Engineering Mathematics - I
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Lecture – 10
Partial Derivatives of Higher Order

Welcome to the lectures on Engineering Mathematics-I.

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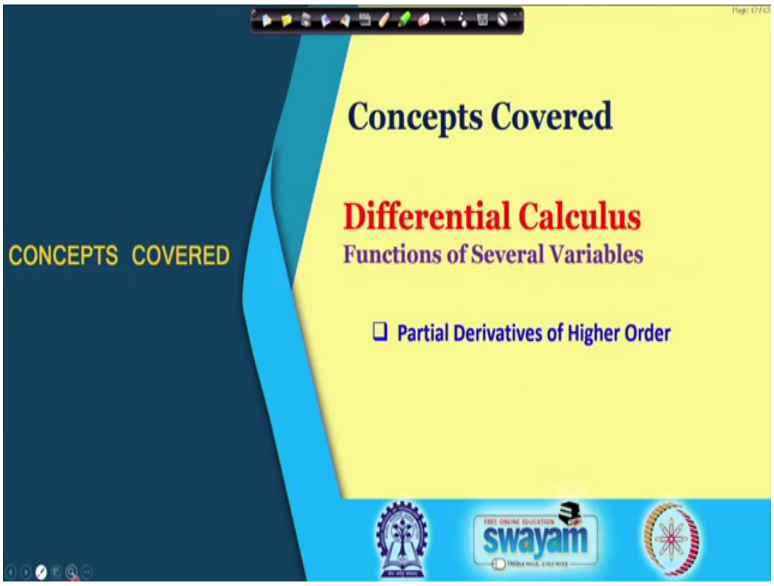

And this is lecture number 10 we will be talking about Partial Derivatives of Higher Order.

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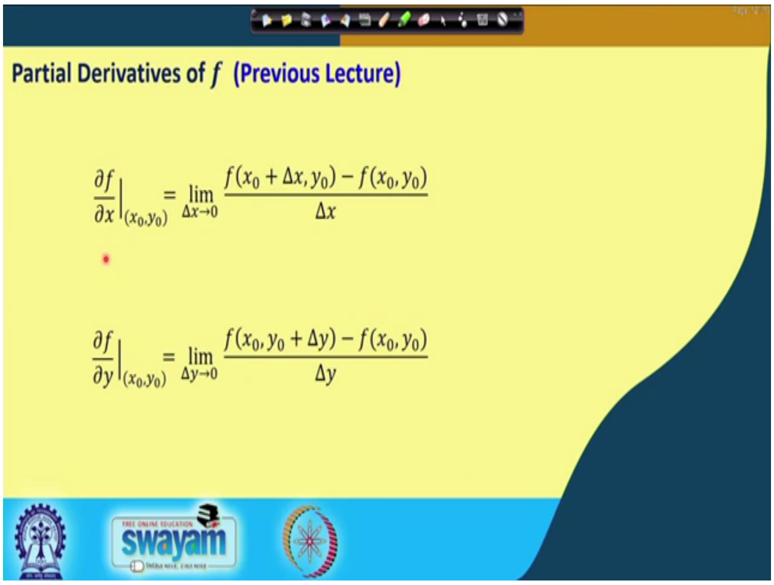

So, in the last lecture what we have seen the partial derivatives of f. So, basically the first order partial derivatives we have done in the last lecture. So, what was? It was the fundamental definition of the partial derivative with respect to x at the point x 0 y 0. So, the increment in x 0 because we are taking or we are talking about the partial derivatives with respect to x and the function value divided by the delta x increment and taking this limit if this limit exists. Then we have the partial derivative with respect to x the first order partial derivative.

And similarly we have the first order partial derivative with respect to y at this point x 0 y 0 when this limit exists. Now we will continue this for higher order derivatives. So, what will happen if we take for example, the derivative of f with respect to x 2 times?

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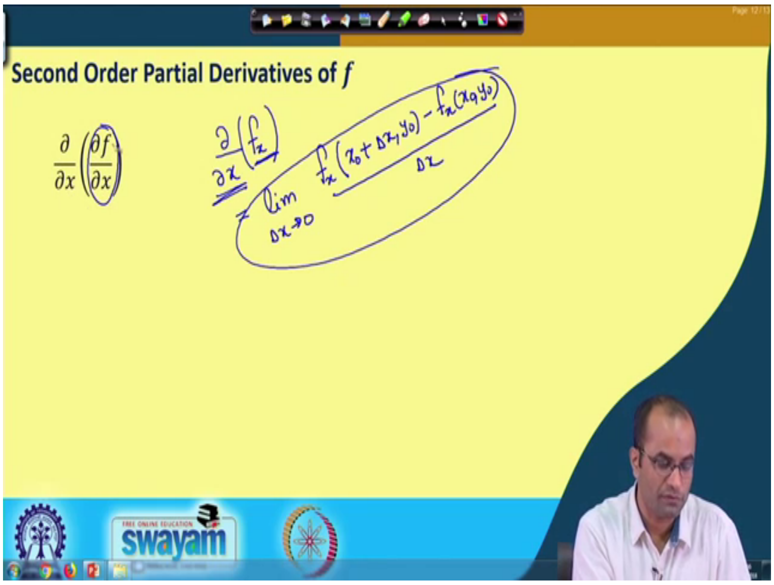

So, in this case this is the notation. So, we have the partial derivative with respect to x and for example, we want to take again the partial derivative with respect to x. So, the idea remains the same because we have this partial derivative with respect to x. So, this is another function and then we are taking partial derivative with respect to x of this function.

So, basically this is like del over del x. And then we have some other function which I am calling as f x which is basically the notation of this a partial derivatives. So, what we are now doing we are taking the partial derivatives of this function f x. And now the same definition what we have learnt already that the delta x goes to 0, we will be talking about this limit so our function here f x. And for example, if we take at x naught y naught points. So, here with respect to x means we will make an increment here in x and then y naught will remain and then we have the f x at x naught and y naught and divided by this delta x.

So, that will be the definition now of the second order derivative here with respect to x.

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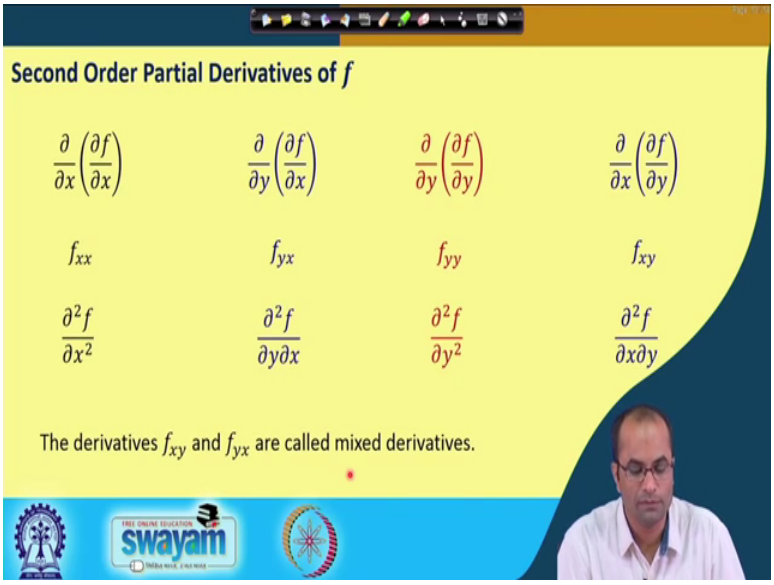

So, here the notation again: so we instead of this we also use f xx the derivative of f 2 times with respect to x or we also use this notation del 2 f over del x square. So, again the second order derivative a partial a derivative of f with respect to x. Similarly, we have the notation when we take the partial derivative of x and then we are taking the partial derivative with respect to y. So, in this case we use this notation f y x or del 2 f over del y del x. In some literature people use this as f x y or some other way around del x del y.

So, we will be following this notation the partial derivative with respect to x and then we take once again the partial derivative with respect to y. So, we will keep this order here yx and f yx and del 2 f del y del x. This means the first 3 partial derivative with respect to x and then with respect to y. So, similar notation for the 2 times partial derivative with respect to y which we will denote as here with respect to x here f yy or del 2 f over del y 2. Or we can have like first the partial derivative with respect to y and then we take the partial derivative with respect to x. So, this we will denote as f xy or del 2 f del x del y. So, this is just the notation. And these derivatives here f xy or f yx are called the mixed derivatives, because here we have used both the variables x and then with respect to y or first with respect to y then with respect to x. So, these are called the mixed derivatives.

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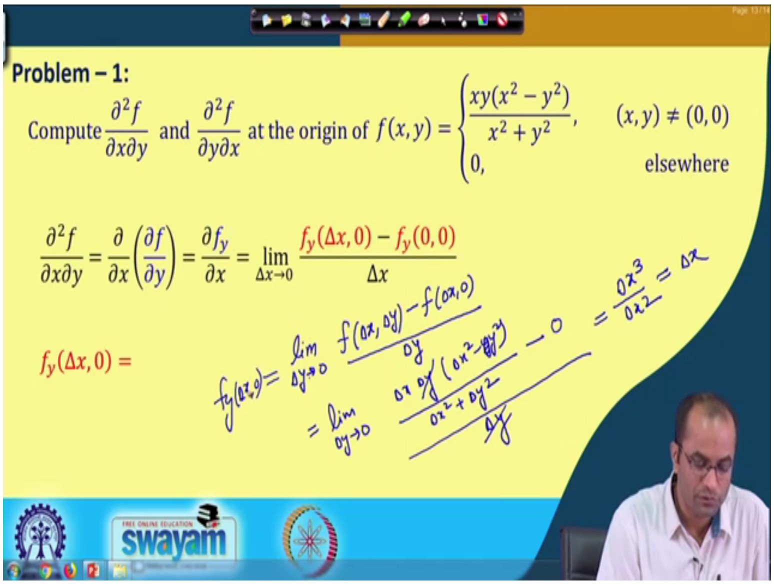

So, let us directly compute these mixed derivative del 2 f over del x del y and del 2 f over del y del x, at the origin of this function. So, as we have discussed the del 2 f over del x del y means that the partial derivative with respect to x of the partial derivative of with respect to y. So, in short we can denote this like f y. So, we have partial derivative f y and with respect to x. So, we want to get the partial derivative of this f y with respect to x. So, note that this f y is also a function of x and y. So, we can talk about this partial derivative with respect to x.

So, as per the definition again with respect to x, so a partial derivative with respect to x. So, we will take the limit delta x goes to 0, f y the function and we are talking about the derivative at origin, so 0 0. So, you have 0 plus delta x, which is delta x 0 minus f y and at 0 0. So now, for the function we have used f y. So, f y delta x 0 fy 0 0 over delta x. The same definition of the partial derivatives with respect to x: so instead of f we have used here f y, because we are taking the derivative with respect to x of the function f y.

So now, in this case we have to know what is f y delta x 0 then only we can use that f y delta x 0 here. And also we should know what is f y at 0 0. Then we can use that value here and compute this mixed order partial derivative. So, let us compute what is f y delta x 0. So, f y delta x 0: now here it is a partial derivative with respect to y. So, we have to use the definition now that delta y goes to 0 and of the function f. So now, we are using the function f, but the partial derivative with respect to y. So, the increment has to be in y argument. So, f the delta x 0: so the delta x will remain as it is because we are not touching delta x. We are taking the derivative with respect to y. So, this delta x will remain and here 0 plus delta y which I am writing just delta y and minus f the function value at delta x 0; so delta x and 0 and divided by then delta y, because we are taking the derivative with respect to delta y.

So now, if we compute this delta y goes to 0 f delta x delta y so we have to substitute there now. The delta x delta y and then we have delta x square minus delta y square and divided by delta x square delta y square minus f delta x and y 0. So, when y is 0 this will become 0 and we have here then this delta y term. So, this delta y again will get cancelled with this one and when we now take the limit delta y goes to 0. So, this term will become 0, this term will become 0. And what we will get delta x 3 there delta x and delta x square will become delta x 3 over this delta x square. So, this delta y when we have taken the limit this is 0.

This is 0 and we got delta x cube over delta x square which is delta x. So, this was this f y and delta x 0. So, this we have computed as delta x. So, let us just use this.

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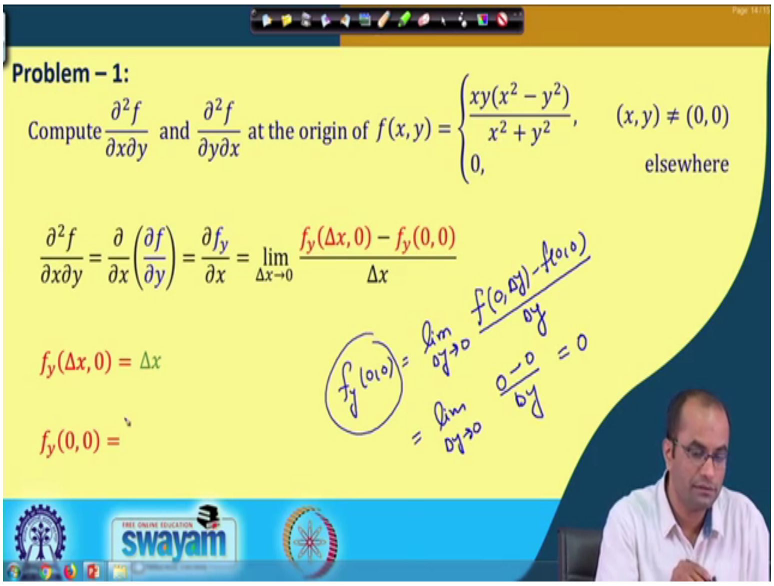

So, this is delta x. Now we have to compute f y 0 0. So, what is f y 0 0? So, f y 0 0 again we have to use the definition. So, delta y goes to 0 and then we have the function f we are taking the partial derivative of the function f so with respect to y. So, the 0 will remain as it is so 0 plus delta y. So, here we have delta y minus this f 0 0 and divided by delta y.

So, this limit delta y goes to 0. And this one argument is 0 and we have the product. So, this will become 0 and minus f 0 0 is 0 that is given and delta y. So, this will be 0. So, the partial derivative with respect to y at 0 0 is 0. So, this is here 0.

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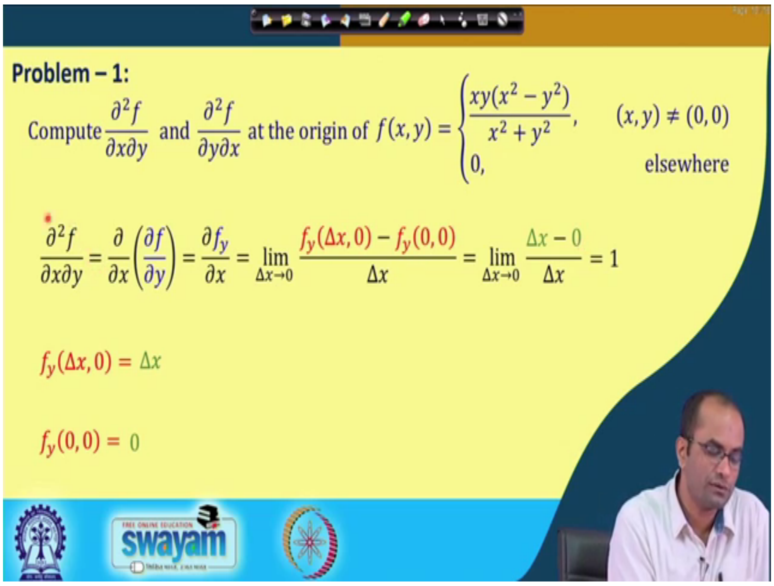

So, we got these 2. So, f y delta x 0 0 or delta x comma 0 is delta x and here f y 0 0 is 0. So, we can substitute now: delta x minus 0 over delta x and this delta x over delta x will become 1. So, we have this limit as 1. So, we got this paths of mixed order partial derivative of this order del x del y as 1.

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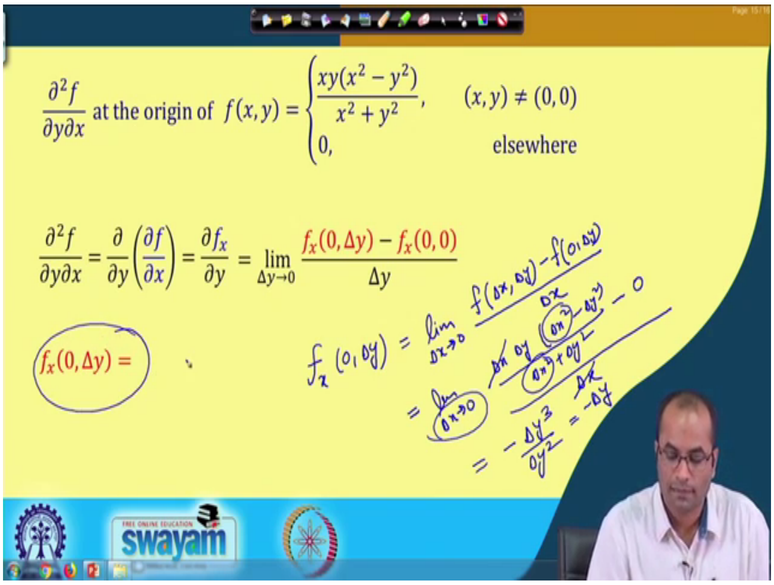

Similarly, now we will compute the second order partial derivative with respect to y and then with respect to x.

So, in this case we have the order now del over del y of del f over del x; that means, the partial derivative with respect to y of the function f x. So, in this case again we use the definition. So, delta y goes to 0 this delta y and we have this function f x. So f x 0, because we are taking the partial derivative with respect to y, so we will get increment in y so 0 plus delta y which is delta y minus f x 0 0 over delta y. So now, as in the previous slide we have to compute this f x 0 delta y and f x 0 0 instead of f y earlier. So, what is this derivative here f x 0 delta y. So, again the limit has to be for delta x because we are taking the derivative with respect to x and f here the increment. So, 0 plus delta x the delta x and delta y as it is minus then we have f and 0 delta y divided by delta x. We are taking the derivative with respect to x.

So, here the limit delta x goes to 0 and this will be again delta x delta y and this term delta x square minus delta y square divided by delta x square plus delta y square minus that will become 0 and then we have delta x here. So, in this case this delta x and delta x will be cancelled, and we are taking the limit delta x goes to 0. So, this term will go to 0 and this term will go to 0 and when we will get here minus delta y cube over delta y square, which will be minus delta y.

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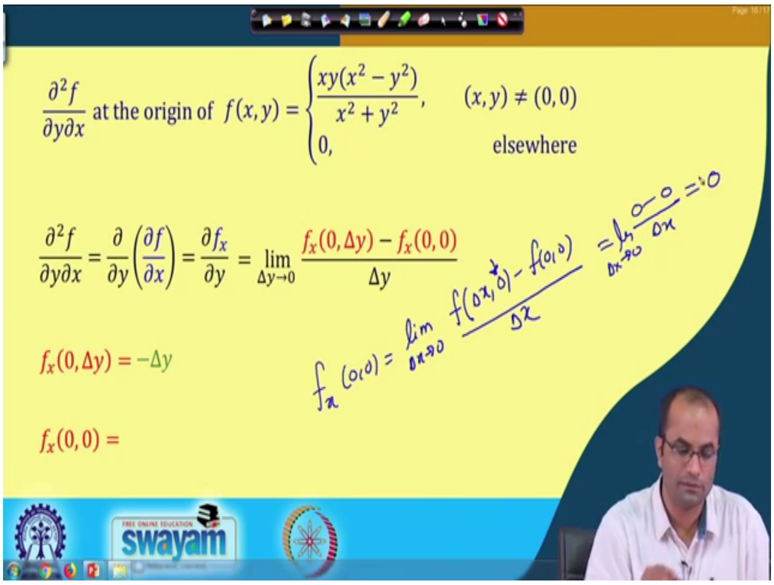

So now this derivative here is minus delta y minus delta y and then fx at 0 0. So, this fx at 0 0. We have to use again this definition. So, the limit delta x goes to 0 and then we have f delta x 0 minus f 0 0 over delta x. And since one of the argument here is 0, because of this product this will become 0 minus 0 over delta x and the limit delta x to 0. So, we have here again this 0. So, we got this 0. Now we will substitute here.

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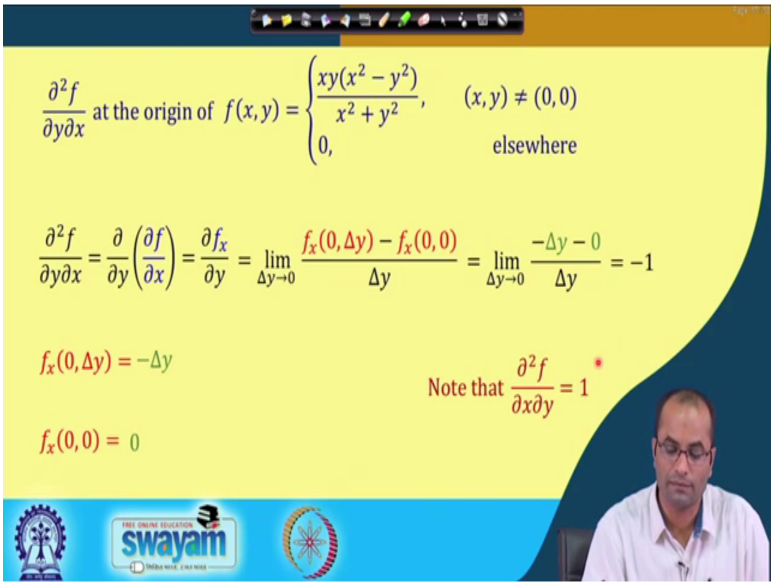

So, minus delta y minus 0 over delta y and this limit this delta y delta y gets cancelled and we will get minus 1. Note that: the earlier we got in the previous slide del 2 f over del x del y which was 1, and now we got del 2 f over del y del x which is minus 1. So, what we have observed that these 2 partial derivatives first with respect to y and then with respect to x or first with respect to x and then with respect to y they may not be equal. Like in this present example here it is value minus 1 and here the value is 1.

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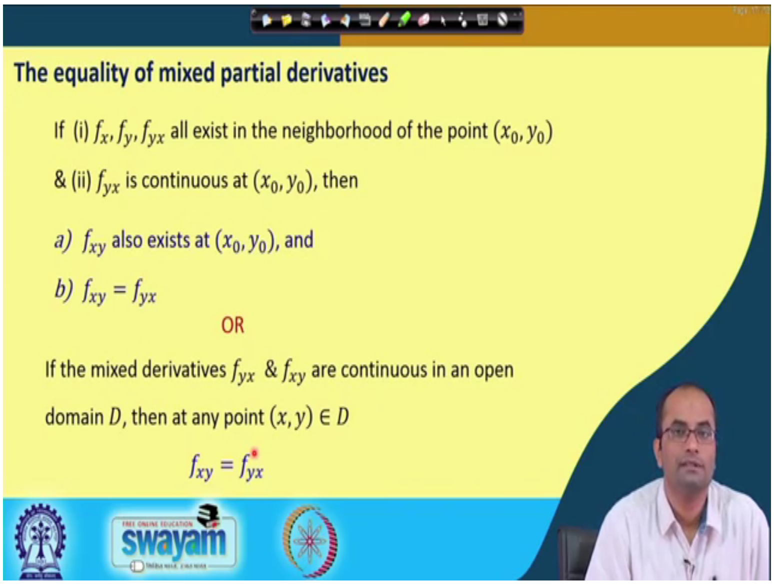

So, when the equality of this mixed partial derivatives happen. So, we have this (Refer Time: 14:02) if f x f y and the f y x all exist in the neighborhood of this point. So, not only at this point, but also in the neighborhood of this point all these 3 exist and this on the top here f y x is also continuous at that point 0 0. In that case we have the result that f xy will also exist at this point and it will be equal to the value of f y x. So, basically the continuity plays here major role of this f y x or we can do this result for x; x y. Or there is another parallel result here.

If the mixed derivatives this f y x and f xy it is easy to remember. So, when the equality happens when these 2 derivatives f y x and f xy are continuous in an open domain d. Then at any point in the domain we have the partial derivatives equal; this mixed order partial derivatives equal. So, in the previous example definitely those partial derivatives were not equal. Otherwise we would have got the equality there that f xy is equal to f y x, because that is a sufficient condition. So, if these derivatives are continuous second order partial derivatives are continuous then they will be equal.

So, what we can conclude from here, if they are not equal, if they are not equal; that means, the derivatives were not continuous because if the derivatives were continuous then they has to be equal. So, if these derivatives are not equal this mixed order derivatives are not equal, then we can conclude that the partial derivatives these mixed order partial derivatives are not continuous.

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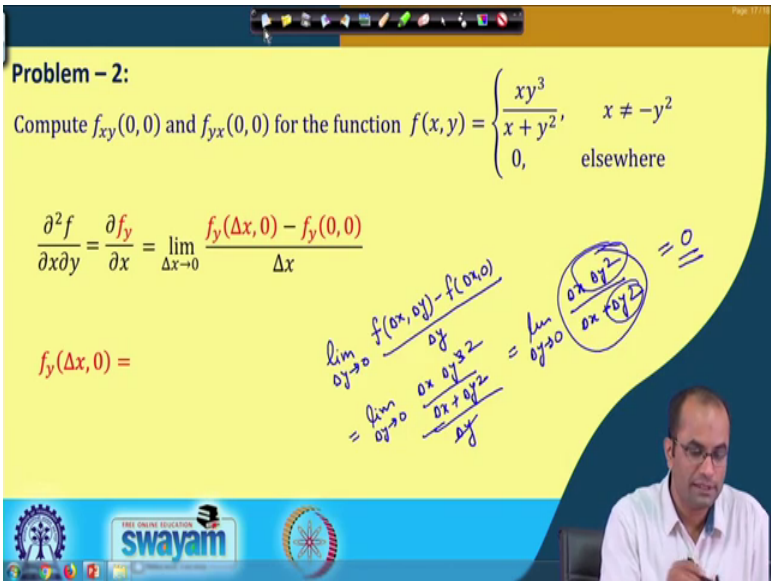

So, let us see in this example. So, we will compute first the fx y and f y x for this problem x y q over x plus y square, when x not equal to because otherwise this will not be defined here and elsewhere the value is 0.

So, we compute the second order partial derivative with respect to x of the function f y. As per the definition we have f y delta x 0 and f y 0 0 over delta x. And similar to earlier example we need to compute now this f y delta x which is again this will be the limit here and the delta y should go to 0. And f we are talking about delta y. So, the delta x will not change and we have the 0 plus delta y then we have f delta x comma 0 and divided by delta y. So, what is this limit? So, if we take this limit here delta y goes to 0 here it will become delta x and delta y cube from here and then divided by delta x and delta y square minus.

So, again this y is 0. So, this will become 0 and delta y. So, this delta y will be cancelled here, and then delta y if we are taking to 0 because this is let us write down again delta y to 0. So, we have delta x delta y square divided by delta x plus delta y square. So, when delta y goes to 0. So, this term goes to 0 and here this goes to 0. So, you have this limit as 0. So, in this case this is 0.

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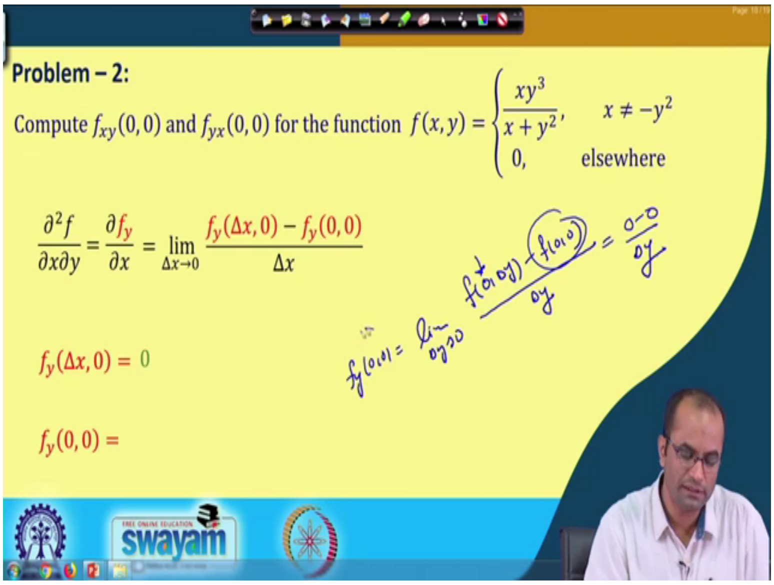

And f y 0 0 will be let us compute here. So, f y 0 0 is the limit delta y to 0. And then we have f, so 0 delta y minus f 0 0 and divided by delta y. So, again in this case here it is 0, and this is also 0, so 0 minus 0 over delta y. So, this partial derivative at 0 0 is 0.

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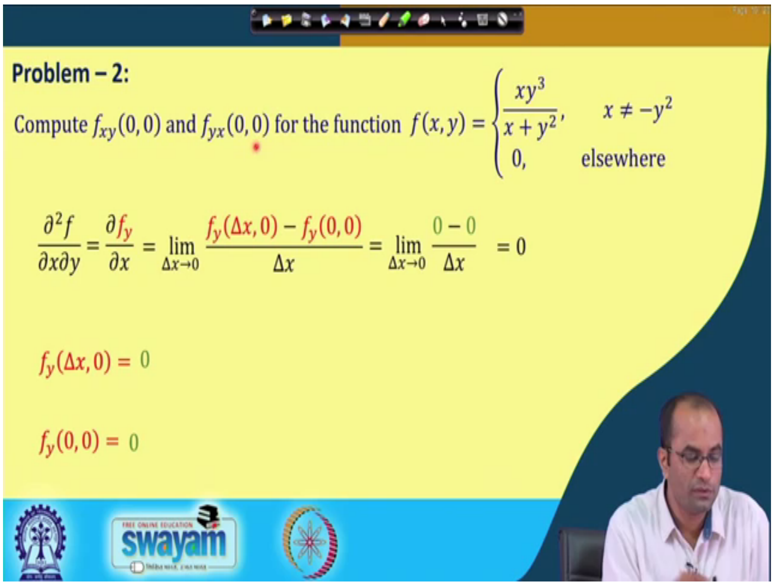

So, in this case we got this 0 both are 0. So, 0 minus 0 divided by delta x. So, in this case we got this partial or a second order derivative as 0. Now you will compute the other one f y x of this function.

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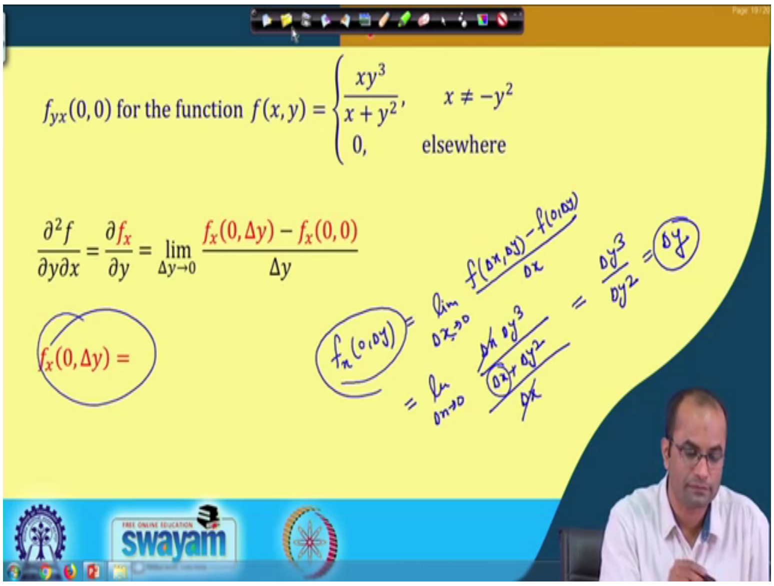

So, here now the partial derivative of y with respect to the y of this function f x again the definition. So, increment in y and then. So, we need to compute again this partial derivatives here at 0 delta y and at 0 0. So, if you compute this at 0 y. So, in this case what will happen? So, we are computing now 0 delta y.

Let us take the limit as delta y to 0 sorry delta x to 0, not delta y to 0 this will be delta x to 0. And we have f. So, the increment in x delta y as it is the function as it is now divided by this increment delta x. So, this will be the partial derivative with respect to x. So, the limit delta x goes to 0. So, this will be delta x delta y cube over delta x and delta y square. And minus this will become 0 and then we have delta x there. So, this delta x will get canceled. And now we will let delta x to 0. So, this term will become 0 and we will get delta y cube over delta y square. So, in this case we are getting delta y this derivative here, we are getting this delta y.

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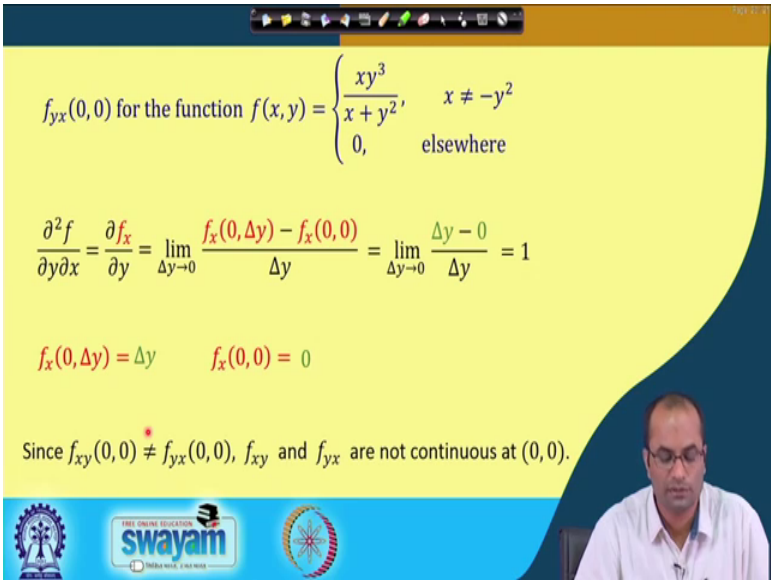

So, in this case: and now f x at 0 0, so this will become 0 because of this product there, so no need to compute again. And now we have delta y for this fx 0 delta y and minus the 0 term f x 0 0 divided by delta y. So, in this case now delta y divided by delta y, so this will become as 1. So, this f xy 0 comma 0 comma 0at these 2 derivatives again at this origin are not equal. So that means, they are not continuous also at 0 0 because we have seen if the derivatives these derivatives are continuous then we will get the equality there.

So, there they are certainly not continuous which we can check.

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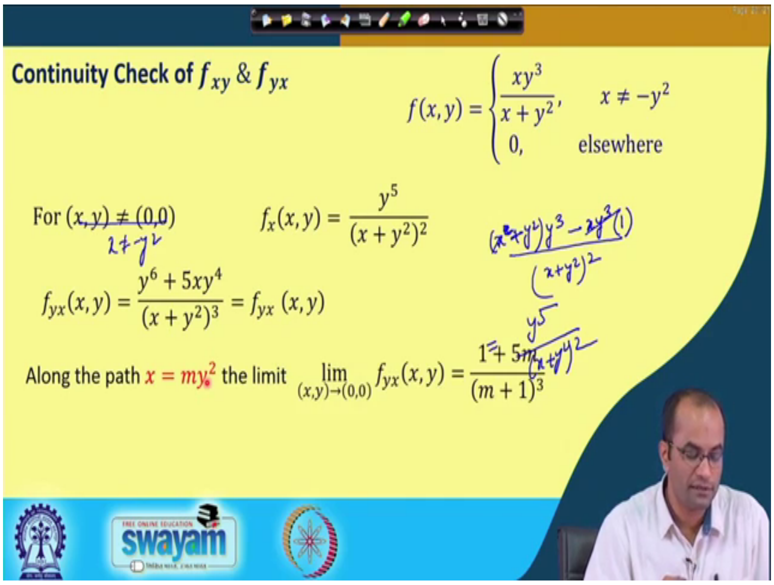

So, it is a continuity check of these 2. So, what to do to con to check the continuity we have to get the these mixed derivatives when x is not equal to minus y square and also elsewhere the both the places. So, here for x y not equal to origin if we compute or rather we will say when x is not equal to. So, here instead of this we will call as x not equal to minus y. So, at all these points where there is no problem. We can just compute this derivative by keeping y as constant.

So, from here if you want to take this derivative from this f x y with respect to x what will happen; so here x plus y square the constant rule the whole square will come. And this term sorry x plus y square and the derivative of this with respect to x will be y cube minus this x y cube and the derivative here with respect to x will be 1 ok. So, what will be this xy cube and xy cube will be canceled and then we have y power 5 over x plus y square whole square. So, exactly this is the term here. So, the partial derivative of f with respect to x we are getting y 5 over x plus y square whole square.

And now, if we compute now again the partial derivative of this with respect to y for those points where we have we do not have any problem when x is not equal to y square. So now we will again differentiate this function with respect to y keeping x constant. So, doing exactly what we have done there now the function is this one. We will get here y 6 plus 5 x y 4 over x plus y square cube. When we take the derivative of this one with respect to y keeping x as constant we will get this term here. So, we have this partial derivative f y x at a point x y when x is not equal to minus y square.

Now we have to also compute at other points. So, first of all we should note that if we compute the partial derivative here first with respect to y, and then with respect to x we will get the same quantity as before. The reason is clear, because when x is not equal to y we have this nice function and which is certainly continuous we can prove that. So, all these derivatives, so they must be equal at these points where there is no problem the problem will be when this is defined as 0. So, when x is equal to minus y square. So, at those points we have to be careful. At all other points certainly the derivatives will be continuous and the value will be equal. So, we do not have to compute this again one can check or one can verify this. So, one has to first compute the partial derivative with respect to y and then with respect to x. And then you will notice that we will get exactly the same expression.

So now for this to check the continuity of this, what we have to now take this limit of this f yx. Now if we take this x is equal to m y square this special path then what will happen? Let me clear this first. So, what will happen now, because x is equal to m y square. So, if we put here x is equal to m y square what will happen.

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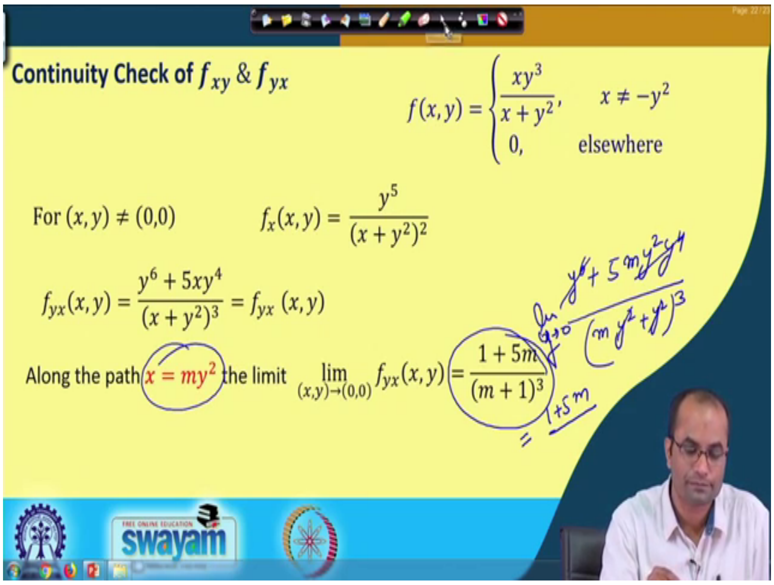

So, we have y 6 plus 5 x is m y square and then we have y 4 and divided by x is again m y square plus y square and power 3 and then we can take the limit as x goes to 0 to approach to the origin because this path will take us to the origin.

And now when x goes to 0, so what is happening here sorry y goes to 0 because we have replaced the x. So, y goes to 0. And now y 6, so here y 6 y 6 and y square power 3. So, everything other than this m will cancel out. And we will get in the numerator 1 plus 5 m and over this m plus 1 cube so this number. So, that is the limit now along this path which depends on m. So, it depends on the path. So; that means, this limit does not exist and now we can just say that the function is not continuous because for continuity this limit should exist and should approach to the derivative at 0 0 point.

So, in this case the partial derivatives either xy or f yx they are not continuous which was clear Because those values were not same as we have observed. And we know the result that if these 2 derivatives are continuous then the value should be also equal. So, the value was not equal at 0 0. So, naturally these functions are not continuous, which is clear again from here we have taken this fy x and then along this path we have seen that the limit depends on path and hence the limit does not exist and the function is not continuous.

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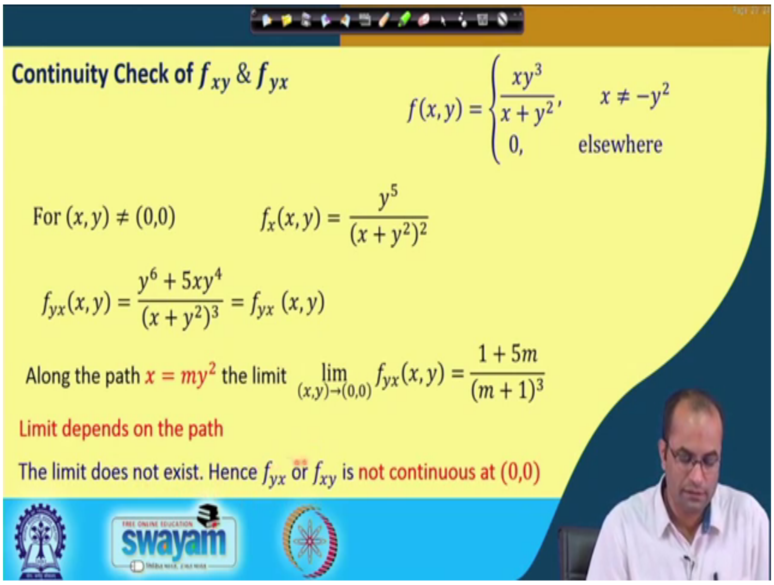

So, the limit depends on the path and hence these two are not continuous at 0 0.

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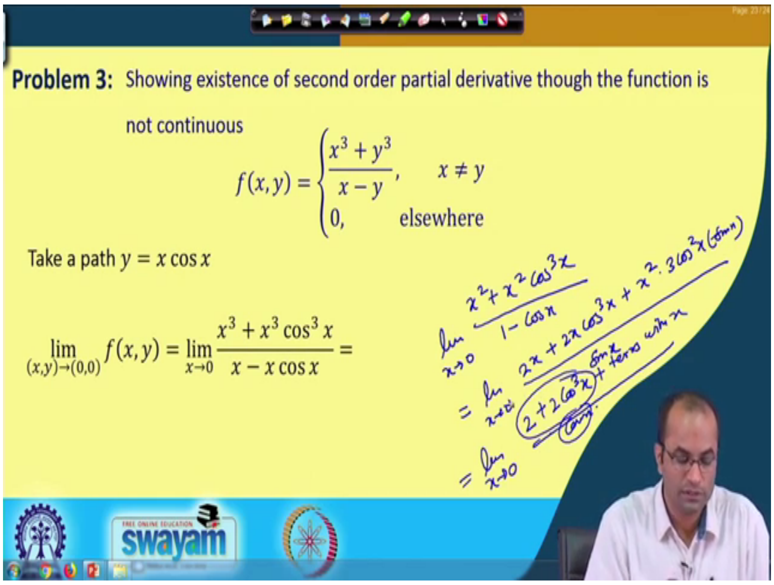

So now the next example it shows the existence of the second order partial derivative though the function is not continuous.

We have seen such results earlier for the first order partial derivatives that the function is not continuous, the partial derivatives, exist and function is continuous partial derivative does not exist. So, here also we have for example, this result that the second order partial derivatives exists, but the function is not continuous. So how: if you take this path y is equal to x cos x this special path, and try to get the limit for the continuity what will happen. So, here x cube and then y cube will be x cube cos cube x and again note that that this will take us to the origin x x goes to 0 this y goes to 0. So, this path is correct it is taking us to the origin and we are checking the limit here as x y goes to 0 0 along this particular path y is equal to x cos x. So, substituting this x cos x we have x minus and the y is substituted as x cos x.

And in this case what is this limit here now. So, this x will get cancelled we will have x square there plus x square. And we will have cos cube x and divided by here 1 minus and cos x. And then we will take the limit as x goes to 0. So, in this case, so 0 by 0 forms we have to use the L'Hospital's rule. So, this will be the limit now here. So, 2 x plus again 2 x here with the cos cube x plus the x square and then this will become 3 cos square x. And then with minus sin x and divided by here the sin x the derivative of this minus cos x will be sin x, but again we see it is a 0 by 0 form.

So, we have to take we have to apply the L'Hospital's rule again. So, here we will have 2 without x. And in this case also we will have 2 without x when we do this derivative here the product rule cube x. And in the rest I see the x will appear. So, terms with terms with x and then we have here cos x. So, in this case now if we take the limit this is 1 and here 2 plus 2 so will become 4. So, this limit is 4.

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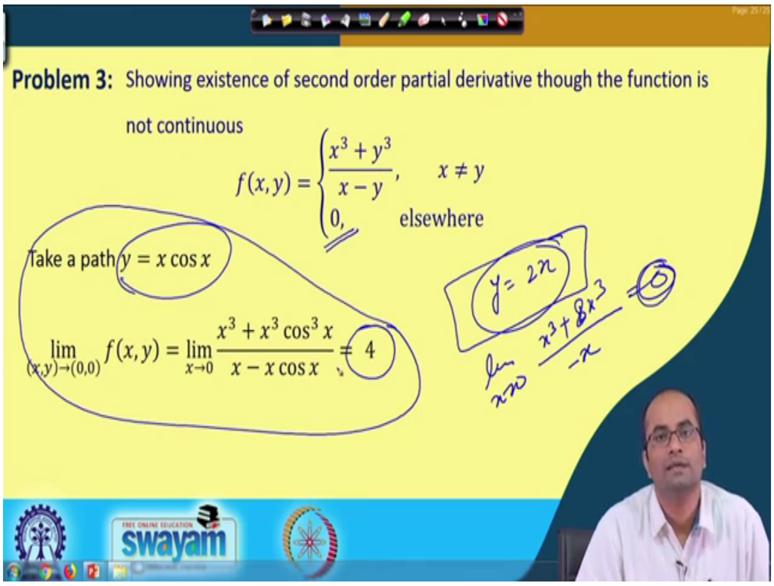

And now, we have seen that this limit as x y goes to 0 is 4 and not 0. So, here the function value is 0, but along this particular path we have seen the value is 4. So, certainly this function is not continuous. One can also take some other path and can show that the limit is different.

For example, if we take y is equal to 2 x if we take this path here and then again we will see that the limit is different. So, if x goes to 0 and we have taken this x cube along this path. So, y is 2 x. So, 8 x is cube and then here we have minus x and minus 2 x. So, minus x and this will become 0. So, we have 2 different path one is y is equal to x cos x another one is y is equal to 2 x here the limit is 0 there the limit was 4. So, that concludes that the limit does not exist. And in any case here the value of the function is 4 and we have already seen along this path the value is 4. So, there itself we can conclude that the function is not continuous.

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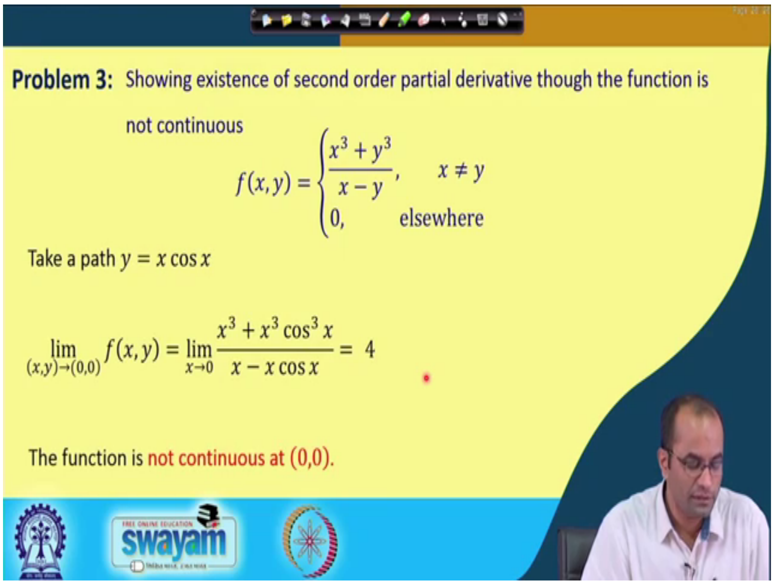

So, in this case the function is not continuous.

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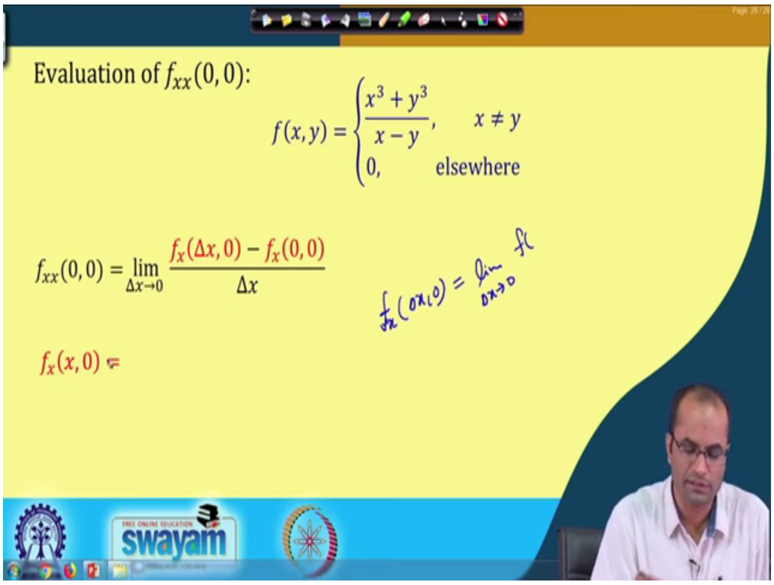

But what we can do we can evaluate f xx at 0 0. How? So, f x at 0 0 as per the definition f x delta x 0 and minus f x 0 0 over delta x the usual definition we have used several times. And then again we have to see what is this function here f x delta x. So, we have to see what is f x and delta x 0. So, this will be the limit delta x goes to 0, because of the derivative here and f. So, better to use because delta x is already there let us compute this f x at general point x 0.

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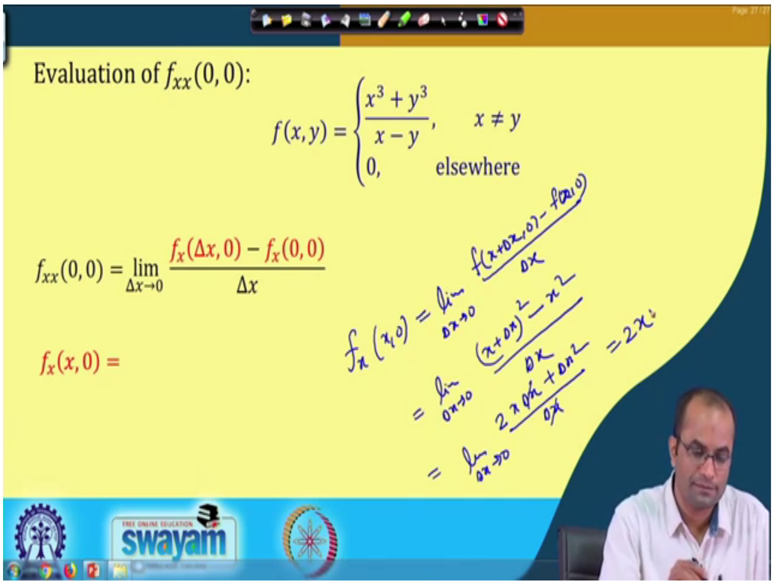

So, instead of this we will choose now here f x x 0 instead of delta x 0 later on we will substitute for this point as delta x 0.

So, at any general point x 0 we are computing the derivative. So, limit delta x to 0 and f. So, x plus delta x; because increment has to be made in the first argument then 0, and minus f x 0 sorry and then we have delta x there. So, in this case this limit delta x to 0 this will become because the second argument is 0. So, there is no y term we will have x square simply there; that means, the x plus delta x whole square and minus when this is x there. So, we will get x square and divided by delta x.

So, this will become x square which will cancel out. So, 2 times x delta x and delta x square divided by delta x and this limit delta x goes to 0. So, in this case we will get this result as 2 times x.

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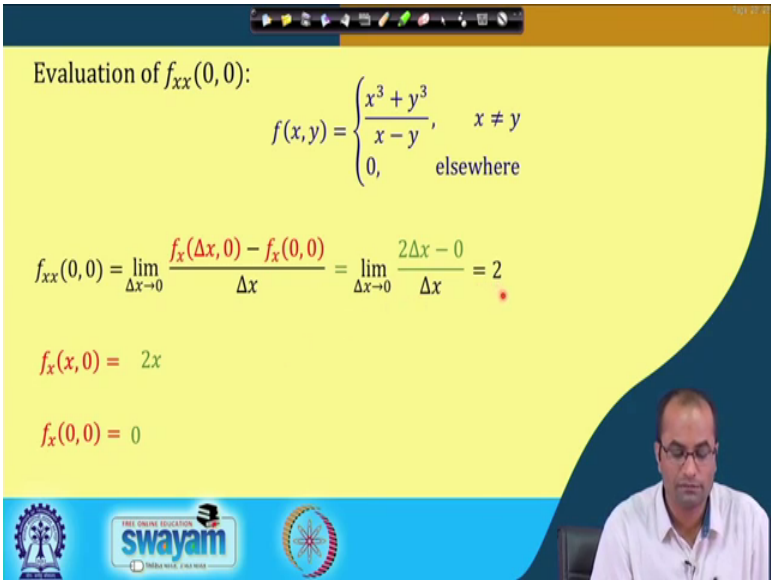

So, this is 2 times x and f x 0 0 it is much easier to compute that this will come as 0. So, in this case we have this 2 times. Now the result was f x at point x 0 is 2 times the x. So, here is delta x. So, 2 times delta x minus f x 0 0 is 0 over delta x and then this value here is 2. So, what we have seen the second order derivative with respect to x of this function is 2.

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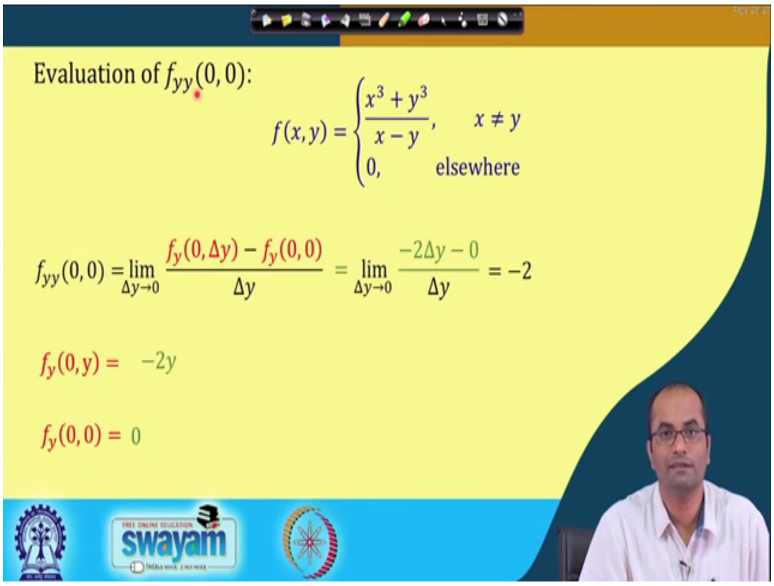

Now, we can also compute the derivative with respect to y like we have done before. So, with now we have to compute this f y 0 delta y as earlier. So, we will get minus 2 y because of this sign here the 1 minus will appear. So, we will get minus 2 y in this case. And the 0 0 will be 0 so now this f yy. So, second order derivative with respect to y will be minus 2 delta y over delta y and this will cancel out and we will get minus 2 as the answer of this f yy at 0 0.

So, in this particular case we have seen the function was not continuous, but we have a second order derivative not only the first order we have the existence of second order derivatives.

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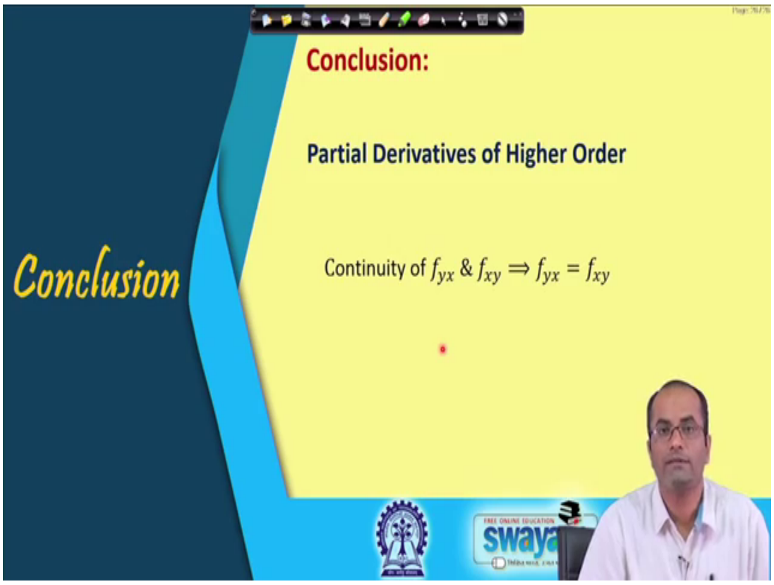

So, what is the conclusion for today's lecture we have now learn the partial order derivative of higher order. And what we have also seen that the continuity of the partial derivative is sufficient to ensure that these 2 mixed order partial derivatives are equal. Or if they are not equal that means these partial mixed partial order derivatives are not continuous in that case.

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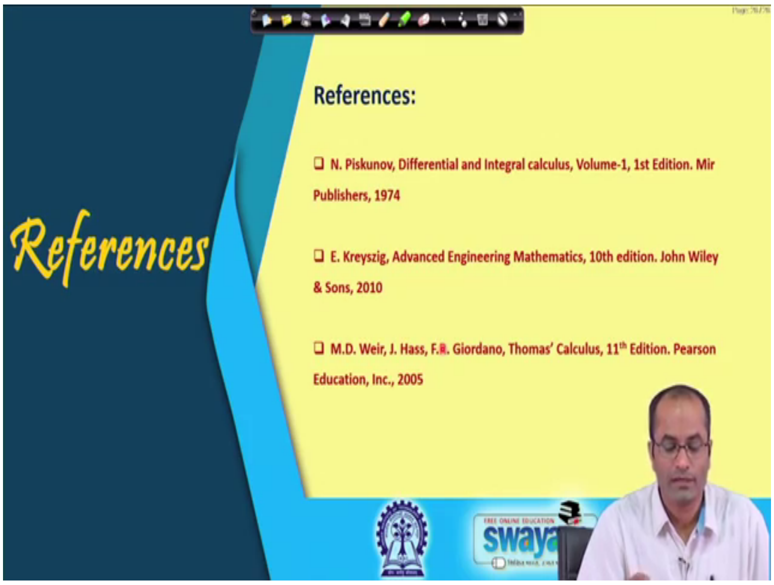

So, these are the references used for preparing this lecture.

Thank you very much.