# Engineering Mathematics - I
Prof. Jitendra Kumar
Department of Mathematics
Indian Institute of Technology, Kharagpur

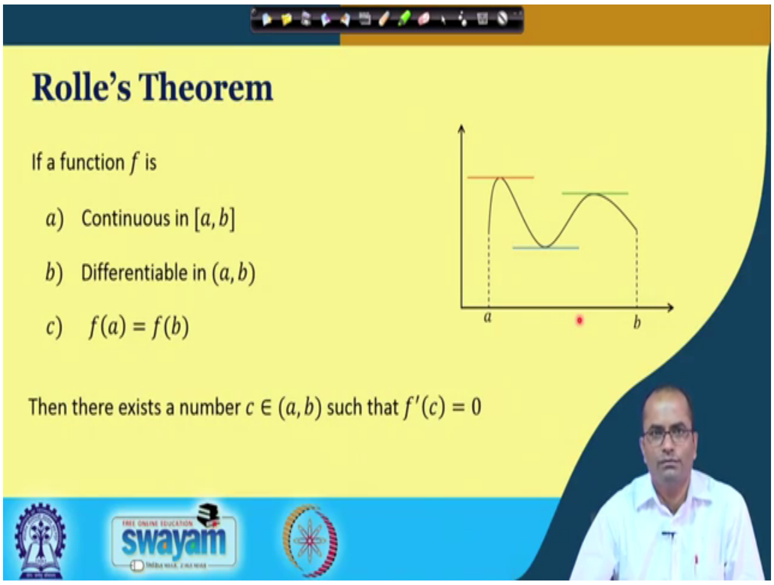
## Lecture – 01
## Rolle's Theorem

Welcome to the first lecture on Engineering Mathematics, I am Jitendra Kumar from the Department of Mathematics. And, today we will be discussing the Rolle's Theorem from differential calculus of variable 1.

(Refer Slide Time: 00:35)


So, these are the topics we will be covering today. So, starting with the Rolle's Theorem. And since it is a very fundamental theorem so we will also go through the detail proof because this will be used for various other results in other lectures and then some worked examples.

(Refer Slide Time: 00:56)

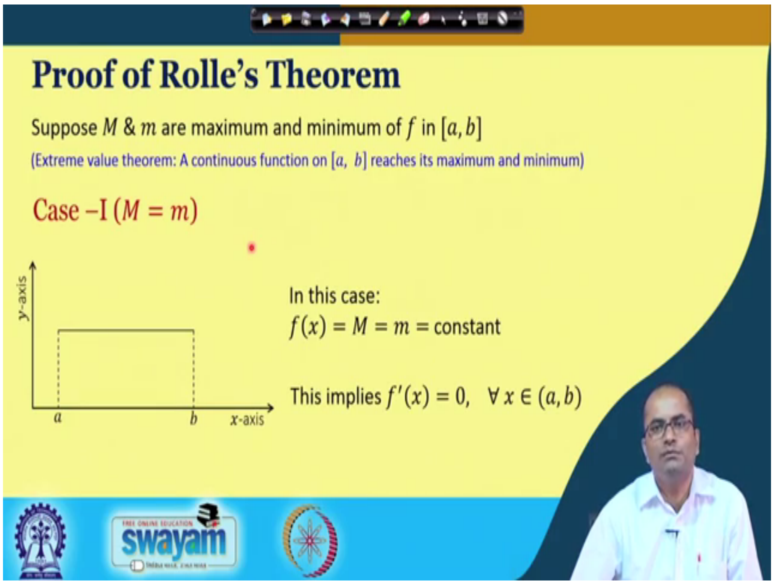



So, what is Rolle's Theorem? So, Rolle's Theorem if a function f of single variable is continuous in closed interval a b and differentiable in open interval a b. And, there is a one more condition which says that the function value at a is equal to the function value at the point b. So, the 2 end points the function is taking same value.

In that case the theorem says that there exist a number c in open interval a b such that f c is equal to 0, meaning that there will be a point c where the slope of the tangent will be 0. So, if we go through the geometrical interpretation; so, let us consider this is the function which is plotted in this x axis and the y axis here. So, this is the point at a so, the function value at this point a is here and the same value the function is taking at b. So, the function is continuous and differentiable everywhere then this theorem says that there will be at least 1 point where the derivative will vanish or the tangent will be parallel to x axis. So, the slope of the tangent is 0 meaning it is parallel to the x axis.

So, clearly we can observe that there is a point here somewhere next to this a, where the tangent is parallel to the x axis. Indeed in this situation there are more points one I can see here where tangent is again parallel to the x axis. And, there is another point where the tangent is parallel to the x axis, but the theorem says that there will be at least 1 point where the tangent will be parallel to the x axis. So, in this particular situation we are getting more than 1 point where the tangent is parallel to the x axis.

(Refer Slide Time: 03:14)

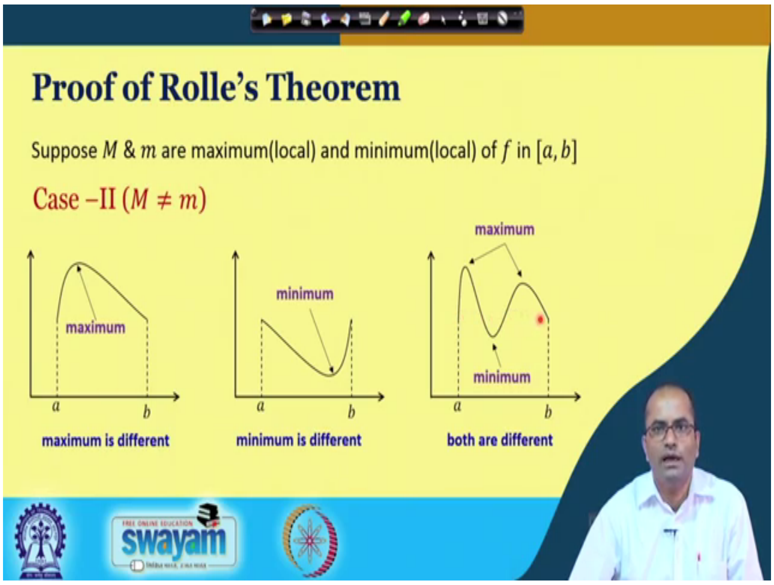



So, if we go through; now the proof which is pretty simple so, let us go step by step. So, here we assume that the function takes the maximum and the minimum value and they are denoted by big M as maximum and small m as minimum in this interval a b, which is guaranteed due to the extreme value theorem because the function is continuous in the closed interval. And therefore, a maximum and minimum will be reached in this interval at some point.

So, now we consider the following situation a particular situation the case I when M is equal to small m. So, the maximum value is equal to the minimum value. Now, think about the situation, then the where the function is having maximum and the minimum value as same. So, in this situation clearly there is no change in the function value and therefore, the minimum value is equal to the maximum value.

So, basically if we plot this it is a constant function and that would be the situation when the minimum value will be equal to the maximum value. So, in this particular case fx is the constant function, and since fx is the constant function naturally whatever point you take the derivative is going to be 0. So, the theorem is proved in this case when M is equal to small m. So, the maximum is equal to the minimum the function value.

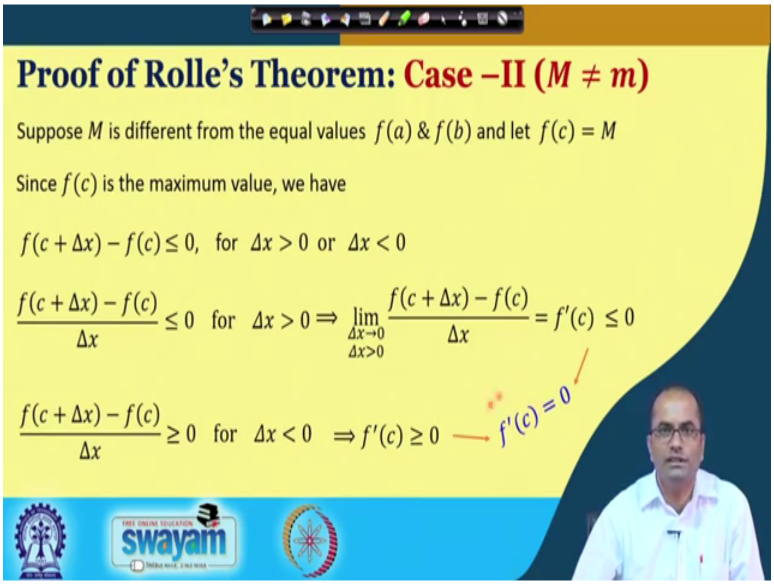
(Refer Slide Time: 04:52)




So, now the second case when the maximum value of the function is not equal to the minimum value of the function. So, in this case we will consider three situations or three cases again. The first one let us assume that the maximum value in this situation here which is clearly can be observed that it is different than the function value at a and b. The function value at a and b are equal as per the assumptions of the theorem.

So, here we assume that the function the maximum value of the function is different than the function value at a and b. The second case when we take the minimum value is different than the function value at a and b and the third situation that for some functions both maybe different. So, in this case the minimum or rather I would say the local minimum in each case or local maximum. So, which is here and this is different than the equal values at a and b. And, here as well the local these 2 local maximum are also different than the equal value at a and b.

(Refer Slide Time: 06:10)




So, both are different in this case. So, in either situation let us consider the case we suppose that the maximum value of the function is different from the equal values of f at a and b which are the same here. So, the M is different the big M the maximum is different. Similarly we will consider later on if M is not different then the small m should be different at least one of them will be different because M is not equal to small m.

So, we take that the function is taking this value M at a point c. So, f c is equal to M, the function is having this local maximum at the point c. So, if this f c is the local maximum then we have f c plus delta x minus f c; just considered the situation f c plus delta x. So, this point here c plus delta x is in the close vicinity of c just assume this delta x is close to 0.

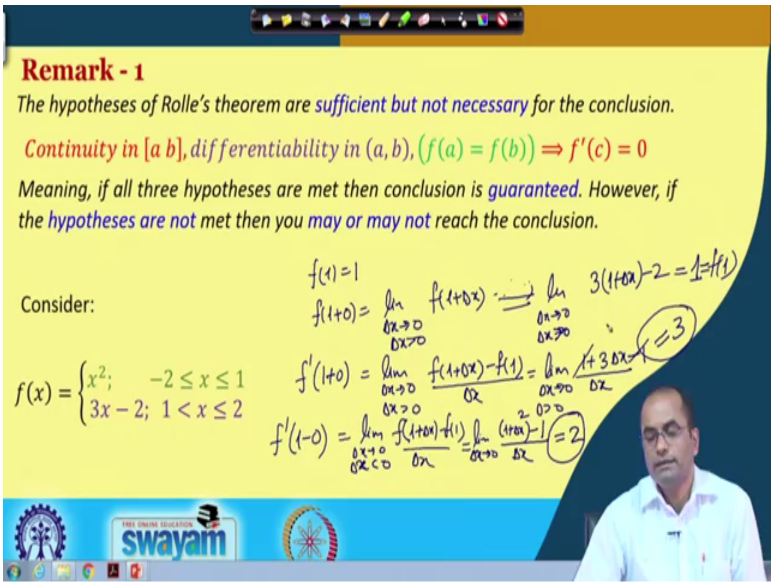
So, in that case since f c is the maximum value of the function then f c plus delta x and this difference; so, f c plus delta x will be smaller than the this f c because f c is the local maximum. So, in this case there will be a such a delta x definitely because f c is the maximum value that, this expression here f c plus delta x minus f c will be less than equal to 0 whether this delta x is positive or delta x is negative. That means, any point you take in the vicinity of this point c then this difference here f c plus delta x minus f c will be less than equal to 0. And, now if I divide this expression here by delta x and if I in the first case I take delta x as positive, then the sign of this expression will not change. And, it will remain as less than equal to 0 if delta x is positive.

On the other hand if I take delta x as a negative number then this expression will change the sign and this f c plus delta x minus f c divided by delta x will become greater than equal to 0. And now, I will take in this first case when I have taken here the delta x positive the limit that delta x goes to 0 and this expression and the less than 0.

So, if you take a close look at this one this is the right hand derivative of the function f and since f is differentiable this will be equal to the derivative of the function. So, we have here this inequality that the derivative will be less than equal to 0 in the situation. On the other hand when you divide this by delta x which is negative and take the limit again the same similar case here. Since the function is differentiable that left derivative will be also positive because this inequality is greater than equal to 0.

So, in this case we got f prime c is equal to 0 whereas, they we have f prime c less than equal to 0. So, out of these 2 we conclude that the f prime c has to be 0 because it cannot be less than equal to 0 or greater than equal to 0 at the same time. So, there only possibility is that f prime c has to be 0. So, in this way we have proved this that there is a point in this interval c, in the open interval c where the derivative vanishes.

(Refer Slide Time: 10:05)




There are few remarks which are of great importance. So, here the hypothesis of the Rolle's Theorem are sufficient, but not necessary for the conclusion. What do we mean by this? So, what we have seen that this continuity of the function in the close interval a b and the differentiability in the open interval a b and there was a third condition that f a is equal to f b. So, if these three conditions are satisfied then there will exist a point c where the derivative will vanish. So, these conditions are sufficient meaning that these conditions here all these three conditions implies that f prime c is equal to 0. But not the other way around that f prime c is equal to 0 does not imply that the function will be continuous, differentiable and we will take these equal values at some points a and b.

So, in other words if all these hypothesis these three hypotheses are met then the conclusion is guaranteed; conclusion means the f prime c is 0 that is guaranteed. However, if the hypothesis are not met then you may or may not reach the conclusion which we will see with the help of some examples now. Let us consider this example f x is equal to x square in the range minus 2 to 1 and 3 x minus 2 in the range 1 to 2. So, this function here the clearly if we see that the function is continuous, the function value at 1 is here f at 1 which is we can substitute directly the function is defined until 1.

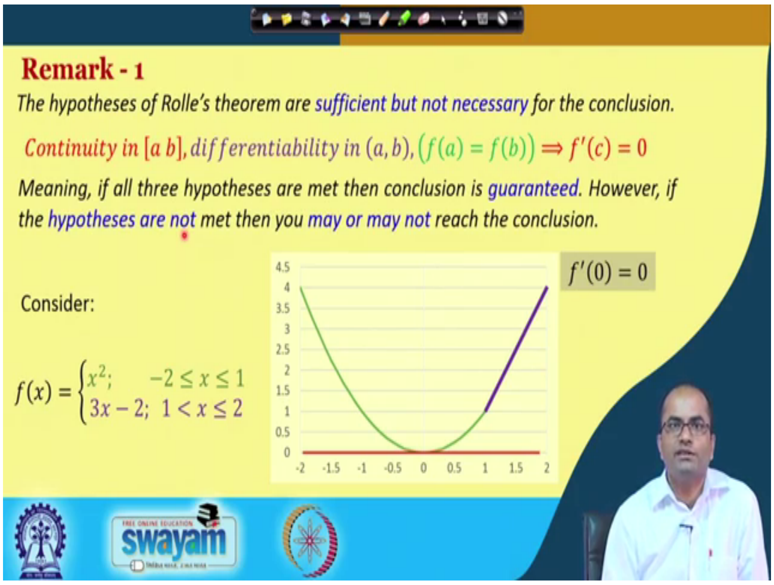
So, f 1 is 1 and then if you take the right limit so, f 1 plus 0 the right limits. So, the limit delta x to 0, and this f 1 plus delta x and minus this f at 1 or just the limits we are not going to get the derivative now. So, this just this expression here f 1 plus delta x and delta x goes to 0 and we take here the delta x positive. So, the right limit of this function as delta x goes to 0. So, this will be simply the limit delta x goes to 0, the delta x we are taking as positive here. And, then since delta x is positive 1 plus delta x we will be calculated from this here 3 x minus 2. So, you have the 3 and x means 1 plus delta x the argument and minus 2 and this is nothing but 3 and minus 2 1. So, 1 plus 3 delta x and delta x goes to 0.

So, this is 1 and which is equal to the f 1. So, the function is naturally continues in this case and if we check the differentiability; that means, the right derivative first. So, the f 1 plus 0 the right derivative means the limit delta x goes to 0 and the delta x is positive. So, f 1 plus delta x minus f 1 and divided by delta x this goes not here. So, in this case the limit delta x goes to 0 the delta x is positive; so, here 1 plus delta x again will be calculated from 3 x minus 2 which we have just done before. So, it was 1 plus 3 delta x was coming and divided by delta x and then here minus this f 1 is 1.

So, this gets cancelled and then this value here is nothing, but 3. So, the right side derivative of this function is 3 where as the left hand derivative. So, f 1 minus 0 which is the notation and here the limit if you compute delta x goes to 0 delta x negative, what will happen to this one. So, here you have again f 1 plus delta x minus f 1 by delta x. But now this f 1 plus delta x and delta x is negative will be computed from x square. So, meaning we have here delta x goes to 0 and this is 1 plus delta x whole square and minus 1 and divided by delta x.

So, this 1 when we expand this there will be 1 plus delta x square plus 2 delta x terms so, 1 1 will get cancel and this 2 delta x and divided by delta x will give you a 2 and the rest because of the limit will go to 0. So, here the derivative is 2 whereas, there the left side derivative is 3 and the right side derivative is 2. So, the function is not differentiable in this case.

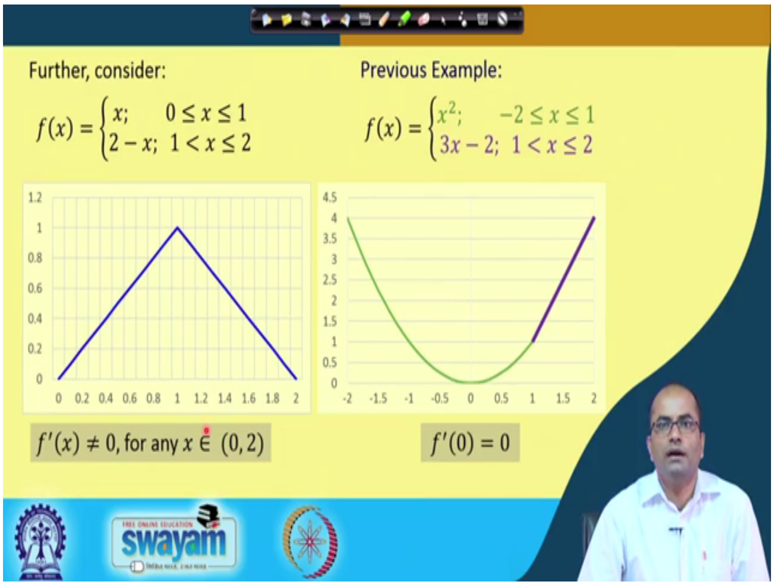
(Refer Slide Time: 15:12)




And we can plot this one and then again you can see that at this point 1 here the function breaks its differentiability. So, the right side derivative which we have just seen was minus the left side derivative so was 2 and the right side was 3. So, there is a point here where the function is not differentiable. But, what is interesting in this case the all the hypothesis are not made because the function is not differentiable at this point.

But there is a point here 0 which you can easily compute again from this x square is a derivative is 2 x and x is equal to 0 the derivative will become 0. So, here the f prime at 0 is 0. So, the derivative vanishes or the tangent is parallel to the x axis in this case though the function was not differentiable here. So, exactly what we have said if the hypotheses are not met the function may or may not reach the conclusion. So, in this case it is reaching the conclusion, but this is not because of the Rolle's Theorem.

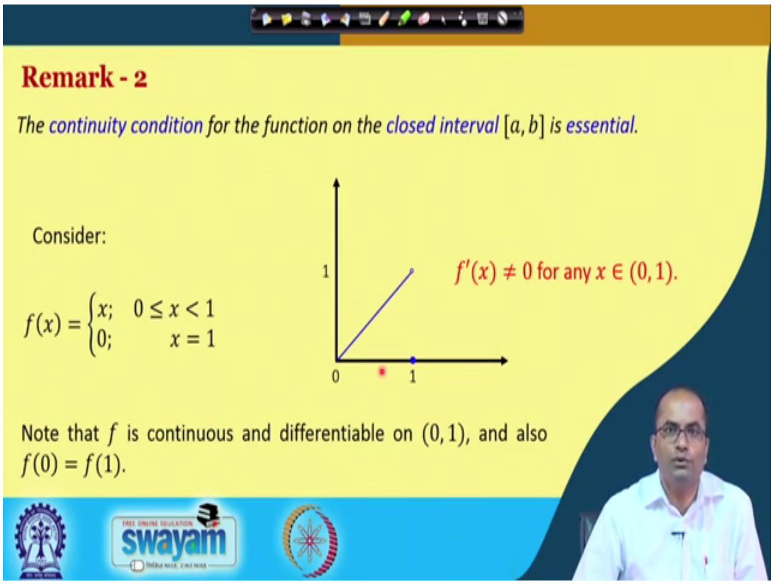
(Refer Slide Time: 16:20)




Another example if we take that we have f x is equal to x in the range 0 to 1 and 2 minus x in the range 1 to 2. So, again the similar situation one can easily float this function and one clearly sees that at this point 1 the function is not differentiable. And, in this case we are not getting any point between this 0 and 1 where the function is taking over the derivative is vanishing. So, in this situation f prime x is not equal to 0 at any point in the given interval. So, we have seen these 2 examples the other one was this one, the previous example where the function was not differentiable, this is also not differentiable.

But in 1 f prime 0 is equal to 0. So, there is a point where the derivative vanishes whereas, in this case the derivative does not vanish at any point in the interval. So, therefore, these conditions these three hypotheses of the Rolle's Theorem are sufficient conditions and they are not the necessary conditions. So, under those conditions it is guaranteed that the function will derivative of the function will vanish at least at 1 point in the open interval a b.

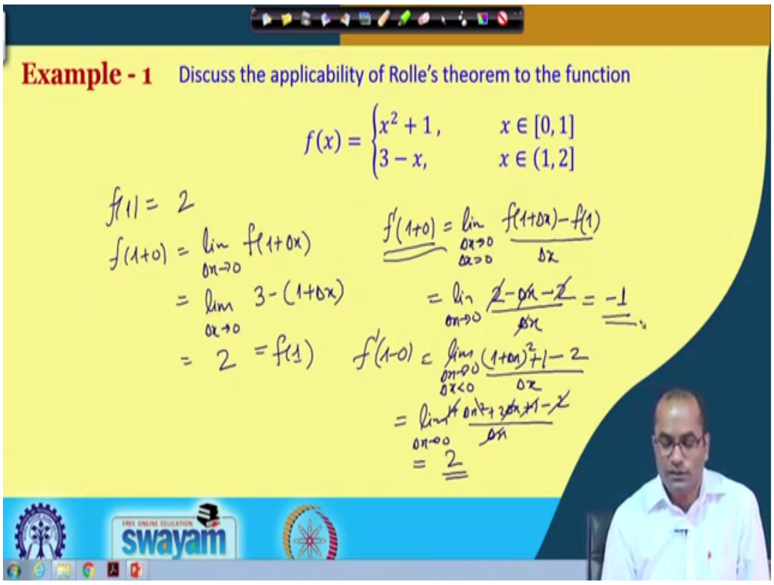
(Refer Slide Time: 17:44)




Another remark that the continuity condition which we have seen the continuity in the closed interval for this function is essential, if it is not met then we may not that the theorem may not guarantee the existence of such a c where prime c will be 0. So, for example, if you look at this function f x is equal to x and then 0 at x is equal to 1. So, what do we see here the function is continuous and differentiable on 0 1 and also f 0 is equal to f 1. So, this condition is met differentiability condition is met, but the function is not continuous at 1. We should note that because the function is x from 0 to 1 and then it is x is equal to 1.

So, there is jump here which we can see. So, at x is equal to 1 the function is taking value as 0 and otherwise its taking here as x. So, the function is not differentiable at oh sorry continuous at 1, otherwise all other conditions are met in this case of the Rolle's Theorem. And, then we clearly see the derivative is 1 everywhere here between these two 0 and 1 open interval 0 and 1. And therefore, the f prime x is not equal to 0 at any point in this interval x 0 to 1.

(Refer Slide Time: 19:12)



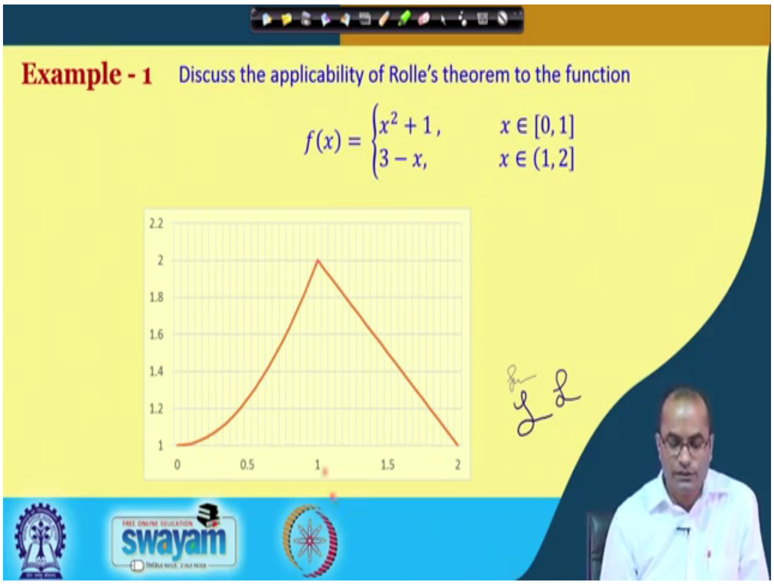

Another example we will discuss now the applicability of the Rolle's theorem for this function f x is equal to x square plus 1 in the interval 0 to 1 closed interval 0 to 1 and 3 minus x in the interval 1 to 2. So, again if the continuity is concerned then the function is continuous because it is taking like f 1 is f 1 is 2 and f if we take the right limit here f 1 plus 0. So, the limit delta x goes to 0, and this f 1 plus delta x will be this is limit delta x goes to 0 and delta x positive because the right limit we are taking here. And, in this case this will be 3 minus 1 plus delta x; that means, it is a 2 minus delta x.

So, delta x goes to 0, this is 2 and the value is equal to 1. So, the function is continuous in this interval 0 to 2 and what is about the differentiability. If you look at the differentiability is pretty similar to the earlier case. So, if you compute the right derivative so, 1 plus 0; that means, the delta x goes to 0 and delta x is positive because the right limit I am talking about. And, in this case again you have take the f 1 plus delta x and minus this f 1 divided by delta x. So, limit delta x goes to 0 and f 1 plus delta x.

So, f 1 plus delta x we have computed here this is 2 and minus delta x and minus f 1 is 2 again and divided by delta x. So, this limit will be coming as minus 1 because this will got cancelled and then you will get minus 1 there. So, the right derivative is minus 1 and the left derivative of 1 minus 0 which is limit delta x goes to 0 again with delta x negative.

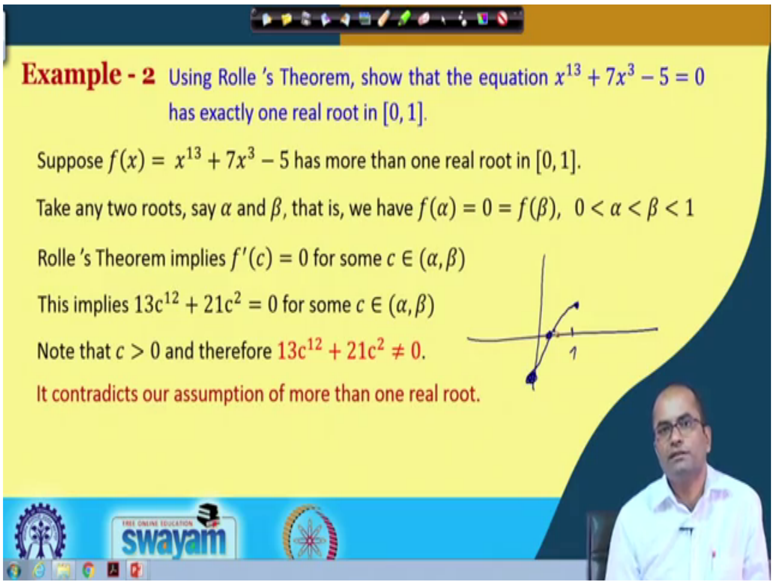
So, in this case f 1 plus delta x will be computed from here. So, 1 plus delta x whole square plus 1 minus f 1 which is 2 divided by delta x. So, delta x goes to 0 and here you will get 1 plus delta x square and 2 delta x; so, 1 plus delta x square 2 delta x plus 1 minus 2. So, this will cancel out and then here also so, you will get and this power. So, delta x goes to 0 this will be coming as 2. So, in this case the left derivative is 2 and the right derivative is minus 1. So, the function is not differentiable at the point 1. So, the Rolle's Theorem is not applicable in this case.

(Refer Slide Time: 22:10)




And if we take a look here at this floor, then you again see that at 1 here the function is not differentiable which we have just seen.

(Refer Slide Time: 22:23)




So, moving further this is another example which says the using Rolle's Theorem show that the equation this x power 13 plus 7 x power 3 minus 5 is equal to 0 has exactly 1 real root in 0 1, in the closed interval 0 1. So, this is another kind of application which where we can use the Rolle's Theorem to show that this equation has exactly 1 real root. So, if we move further suppose that this f x this function here x power 13 plus 7 x minus 5 has more than 1 real root in 0 1. So, we assume that this function f x has more than 1 real root. So, if it has more than root then we can take any 2 roots let us say alpha and beta.

So, you have taken 2 roots and since this alpha and beta are the roots so, f alpha will be 0 and that will be also equal to f beta. So, alpha and beta both are roots so, the function will be 0 at alpha and as well as at beta. So, here we just for the convenience we have assume that alpha is smaller than beta and naturally these 2 will fall between 0 and 1; because 0 and 1 are is not the root of the equation which clearly we can see there. So, this alpha beta these 2 roots because, we have assume that this function has more than 2 roots so, these alpha and beta will be between less between 0 and 1.

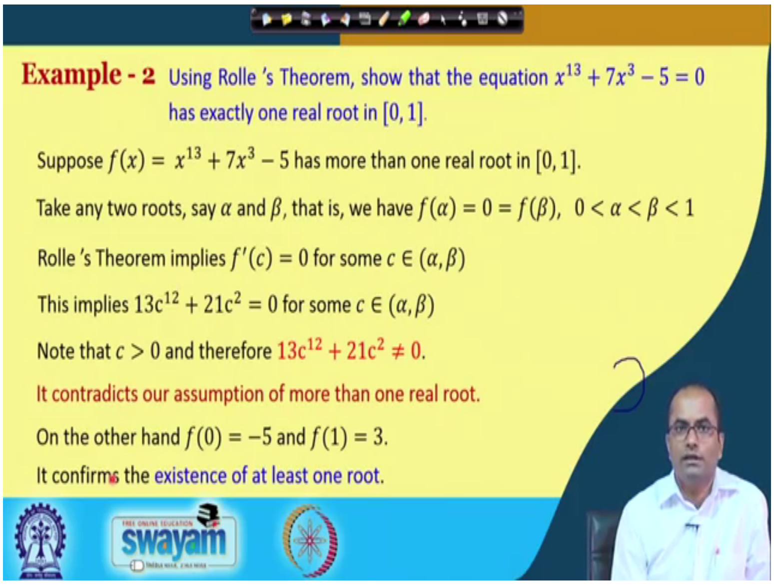
So, both have the positive number here alpha and beta and less than 1. So, what Rolle's Theorem says, if we apply the Rolle's Theorem to this interval alpha and beta. If we apply we apply this Rolle's Theorem to the interval alpha and beta in that case the Rolle's Theorem says that there will be a point of prime c will be 0; there will be a point c where f prime c will be 0. Because, of the reason because the function is taking now equal value at alpha and beta, function is differentiable, it is a polynomial function, there is no problem, the it is continuous naturally and it is taking the same value at alpha and beta.

So, if we apply in this interval Rolle's Theorem that will give us that f prime c is equal to 0 for some c in the interval alpha and beta. So, this implies so, what is this f prime c? So, f prime c is 13 c power 12 plus the 21 and c power 2 and is equal to 0; for some c in the interval alpha and beta. Again note that the alpha and beta both are positive number and now which you see because the c is positive here, then this expression here 13 c power 12 plus 21 c square cannot be 0, because this is a power 12, here the even number also c power 2 and this c is positive.

So, this is a positive quantity, this is a positive quantity. So, it cannot be equal to 0, but the Rolle's Theorem says that it will be equal to 0; that means, we have a assumption which was that the function has more than 2 real roots is wrong. So, it contradicts our assumption of more than 1 real root. But, now the question is whether there is a root in this case, because we have just proved that there cannot be more than 2 roots.

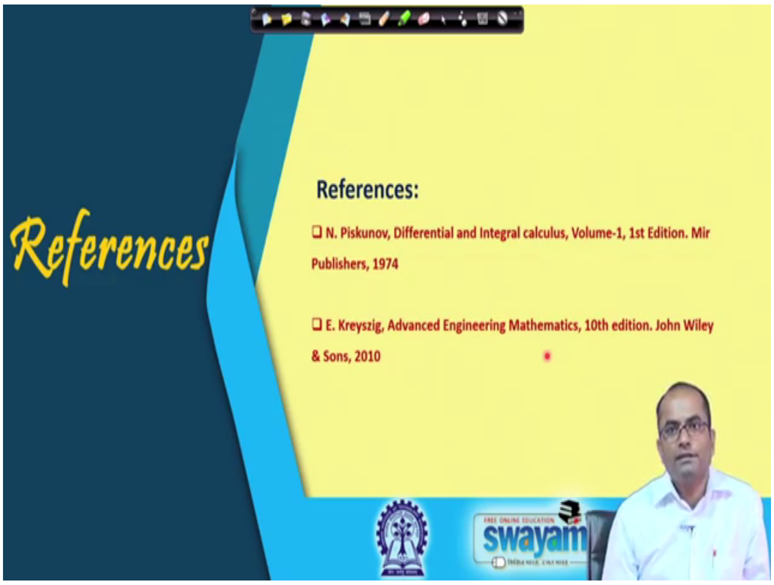
So, if you take a close look at this function here at 0 the value is a minus 5 somewhere here and if you put this 1 there the other end then we will get 3. So, the value will be 3 at 1 so, if this is 1 here. So, at 1 the value is 3 and the 0 the value is minus 5 and function is continuous. So, definitely to reach to this point it will cross somewhere the real axis and so, that proves the existence of 1 root in this case which confirms the existence of 1 root because this is changing its sign.

(Refer Slide Time: 27:00)



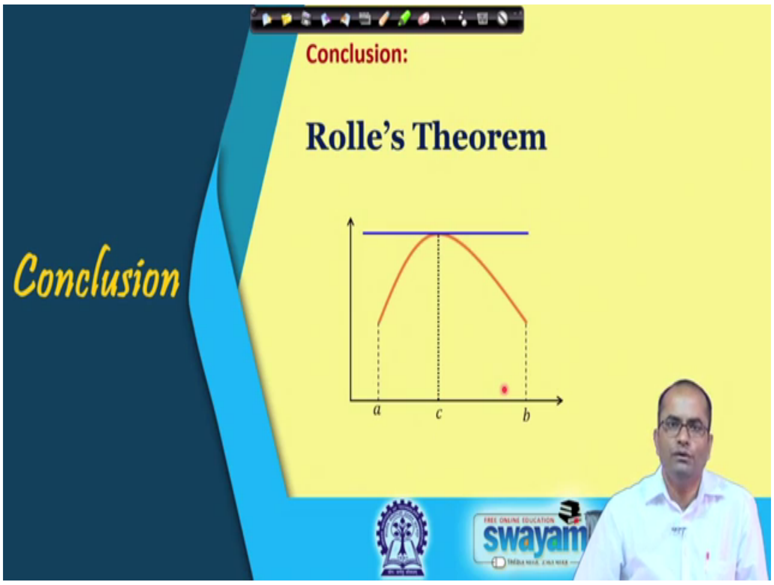

So, f 0 is minus 5 and f 1 is 3.

(Refer Slide Time: 27:14)




Now, there are the references which we are used to prepare this lecture, the book by the Piskunov, Differential and Integral calculus, Volume 1 and also the Kreyszig Advanced Engineering Mathematics.

(Refer Slide Time: 27:28)




So, again the conclusion here we have a studied the Rolle's Theorem which says that if the function is continuous and differentiable having the same value at this a and b, then there will be a point c somewhere in the open interval a b, where the tangent to this function will be parallel to the x axis.

So, this is the Rolle's Theorem which is a particular case of the mean value theorem which we will discuss in the next lecture. And, basically this assumption of having the equal values will be removed and then we will get more general results. And, those are the mean value theorem the topic of the next lecture.

Thank you.