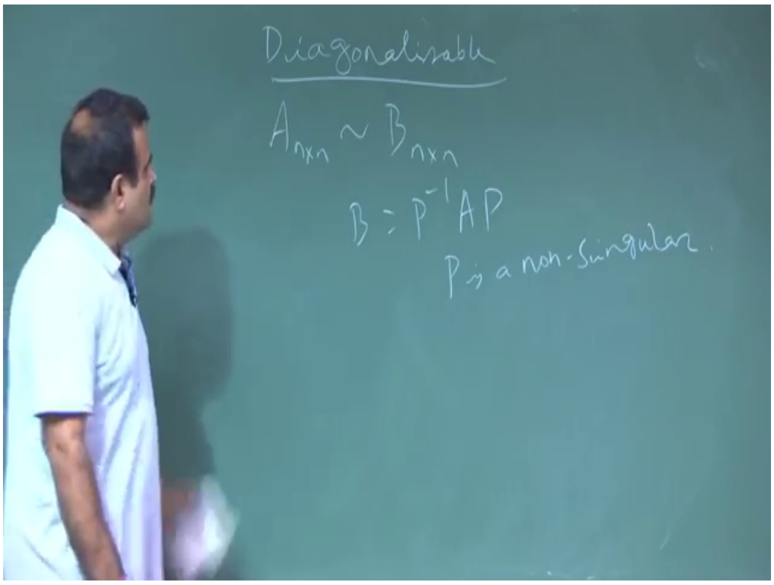
Introduction to Abstract and Linear Algebra
Prof. Sourav Mukhopadhyay
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture - 40
Diagonalisable

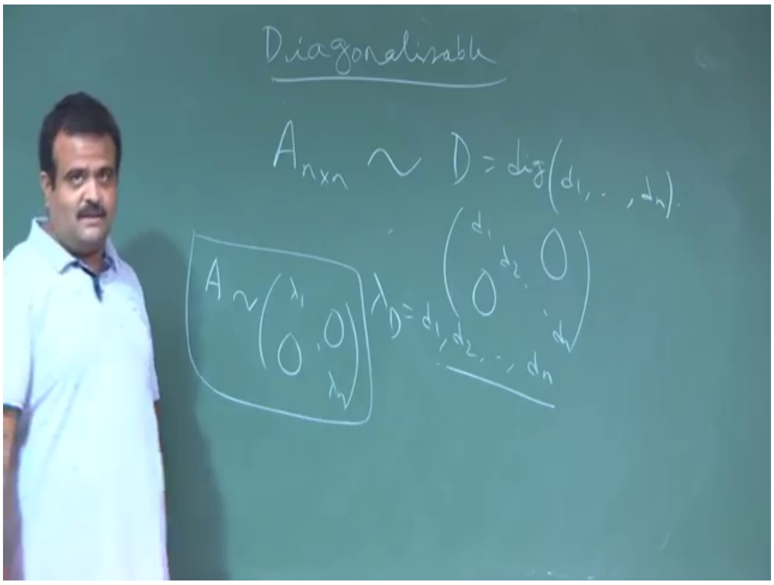
So, we are talking about Diagonalisable of a matrix. So, we shall in the last class we defined similar how to define the relation similar between 2 matrix just to recap.

(Refer Slide Time: 00:29)



We have 2 matrix of size n cross n, we say this 2 matrix are similar; if one is written as some P inverse A P where P is a non-singular matrix, P is a non-singular matrix, then we say A is similar to B. Now, we define the diagonalisable of a matrix.

(Refer Slide Time: 00:57)

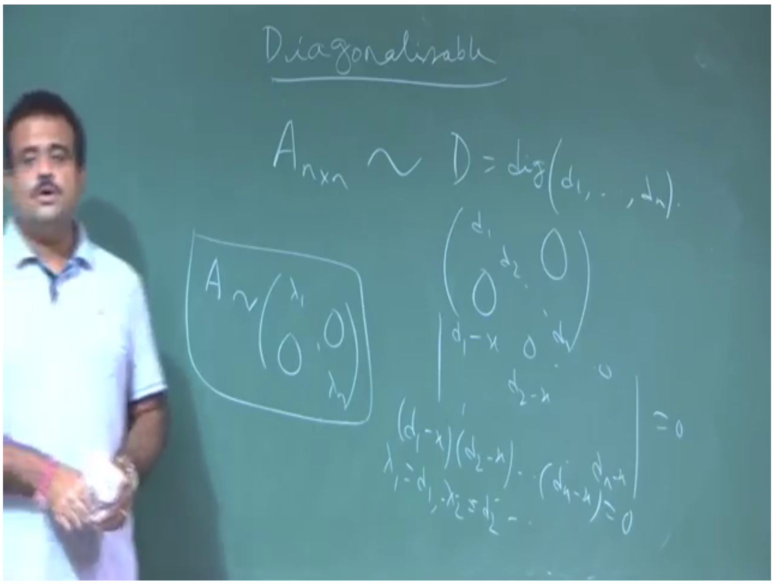


So, a matrix is diagonalisable if it is similar to a diagonal matrix D, a diagonal matrix where diagonal entries are there. So, say they are d 1, d 2, d n.

So that means, D is this matrix d 1 d 2 all diagonal elements are there, off diagonal elements must be 0. Or this di’s may be sum of the di’s may be 0. They are coming from the field element, ok. So, if A is similar to a diagonal matrix, then it is called A then A is said to be diagonalisable. Now in fact, if A is diagonalisable then A and so, A is similar to the diagonal matrix; that means, A and D has the same Eigen values. Now what are the Eigen values of D? Eigen values of the diagonal matrix are the diagonal elements. So, the Eigen values of D’s are, this one d 1, d 2, d n. So, they are the basically the Eigen values of A; so; that means, if A is diagnosable, then the A is because they have the D and A have the same Eigen value. This we have seen in the last class.

So, that means, the A is diagonalizable to this matrix where we have the Eigen values over here in the diagonal element; provided A is diagonalisable. Then it is the diagonal element will be the Eigen values, because it will be the it will be similar to a diagonal matrix, now for the diagonal and these 2 matrix of the same Eigen value. Now, we know for the diagonal matrix the Eigen values are basically the elements; because if you have a diagonal matrix what is the characteristics equation?

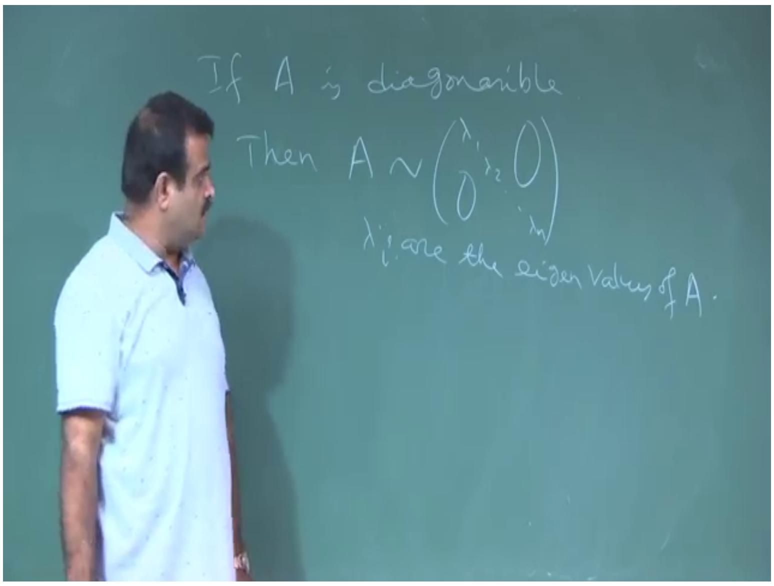
(Refer Slide Time: 02:59)



Characteristics equation is nothing but d 1 determinant of d 1 minus x d 2 minus all are 0's all are 0's like this, d n minus x; so, this is 0.

So, this will give us d 1 x d 2 x d n x equal to 0. So, that that means Eigen values are nothing but d 1 d 2 so on, lambda 2 d 2 like this. So, these are nothing but these are nothing but the Eigen values of A, because they have the same Eigen values. So, if A is diagonalisable then it will be similar to the diagonal matrix, where the diagonal elements are the Eigen values ok.

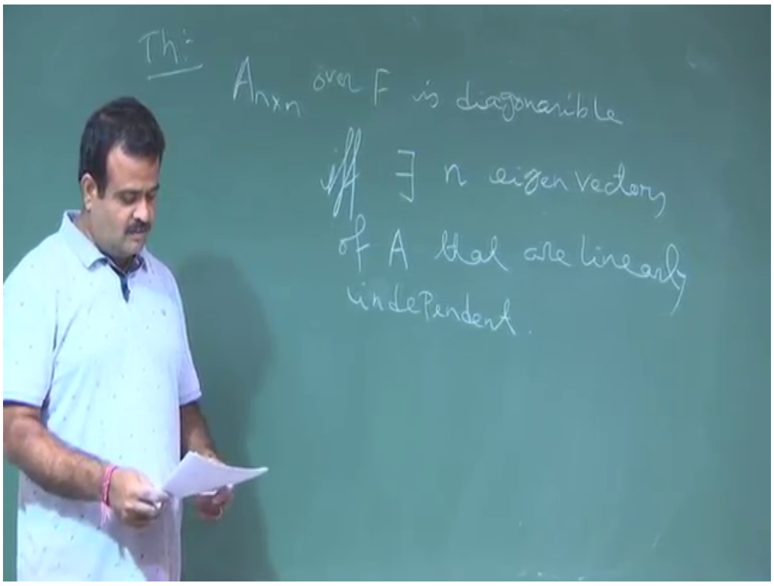
(Refer Slide Time: 03:57)



So, if A is diagnolisable, then A will be similar to the diagonal matrix lambda 1, lambda 2, lambda n, all other elements are 0. So, this is a diagonal matrix of this and this lambda i's are the ith values of A; provided A is diagonalisable then A will be similar to this ok.

So, that means, so now, we will have some properties, I mean necessary and sufficient conditions to be a matrix is diagonalisable.

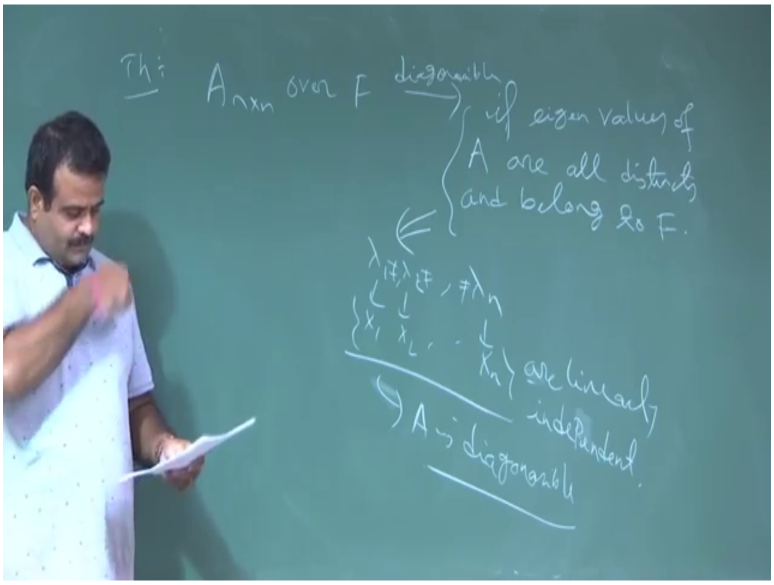
(Refer Slide Time: 05:05)



So, this is a necessary and sufficient condition to be a matrix A diagonalisable. A is a matrix over a field, over F is diagonalisable if and only if this is necessary and sufficient condition, if and only if there exists n Eigen vector of A which are linearly independent. There exist n Eigen vectors of A that are linearly independent. So, this is this condition is the necessary and sufficient condition.

So, if it is diagonalisable, then we have n set of linearly independent vectors Eigen vectors, or the sufficient condition is if it is if there are n set of linearly independent vector, then it is diagonalisable, ok. This is one theorem, we will see one example on this. So, if we cannot find the n set of linearly independent vector, then it is not a diagonalisable matrix, ok.

(Refer Slide Time: 06:47)

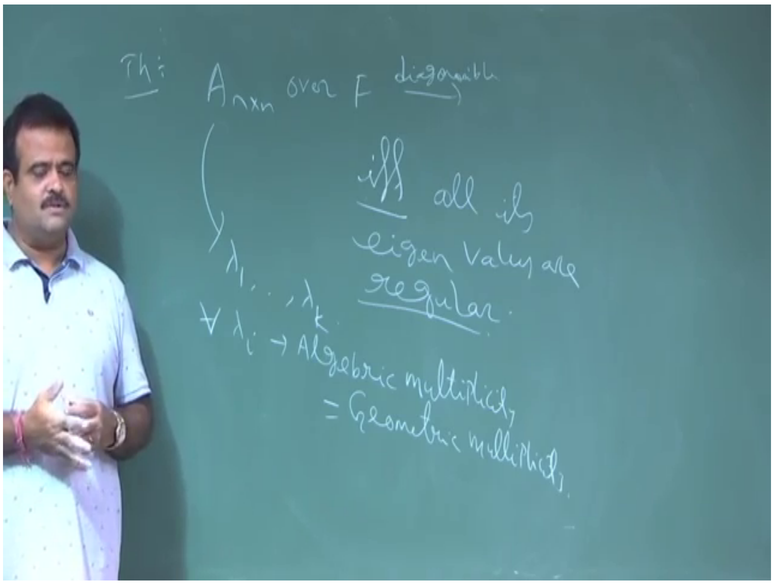


Another theorem is telling so, A is a matrix over a field then it is diagonalisable if this is sufficient condition. If the Eigen values of A are all distinct, this is not the necessary conditions, this is sufficient condition. If the if Eigen values of A are all distinct and belongs to F, belongs to the field F ok. This is a another sufficient, because if this is true if we have this; that means, we know if we have Eigen values are distinct say lambda 1 there are n Eigen values they are distinct, then the corresponding Eigen vectors lambda 2, then we have seen this result. Corresponding Eigen vectors these are, these are linearly independent set independent linearly independent set of vector.

Now, once it is linearly once we get linearly independent set of vector; that means, this implies A is diagonalisable ok. Now, if they are distinct then we have this result we have seen earlier. So, if we have distinct Eigen values and the corresponding Eigen vectors are linearly independent, then they are diagonalisable. So, this is another property, this is another sufficient conditions. And there is another condition which is also necessary sufficient condition for the diagonalisable of the matrix A, this is also if and only if.

So, so A is diagonalisable if and only if this is necessary and sufficient condition.

(Refer Slide Time: 09:22)

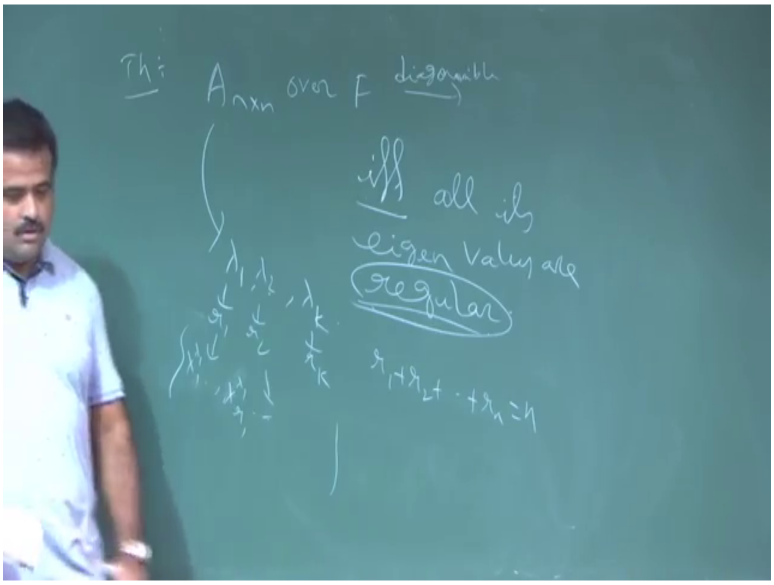


If and only if all the all it is Eigen values are regular, regular all it is Eigen values are regular. So, what do you mean by regular? We know the definition of regular. Suppose all suppose A is a matrix so, it has Eigen value say lambda 1; say, there are k Eigen values lambda to lambda k ok. The distinct Eigen values, now regular means if you take a Eigen value lambda i. So, it has a geometric multiplicity algebraic multiplicity, suppose it is algebraic multiplicity is so, for every for all Eigen values algebraic multiplicity must be same as geometric multiplicity. Algebraic multiplicity must be equal to geometric multiplicity.

So, we know the algebraic multiplicity, algebraic multiplicity means how many times it is coming in the root of the characteristics equations. And geometric multiplicity is if you consider the vector space for this Eigen values, I mean Eigen vectors that will form a vector space with the null vector, and the dimension of that. So, that means, number of linearly independent vector in that set. So, if the number of linearly independent of that set is same as the algebraic multiplicity of that then; that means, if you combine all the Eigen values we have we have n such linearly independent Eigenvectors.

For example, suppose we have; how prove this? So, suppose we have say this is true for all, algebraic multiplicity the regular means algebraic multiplicity and geometric multiplicity are same. Suppose we have these are the distinct Eigen values of A and their multiplicity say r 1, r 2, r k.

(Refer Slide Time: 11:56)

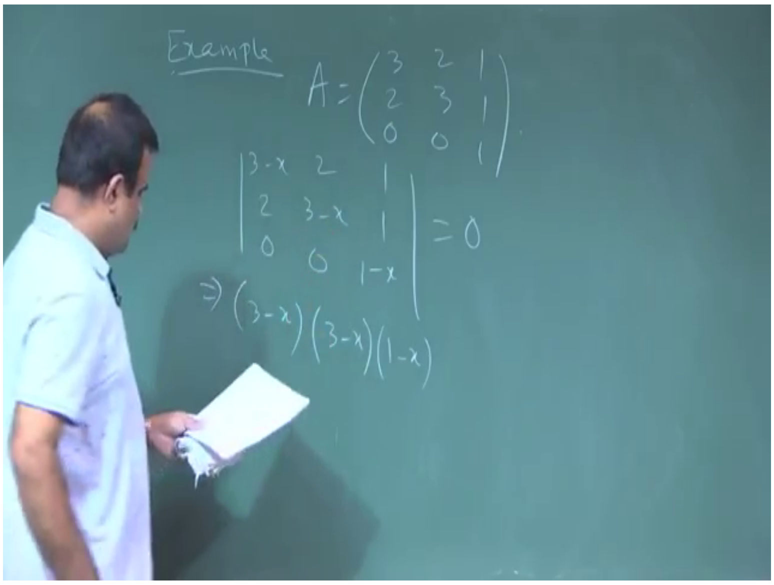


So, that means, lambda 1 is r 1 time; so that means, lambda 1 is coming r 1 time in the characteristics equation I mean root of this characteristics equation lambda 2 is coming r 2 times like this. So that means, total root is n r 1 plus r 2 plus r k is equal to n. These are the algebraic multiplicity.

Now, we have we the regular means the geometric multiplicity is also same. So, that means, the Eigen vector space of corresponding to lambda 1 it has dimension r 1. So, that means, it has r 1 independent vectors. So, r 1 independent vectors so, this is x corresponding to lambda 1. So, x lambda 1 1, x lambda 1 x lambda 1 r 1, like this. So, these are independent, similarly for this these are independent. So, eventually we have all are n, eventually we have n linearly independent set of Eigen vectors. Once we have that, then we know the earlier theorem necessary and sufficient condition that this is diagonalisable. So, this is another way to look at it.

So, let us take an example let us take an example of a matrix A and we will see whether this is a diagonalisable or not. So, we have to find the matrix P.

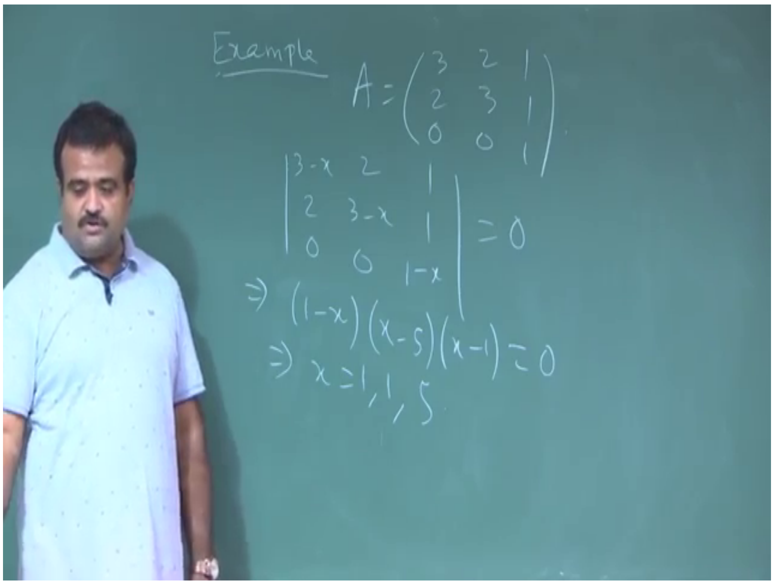
(Refer Slide Time: 13:35)



So, again how to get the matrix P that is important. Suppose we have a matrix like this 3, 2, 1, 2, 3, 1, 0, 0, 1 ok. Now we are going to check whether this matrix is diagonalisable or not. So, that means, we need to find out the P such that it will be if we multiply P inverse both side it will it should give us the diagonal matrix, and that diagonal elements nothing but the Eigen values. So, basically first we need to find the Eigen values of this.

So, to find the Eigen values we need to consider the characteristics equation of this matrix. So, this is 3 minus x 2 1 2 3 minus x 1 0 0 1 minus x, this is the characteristics equation of this matrix. Now if you simplify this, we will get 3 minus x into this into this. So, this will be 3 minus x times 1 minus x this is 0, then minus 2 this. So, anyway so, you can do this calculation eventually we will be getting I have the calculation but you can work out by your own 1 minus x into now this will be there in the note x minus 1 equal to 0.

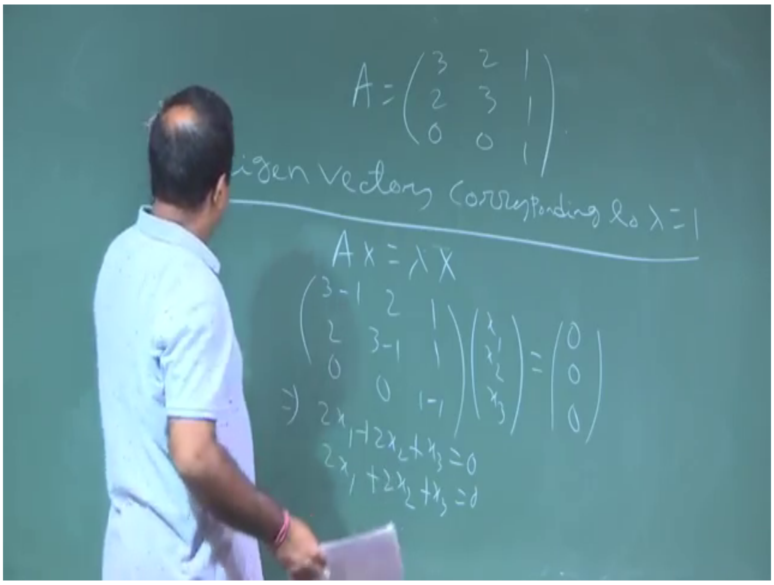
(Refer Slide Time: 15:05)



So, this will be giving us x equal to 1 1 5. So, we have these are the Eigen values. So, we have 2 distinct Eigen value 1 and 5. And they are all coming from the same field R; here Eigen value field is the real number. So, one is the algebraic multiplicity is 2 and 5 is the algebraic multiplicity is 1 1, 5 is coming 1. So now, we have to find their geometric multiplicity. If the geometric multiplicity, if they are regular so, that means, if the geometric multiplicity; the geometric multiplicity of 5 is 1, because geometric multiplicity has to be greater than 1. So, for 5 we have no issue that we have a we have a Eigen vector. Now consider this we have to check with the 1, if for one we have we need to have a geometric multiplicity also 2. It must be less than 2, because geometric multiplicity is less than algebraic, less than equal to algebraic multiplicity. So, we have to find the Eigen vector corresponding to one.

So, let us try that, and if we can get 2 distinct Eigen vector corresponding to 1, then we are done.

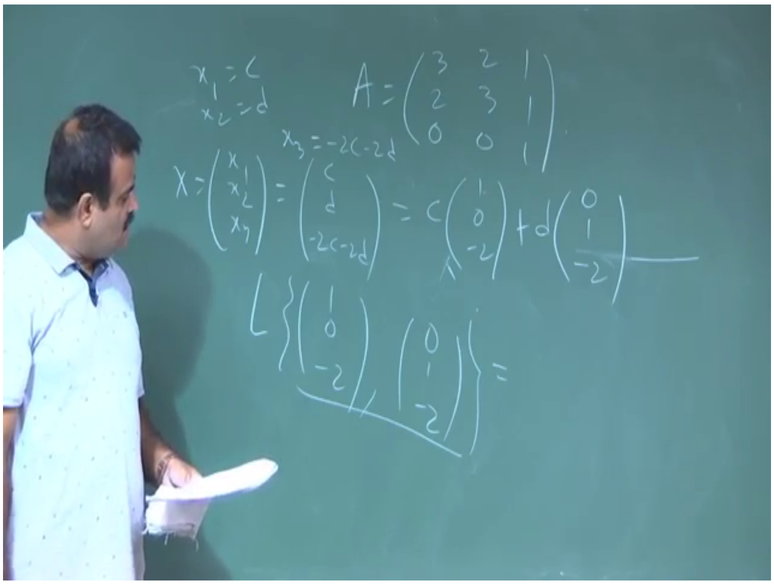
(Refer Slide Time: 16:32)



So, let us try to get the Eigen vectors corresponding to the Eigen value lambda is equal to 1. So, how to get that? So, for that we need to solve this equation A minus. So, that is basically A x equal to lambda x. So, lambda is 1. So, if we take this side so, it will be giving us. So, 3 minus lambda is 1 1 2 1, then 2 3 minus 1 0 0. Then 1 minus 1, and x is say x 1, x 2, x 3 this is 0 0 0, this is the this is our x.

Now, if you solve this we will be getting like 2 x 1, 2 x 1 plus 2 x 2 plus x 3 equal to 0. And then we have only 2 equation, because this becoming 0. 2 x 1 plus 2 x 2 plus x 3 equal to 0. So, we have these 2 are same determinant. So, that means, the solutions are coming as you keep this.

(Refer Slide Time: 18:08)

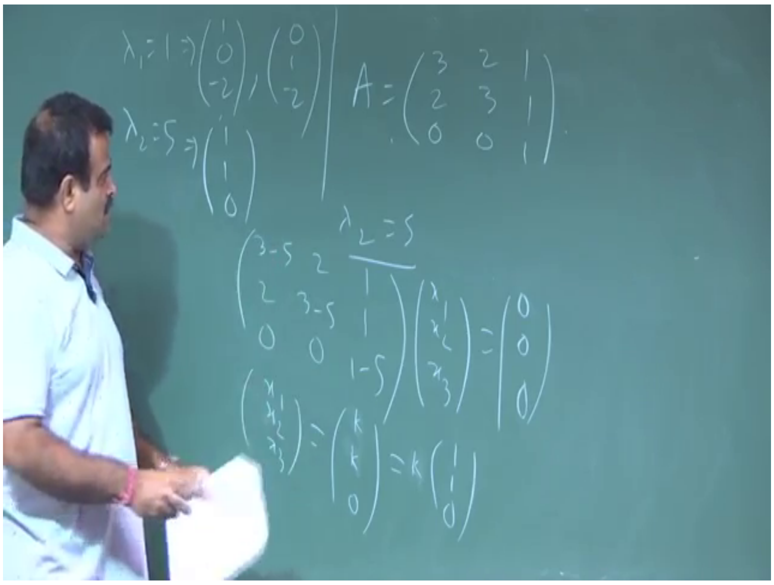


So, the solutions are coming as x equal to x 1, x 2, x 3. So, if we choose to be 2 variable we have to choose x 1 is say c x 2 we are choosing d, then x 3 is x 3 is decided by this equations. So, x 3 is minus 2 c minus 2 d.

So, this is becoming like c d minus 2 c minus 2 d. So, that means, it has 2 dimension, c into 1 0 minus 2 plus d into 0 1 minus 2. So, this 2 are the linearly independent vector; like 1 0 minus 2 0 1 minus 2. So, linear span of this is the Eigen set of Eigen vectors for this Eigen value. So, this is so, the algebraic multiplicity, these are linearly independent. So, geometric multiplicity of this is 2, ok. So, these 2 are linearly independent set of vector so, this is 2.

Now, we have to get a another vector which is corresponding to the 5, and we know that they are distinct. So, lambda 1 is 1, lambda 2 is 5. So, they are distinct definitely the Eigen vector, these 2 will be independent with the Eigen vector corresponding to 5. So, just we need to get a Eigen vector corresponding to 5. So, these are the so, we keep it here.

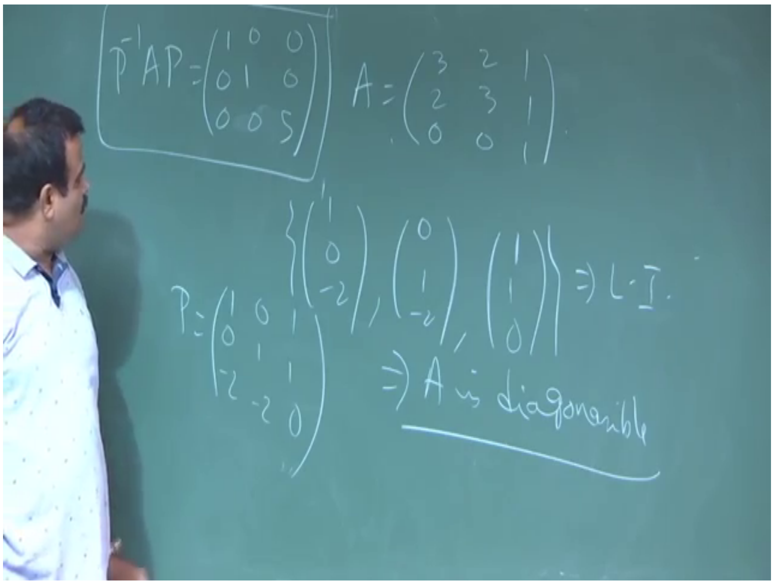
(Refer Slide Time: 19:55)



So, for lambda is equal to 1, we are getting this Eigen vector 1 0 minus 2 0 1 minus 2. Now we need to find the Eigen vector corresponding to another Eigen value which is 5. So, how to get that? So, lambda 2 is 5 so, for that we need to get solve this equation, 3 minus 5 2 1 2 3 minus 5 1 0 0 1 minus 5 x 1 x 2 x 3, this is 0 0 0 ok. So, if you solve this equations you will be getting like, we will be getting x 1 x 2 x 3 as c c 0. So, that means, this or k k 0, k is the another constant. So, k we take common so, 1 1 0.

So, this geometric multiplicity of this has to be 1. So, that is the vector which is generating the Eigen, Eigen vector apace. So, then for lambda 2; which is 5 we are getting this Eigen vector 1 1 0. And if you check if you take any 2 vector if you take a vector from here, and here this they will be independent because they are coming from 2 distinct Eigen values, and we know the result. If you have 2 distinct Eigen value, their Eigen vectors must be independent. So, this is these are the set of independent Eigen values. So, vectors so, we have 3 independent Eigen vectors so; that means, this a must be diagonalisable. So, this implies A is diagonalisable.

(Refer Slide Time: 22:02)

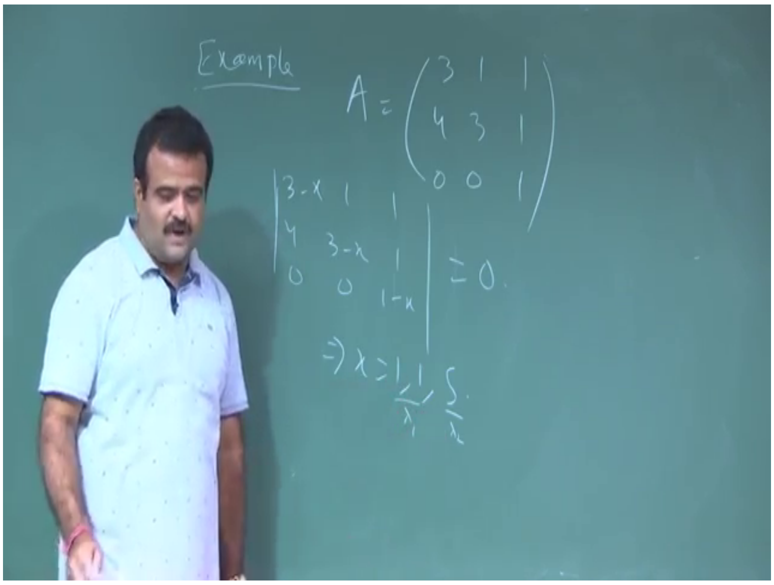


So, we have 3 set of so, this is 1 0 minus 2 0 1 minus 2. So, we have 3 set of linearly independent Eigen vectors. So, this implies A is diagonalisable, so, A is diagonalisable. So now, how to get that P? If A is diagonalisable, then it must be similar to the diagonal matrix where identity where the Eigen values are residing. So, for that, if you multiply P inverse A P, then we must get this diagonal elements. So, P are nothing but this matrix. So, so the P will be this independent matrix; so, this I can write this here. So, if you consider this P to be if we just take the column of this, 1 0 minus 2 0 1 minus 2 1 1 0. If you consider this to be P, then we can easily verify that if you multiply this P inverse A P, then it will be similar to a diagonal matrix. And that will be the diagonal element will be the Eigen values.

So, this will be the what are the Eigen values 1 1 5. So, 1 1; so, this 1 1 will corresponding to this 2 row and 5. When all the elements will be 0's other elements, ok. So, this P we can get from this we can easily verify this. If we just take this P and that A you multiply this and then again you multiply the with the P inverse of this, we have to find the inverse of this, then we will get the this diagonal matrix; where the this is lambda 1, where the Eigen values are sitting there, ok. So, that is a result so, if it is diagonalisable, then we have A P, P is nothing but the diagonal elements which are becoming this ok.

Now, if the so, we take another example; where Eigen values are not regular. In that case it is not diagonalisable.

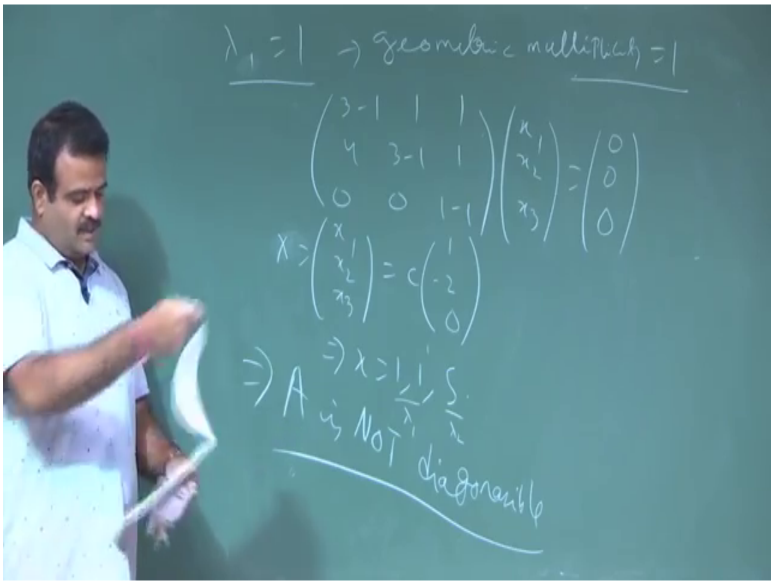
(Refer Slide Time: 24:44)



So, that example I can take quick example, we take another example. So, suppose we consider this matrix A which is again 3 by 3 matrix 1 1 3 1 1 4 3 1 0 0 1 ok. We want to check whether this is diagonalisable or not. So, for that we have to find the first the Eigen values of this. So, to find the Eigen values we need to take the characteristics equation 1 1 4 3 minus x 1 0 0 1 minus x equal to 0. So, this will give us the 3 minus 1 0 0 if you say solve this, it will be x is 1 1 5. So, the Eigen values are same as the earlier matrix, but the problem is here we will take we will try to find the; so, this is lambda 1 this is lambda 2. We have 2 distinct Eigen values lambda 1 and lambda 2 lambda 1 is 1, whose algebraic multiplicity is 2 and lambda 2 is 5.

Now, we just need to check the Eigen geometric multiplicity of this lambda 1 1. So, if it is not 2 then this A matrix is not diagonalisable.

(Refer Slide Time: 26:10)



So, for that we need to find the Eigen vector corresponding to lambda 1. So, for that we need to solve this equation 1 1 4 3 minus 1 1 0 0 1 minus 1 x 1 x 2 x 3, this is 0 0 0. So, if you solve this equation you will be getting this x to be x 1, x 2, x 3, this is nothing but some c into 1 minus 2 0.

So, this is bad luck. So, this is basically the dimension of this vector space, I mean Eigen space corresponding to this Eigen value is 1. So, we have only one independent vector. So, that means, this is the geometric multiplicity of this is multiply is 1; which is not same as the algebraic multiplicity. So, that means, we cannot get 2 independent vector, where we can constant the P. So, this implies A is not diagonalisable, we cannot get a P for which this is happening. So, we can have more example, so, in the note we will be giving some example to work out.

Thank you very much.