilon goes to 0 as delta x goes to 0. So, this concept we will also use later on in many lectures.

So, having this now out of this limit we have in introduce this epsilon which has this property that it will go to 0 as delta x goes to 0, because we know that this limit of this quotient is nothing but A. So, this epsilon must be 0 as delta x goes to 0.

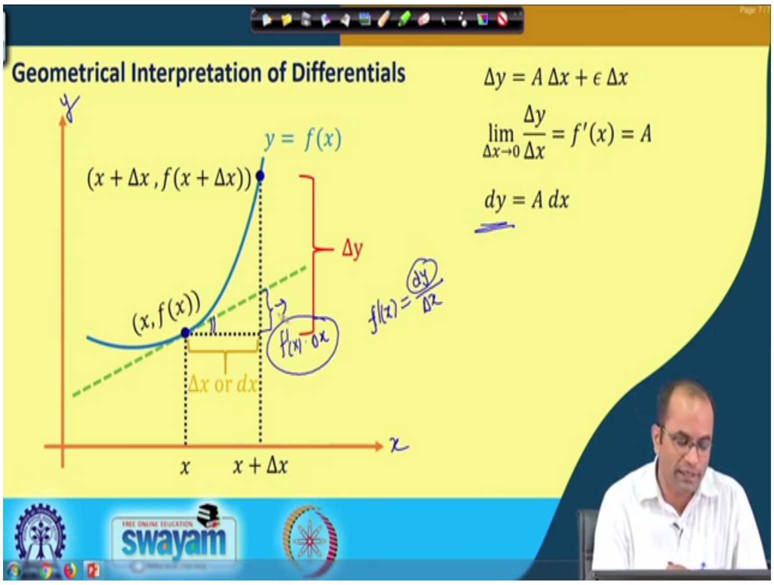
(Refer Slide Time: 12:10)



So, in this case now we can rewrite this as f x plus delta x minus f x and this delta x we can take to the right hand side. So, we get A into this delta x and plus epsilon into delta x and epsilon has this property that it goes to 0 as delta x goes to 0. And, now this implies because we got this expression here that this difference or this is we can also call like delta y. So, the delta y we can write down as A into delta x plus epsilon delta x and this A was independent of delta x because this was the limiting situation here of A. So, A was free from delta x, and now we got that this f is differentiable because that was the definition of the differentiability. So, this implies that f is differentiable. So, what we have learnt now that the differential of the function which we have introduced as dy which was this term A into delta x.

So, this differential of the function is the product of its derivative because A is nothing, but the derivative of this f and the arbitrary increment delta x. So, this term here is delta y sorry dy. So, the dy which is called differential is nothing, but the product of its derivative and the increment delta x of this independent variable or we can write down simply that dy is nothing, but the f prime x and delta x. What else we have learned that having the derivative or having the differentiability they are basically the same concept when we are talking about function of one variable. Well so, moving next to the geometrical interpretation of differentials.

(Refer Slide Time: 14:05)



So, we are talking about the differential. So, what do we mean here in geometry now? So, what we have introduced already that the delta y when the function is differentiable then delta y we can write down as A time delta x plus epsilon times delta x and we have also seen that when we take this limit so, we got that this A is nothing, but the f prime x. This is also we have observed and this we have introduced that the differential y is nothing but the A times dx or A is nothing, but the f prime so, f prime dx.

Now, we will see here what do we mean by this increment and this differential in terms of the geometry. So, let us consider here we have the function y so, this is x axis and this is your y axis. So, we have this curve y is equal to f x or the function of one variable and let us take a point x here so, the value on this curve. So, the point here is x and fx so, this value here the height is f x.

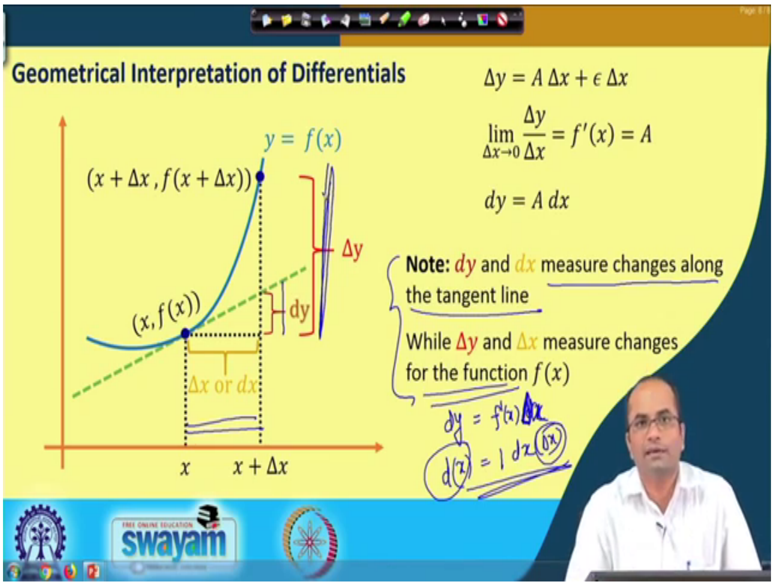
So, this point is x into f x and then we make an increment in x by delta x. So, we have another point here x plus delta x and we take the value of the function as f x plus delta x. So, this point becomes x plus delta x because of this x coordinate and the y coordinate. So, it is the value of this function at this point f x plus delta x.

Now, we draw an tangent here because we are talking about the derivatives so, that we will come into the pictures. So, we draw a tangent at this point x which is denoted by this dotted line and then this is naturally delta x the increment we have given in x. So, this is delta x or dx term, we will come to this point why we have written this dx and then this is the increment in delta y. So, when we have made an increment in delta in x by delta x then this is the change in delta y.

So, that is a change in delta y because earlier the y was this one and now the y has become this one here at this point. So, now, the difference here is the delta y or the increment in delta y this one and what is this dy, dy is this first derivative here f x f prime x and d x. So, if this is the derivative so, the tangent f prime x is nothing, but the tangent of this angle. So, that is equal to this divided by this one.

So, here this distance from here to here will be nothing, but the f prime into this delta x because in this triangle you can figure out that this tangent of this angle here is this f prime at this point x. And, this will be equal to this distance which we will call as d y and is equal to this delta x. So, this d y is nothing, but delta x f x or this distance here is f prime x into delta x. So, this distance is f prime x into delta x this is what we are calling as d y. So, let me erase this.

(Refer Slide Time: 17:50)



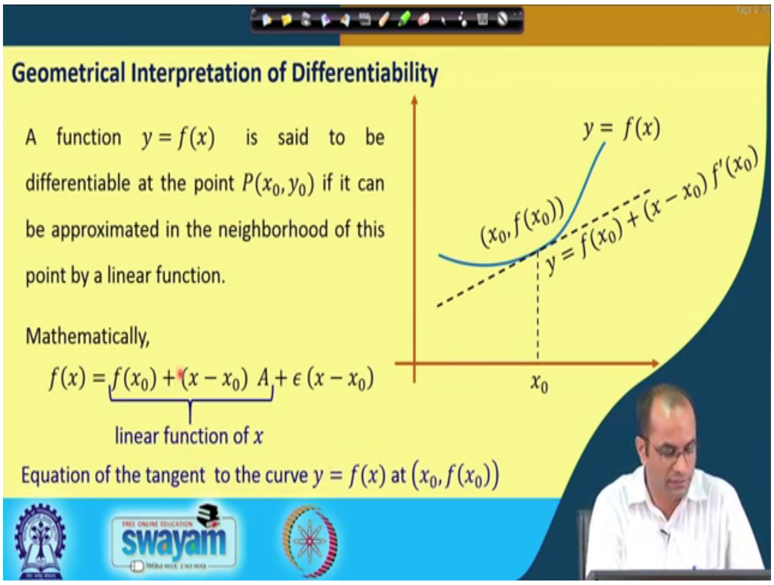
So, this is dy in our definition. Now, one can observe that this was delta y which was the increment in y increment in the function when we made an increment in x by delta x and this is dy the differential of y in our definition. So, what do we observe here that this delta x and the delta y are the changes in the function when we changes x by delta x then there was a change of delta y in the function and this d y was the change in the tangent.

So, along this is change along the tangent and this is changed along the along the function itself when we make an increment in x by delta x. So, we have also noted here that d y and d x d y and this d x measure changes along the tangent line. So, this is here along the tangent line; we have made the change here by delta x or we can call it d x both are same and here that is the change in the tangent line with dy. So, that is the notation here dx and dy; on the other hand the delta y and delta x measure changes for the function f x.

So, here when we make an increment delta x then there was a increment delta y in the function y. So, this is we have denoted delta y and this delta x and delta or dx they are the same because change in the x whether we denote by this delta x or delta y they are the same. But, along this y axis this is the change in the function and this is the change in the tangent line of the function. So, there is a difference or from the formula itself we can observe because we have this definition dy is equal to the derivative of this f prime x and dx and if we substitute for y is equal to x suppose y is equal to x.

So, y is equal to x and then the derivative will be 1 into dx or delta x; so, because this was originally taken as the linear term which was A times the delta x. So, originally this was defined as a f prime x delta x. So, the dx will become just the delta x or we can understood from this note that dy and dx we take the notation the measure changes along the tangent line. So, we have the dx and we have the del dy and then we have again this delta x and then we have here delta y ok.

(Refer Slide Time: 20:59)



So, now we have the geometrical interpretation for the differentiability again just though we have rewritten that definition. So, a function y is equal to f x is said to be differentiable at the point x naught y naught; if it can be approximated in the neighborhood of this point by a linear function. So, this is another interpretation of the differentiability we have written earlier in terms of that linear expression and then epsilon delta x.

So, here also we have another interpretation that this function is said to be differentiable, if it can be approximated in the neighborhood of this point by the linear function. So, what do we mean by this mathematically that fx, if we can approximate this function in the neighborhood of this x naught y naught point by the linear function. So, this is the linear function here f x naught plus x minus x naught A plus epsilon and again this x minus x naught. So, this is a similar definition what we have earlier.

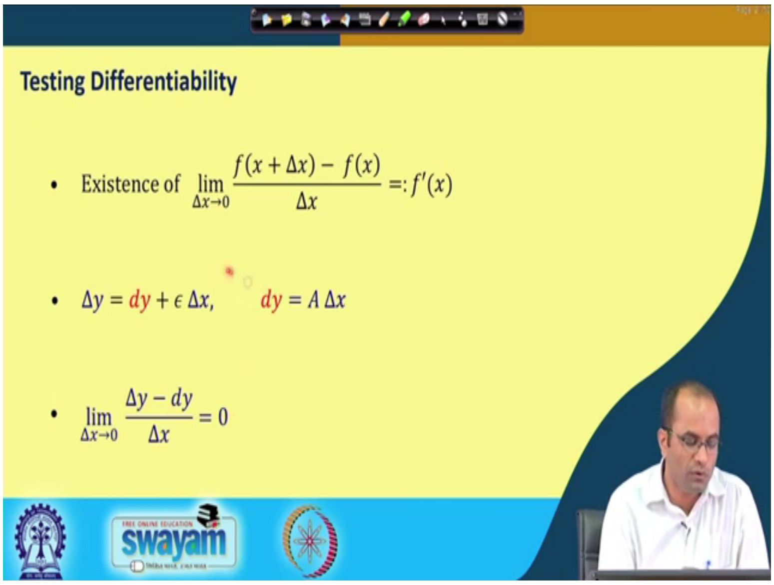
So, if we bring this f x naught to the left hand side this will become like delta y and we have here delta x here and A terms plus epsilon and again delta x the change in x. So, but here now we are talking about that if we can express this f x by a linear function in the neighborhood of x naught y naught point. So, what does that mean? This is the linear function and plus we have epsilon and this x minus x naught note that this epsilon goes to 0 x as x goes to x naught. So, this term is pretty smaller because this is also going to 0 and this is also going to 0.

So, we have somehow the quadratic term in terms of x the difference x minus x naught and we have the linear term here in terms of x minus x naught. So that means, this function we can approximate by this linear function at least in the neighborhood of this point x naught. So, this is a linear function of x and the equation of the tangent of the curve this is nothing, but the equation of the tangent at the point x naught f x naught of this function y is equal to or curve y is equal to f x.

So, again like this is x axis y x is here and then we have some curve y is equal to f x and at this point x naught this is a point x naught f x naught, if you draw the tangent then what we want to say here that if the function is differentiable then we can approximate in the neighborhood by a linear function. And, the accuracy of this linear approximation will depend that where we take x for example, if x is this point this is very well approximated and if it is a far point from this x naught then we are not approximating well.

So, at least in the neighborhood of this point if we can approximate by a linear function then we call this function as differentiable or equivalently or mathematically, if we can write down this f x in terms of the linear function and plus the epsilon and this is the difference x minus x naught and this epsilon also goes to 0. So, then this is not the linear term in terms of this difference it is a non-linear term here. So, we have this linear term plus this non-linear term and this is the equation of the tangent which we have written down in this case.

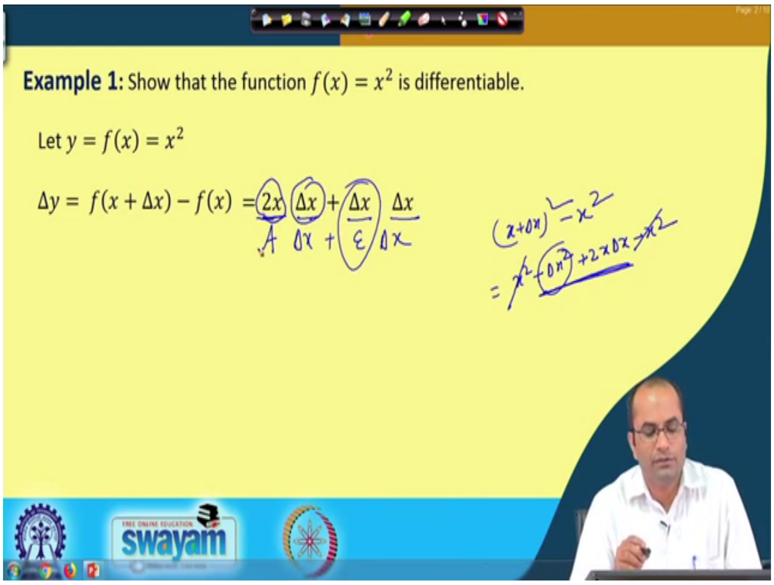
(Refer Slide Time: 24:41)



Now, the testing of the differentiability what we have learned so far. So, we have basically the 3 concept one was the existence of this derivative which is evaluated by this quotient and taking the limit. So, if this limit exists we call that the function has derivative or the function is differentiable because that was equivalent to the differentiability of the function. And, we have denoted this by f prime x or d y d x or y prime. The second concept we have seen that the delta y is equal to d y plus epsilon delta x that we can write down this expression in terms of dy and epsilon delta x and the dy was A into delta x the increment and this is called the differential term.

So, this is another way of testing differentiability that we can write down this delta y in this form; the third one was which is which can be deducted from this term itself. So, the delta y and we take this d y to the left and divided by this delta x and then take the limit since this epsilon goes to 0 as delta x goes to 0. So, this expression must be 0. So, we can use either this one to test the differentiability that this quotient must be 0 or we can use this one or we can use the derivative one.

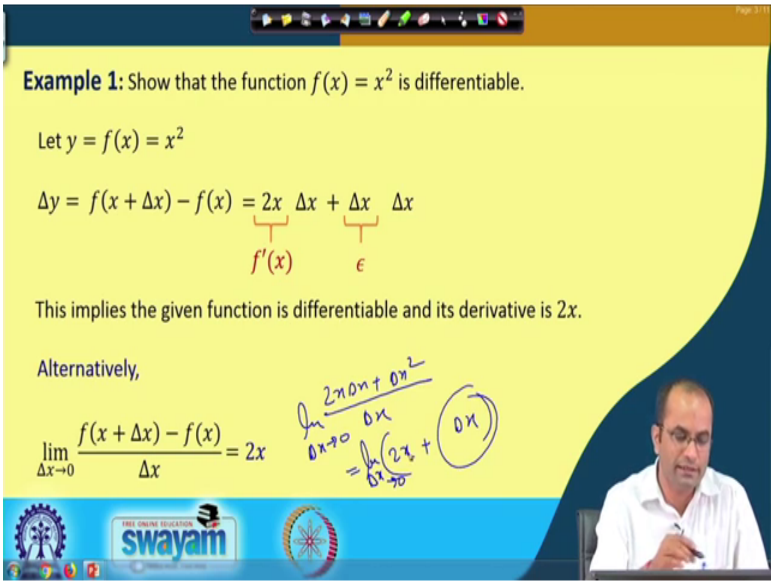
(Refer Slide Time: 26:03)



We have the 3 concepts. So, the example 1 we will show now that the function fx is equal to x square is differentiable its a pretty simple. So, we have y is equal to x square and then the delta y; that means, f x plus delta x minus f x this will be equal to so, here fx plus delta x; that means, x plus delta x whole square and minus f x. So, minus x square if I expand this you will get x square plus delta x square plus 2 times x delta x minus x square.

So, this term get cancelled and we will get a 2 x into delta x and delta x square. So, we get precisely here 2 x delta x delta x and into delta x. So, we have this form this is A times delta x and the epsilon times delta x. So, here this epsilon naturally it goes to a 0 as delta x goes to 0 here this term 2 x is A, the delta x is given. So, we have that format which we have used for the differentiability.

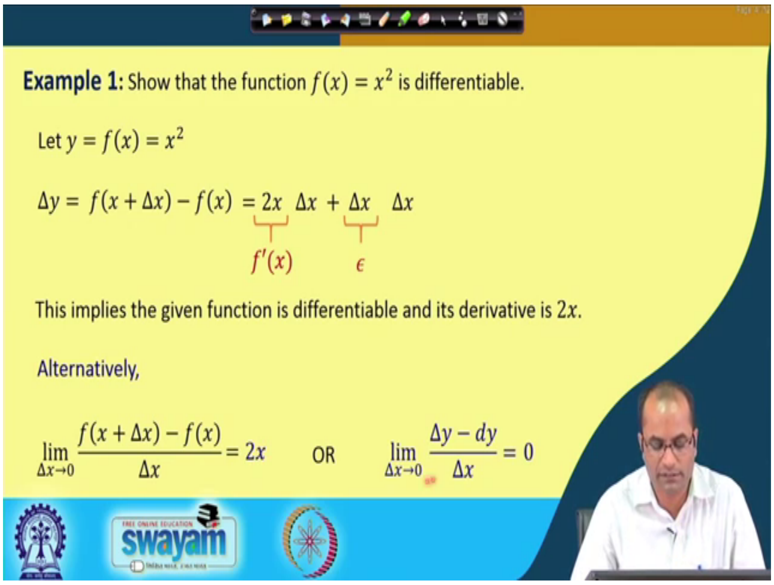
(Refer Slide Time: 27:11)



That means this function is differentiable and the derivative value which is this A the term sitting with this delta x in the linear part is f prime x and this is epsilon. So, this implies that the function is differentiable and its derivative is 2 x; alternatively we can show that this quotient here f x plus delta x minus f x over delta x by taking this limit delta x goes to 0.

So, again here f x minus f delta x is given already there. So, we have the 2 x delta x and plus delta x square term there and divided by delta x and then we will take the limit as delta x goes to 0. So, we have here this 2 x term and plus the delta x term and then we take the limit as delta x goes to 0. So, this will go to 0 and we will get only 2 x. So, we have this showing that these limits exist for whatever for any value of x.

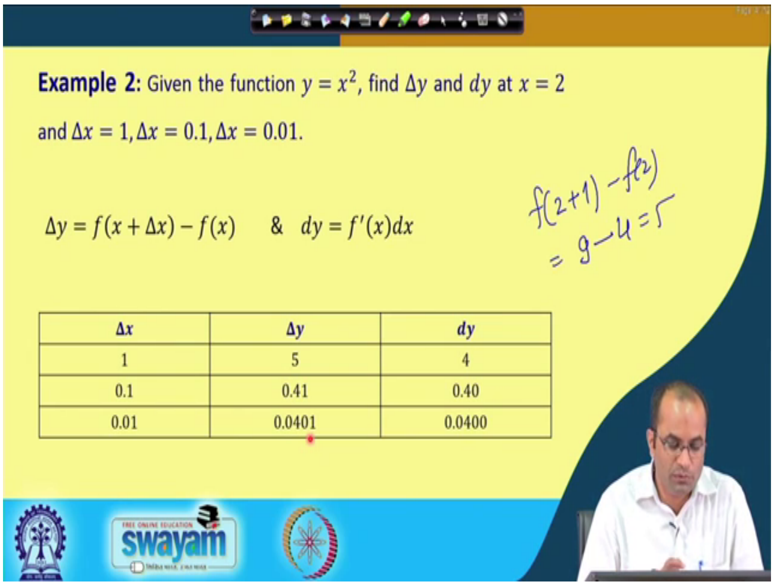
(Refer Slide Time: 28:13)



So, the function is differentiable in the whole domain or we can also show that this here the ratio. So, when delta y minus dy if we take so, this is delta y and minus this was dy. So, we take this will be delta x square and when we divide by delta x. So, we will have delta x only and taking this limit delta x goes to 0, this will go to 0.

So, that was another the third concept of showing the differentiability either by expressing this delta y in this format or alternatively taking this quotient here and limit if this exists or this one.

(Refer Slide Time: 28:50)



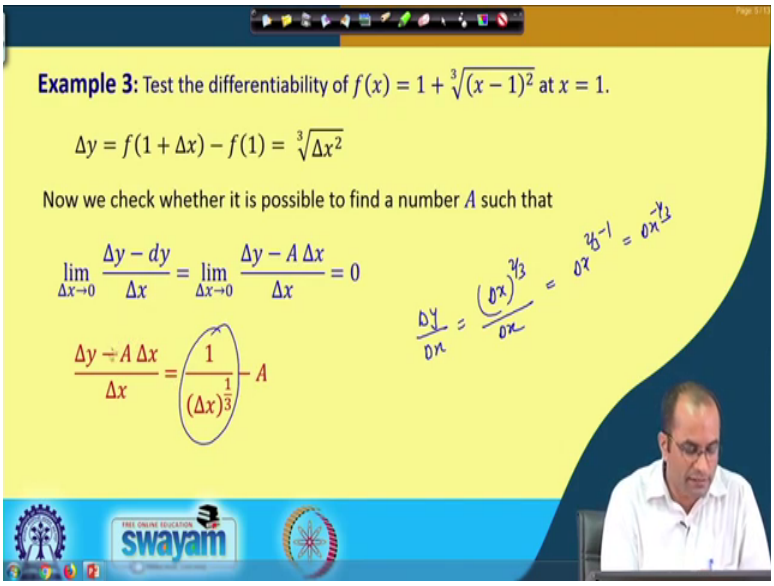
So, here this example shows that y is equal to x square we find delta y and dy at x is equal to 2 when delta x is equal to 1 delta x is equal to 0.1 and delta x is equal to 0.01. So, the point here x is equal to 2 and these are the increment. So, we have the delta y is equal to f x plus delta x minus f x and the d y is given as f prime dx.

So, if you make this table here. So, for example, this value we can evaluate here delta y delta y is f the x is 2 plus delta x we take first as 1 so, 1 and minus this f 2. So, here this is f 3 and x x square so, 9 and f 2. So, x square this will become 4. So, this value is 5 and so, here it is 5 and now d y. So, simply from here f prime is 2 x. So, 2 x into dx or dx is delta x is the same. So, here 2 and this 2 x so, this will be 4 and dx is 1. So, we will get 4.

Similarly, while taking delta x we will get this 0.41 and dy will be 0.40 and while taking a smaller delta x we will get delta y as 0.0401 and here we will get 0.0400. So, in this case what we have observe when delta x is large; obviously, these 2 the delta y and dy they are not the same. So, the difference is large between the 2 its it is not approximating this dy is not approximating this total change in y, but when the when this delta x is smaller. So, if you make a smaller increment then this dy is well approximating this delta y term and that was the meaning of that, if we can approximate by the linear term at least in the neighborhood of the point.

We can also test the differentiability of this function fx is equal to x square 1 plus the cube root of x minus 1 whole square at x is equal to 1.

(Refer Slide Time: 31:05)

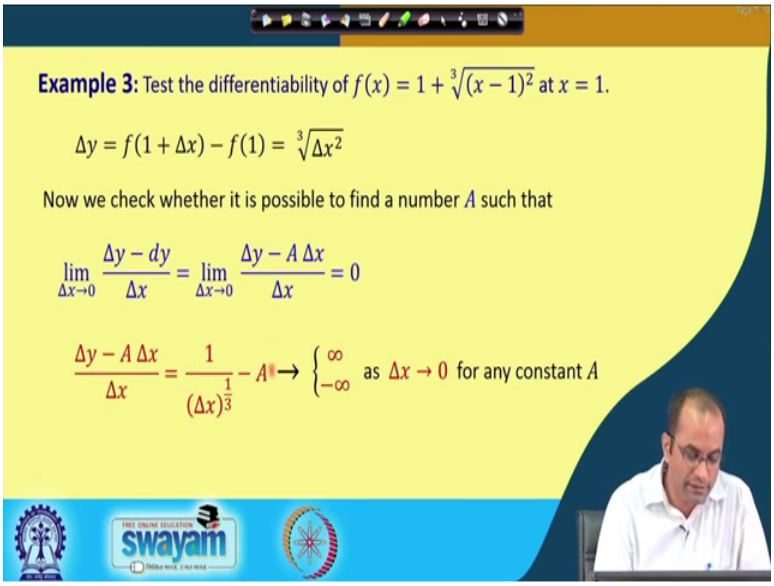


So, this function so, if we take this delta y as this difference because at x is equal to 1 so, 1 plus delta x minus f 1. So, 1 plus delta x we will submit here. So, it will be like 1 plus and this will become delta x square and the cube root minus f 1 will be 1.

So, we will get only the cube root of delta x square and now we will check whether it is possible to find a number A such that this quotient here which we have just talked, that if we can get this quotient and the limit is 0 then we call the function is differentiable. So, we will check whether we can find such a A. So, that this quotient here the dy is A into delta x over dx over delta x, if this quotient is 0 then we call the function differentiable and if we cannot find such a A then we will call the function is not differentiable.

So, we will take this quotient here delta y minus A delta x over delta x. So, we can divide by this delta x so, we will get A there. So, you have delta y over delta x what is delta y over delta x. So, delta y is here cube root delta x square and when we divide by this delta x so, here its power 2 by 3rd and this will be minus. So, let me just compute this one, this term here delta y over delta x term delta y was delta x two-third and then we have delta x here. So, this will be delta x and two-third minus 1 so, delta x power minus 1 by 3. So, this is here and then minus A in this case and we take.

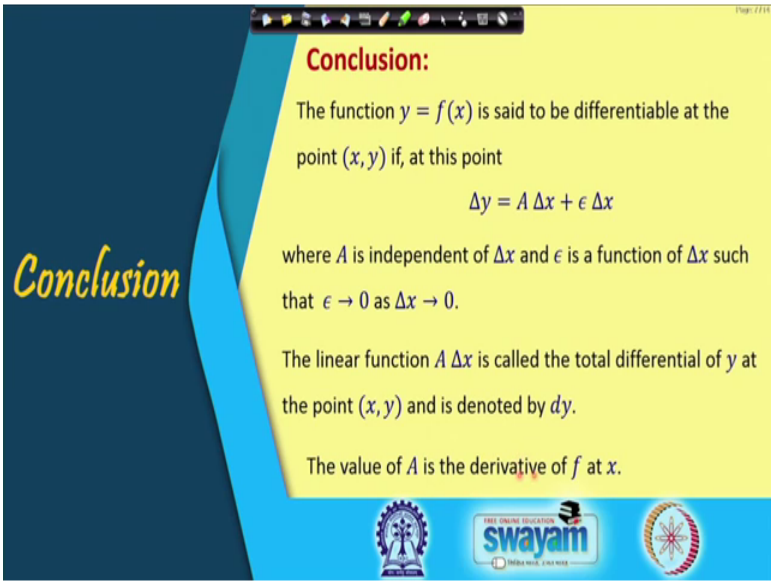
(Refer Slide Time: 32:53)



Now, what we observe that this function here is approaching to plus infinity when delta x is approaching to 0 from the right side for example, when delta x is positive or this is approaching to minus infinity when this delta x we are approaching from the left side when delta x is negative and approaching to 0. So, in that case this limit whatever A we choose this will be plus infinity when delta x approaches from the right side when delta x approaches from the left side this will become minus infinity irrespective of this number A.

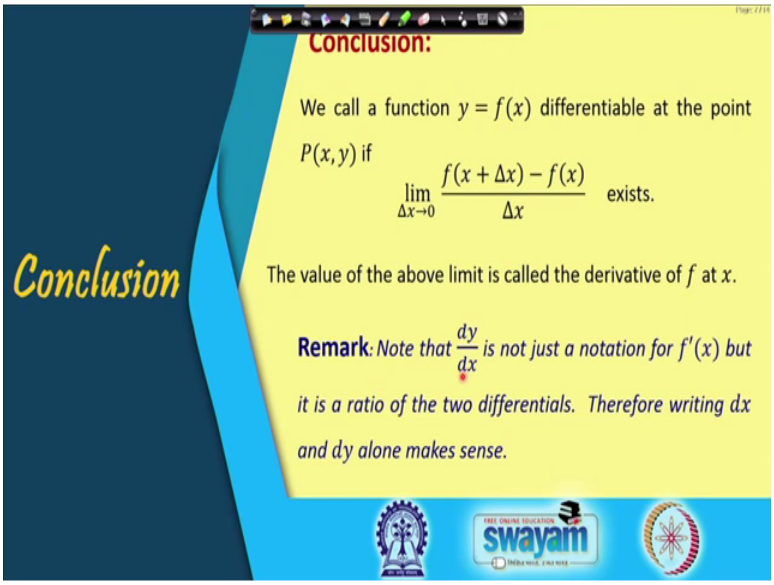
So, meaning that we cannot get such a constant so, that this quotient has a finite limit oh sorry the 0 limit this should be 0 for the differentiability. So, we do not have such a A a finite number A. So, that this quotient has the limit 0 as delta x approaches to 0. So, this limit in fact, does not exist. So, in this case the function is not differentiable at x is equal to A.

(Refer Slide Time: 33:59)



So, coming to the conclusion we have seen that this function is said to be differentiable, if we can write down delta y in terms of this expression A delta x plus epsilon delta x, where A is independent of delta x and epsilon is a function of delta x such that epsilon goes to 0 delta x goes to 0 this was one definition we have discussed. And this term A delta x is called the total differential of y at this point x y and is denoted by delta y and the value of A is nothing but it is the derivative of f at x.

(Refer Slide Time: 34:38)



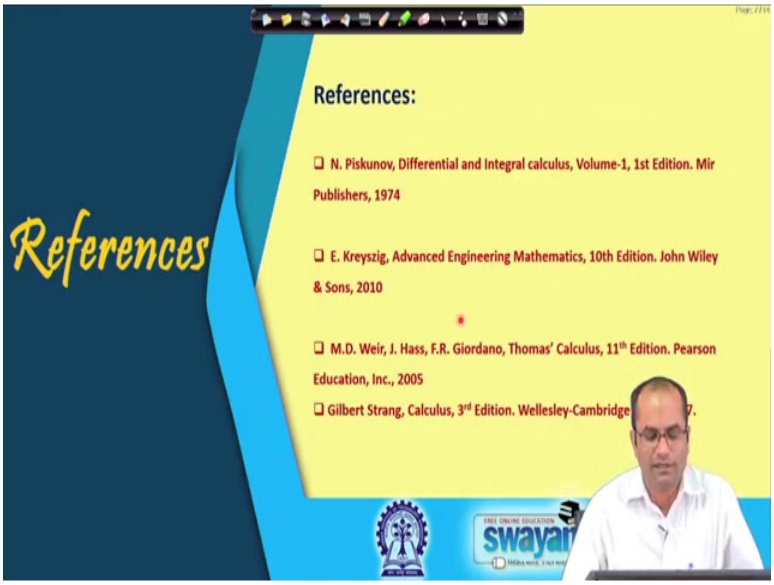
Again we call a function y is equal to f x differentiable if this quotient exist which we call derivative, but both the definitions were equivalent we have seen. And, what we also realized now introducing that differential d y is equal to f prime d x that this is not just a notation for f prime, but this ratio of the 2 differential because d y was f prime into d x makes sense.

So, writing this d y and d x separately makes sense and that we will also use now in the in the in the next lecture which will be on the differentiability of the two functions. And, very important that we first understood this differentiability of the function of one variable mainly these 3 concept having the derivative and then this linear expression and also writing that limit to 0 gives us differentiability.

So, there in case of the two variables what we will see that just having the derivatives, where the partial derivative with respect to x and partial derivatives derivative with respect to y will not help or will not be equivalent to two differentiability. The differentiability will be a different concept than having just the partial derivatives, though in case of one variable if we have the derivative of the function, then the function is differentiable or the other way around.

So, that will be in the next lecture..

(Refer Slide Time: 36:13)



And these are the references we have used to prepare these lecture.

Thank you very much.