**Introduction to Abstract and Linear Algebra**
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**Lecture - 39**
**Similar Matrices**

We are talking about eigenvector and eigenvalues and some properties of the eigenvalues. Today, we will discuss the more on the Eigen some properties of eigenvalues, and then we will go for the, define the diagonalizable of a matrix. So, let us discuss another properties of eigenvalue.

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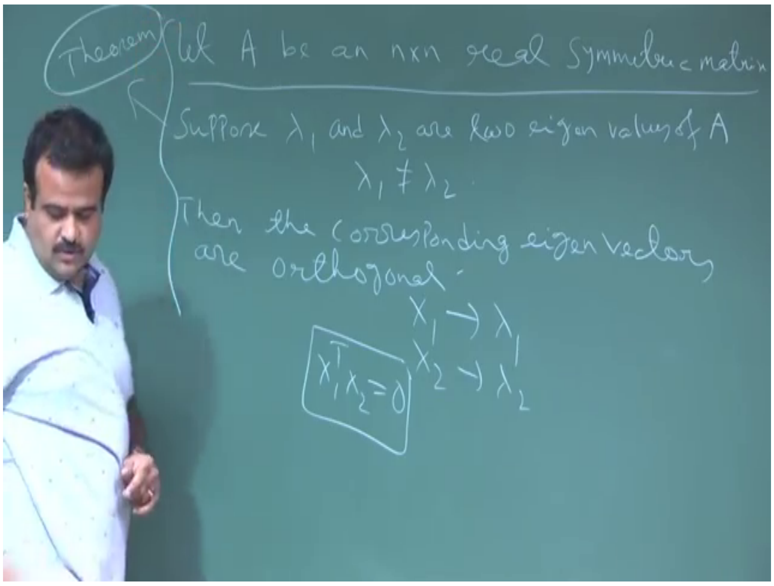


So, let A be an n cross n real symmetric matrix symmetric matrix. Then suppose, suppose lambda 1 and lambda 2, which are not equal, lambda 1 and lambda 2 are two eigenvalues of A, which are distinct to which are district that means, lambda 1 is not equal to lambda 2. Consider two eigenvalues lambda 1, lambda 2, which are not district. Then we want to know their corresponding eigenvector, their orthogonal, so that is the theorem.

So, then the corresponding then the corresponding eigenvectors are orthogonal, that means suppose X 1 is the eigenvector corresponding to lambda 1 and X 2 is the eigenvector corresponding to lambda 2, then this theorem is telling that X 1 transpose X 2 is equal to 0. So, this is the property that two vectors will be orthogonal ok.

We have to prove this; we are going to prove this. This is the theorem. This is the theorem or result, and we are going to prove this theorem. So, how to prove this, so we will just use the property of will just do use the definition of eigenvector and eigenvalue, and then we will take the transpose, and then the, this will this will be ok.

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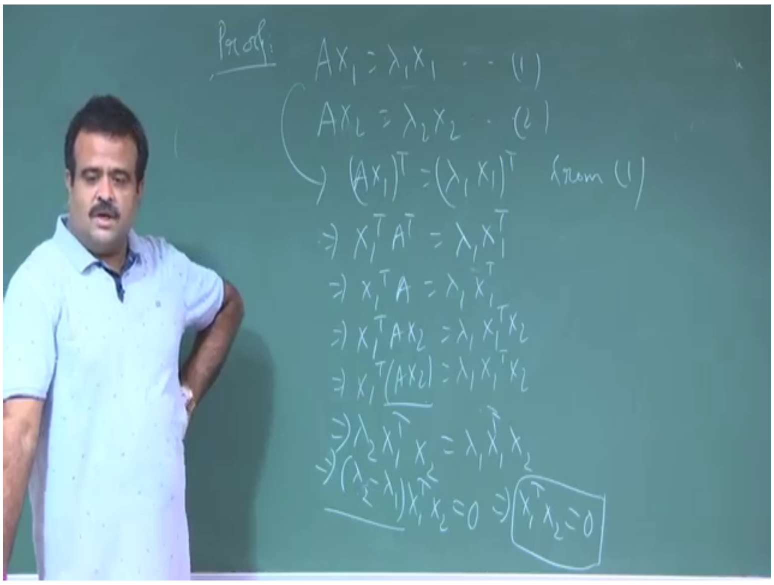


So, let us start with so, since X 1 is a eigenvector of corresponding to lambda 1. So, AX 1 equal to lambda 1 X 1, this is equation 1. And X 2 is the eigenvector corresponding to lambda 2. So, AX 2 equal to lambda 2 X 2, so this is equation 2 ok. So, we have these two equations. So, now from 1, we are going to take the transpose on this. So, AX 1 transpose equal to lambda 1 X 1 transpose, so from 1.

And then this will give us this implies X 1 transpose A transpose is equal to lambda 1 transpose lambda 1 transpose, where lambda 1 is a just a scalar, so X 1 transpose. And a A is a symmetric matrix real symmetric matrix, so A transpose is same as A. So, this implies X 1 transpose A is equal to lambda 1 X 1 transpose sorry X 1 transpose ok. So, now we want to bring X 2 over here, so AX 2 is equal to lambda 1 X 1 transpose X 2.

So, now we can take this, this is associativity property of the matrix multiplications lambda 1 X 1 transpose X 2. So, this implies now this we can use the 2, so this will give us lambda 2 X 2 ok. So, lambda 2 we can take outside, so lambda 2 X 1 transpose X 2 is equal to lambda 1 X 1 transpose X 1 transpose X 2. So, now we can bring this, this side, so lambda 2 minus lambda 1 X 1 transpose X 2 equal to 0, but lambda 1 is not equal to lambda 2. So, this is non-zero, so this has to be 0. So, this implies X 1 transpose X 2 is equal to 0, so that means, they are orthogonal. So, X 1 and X 2 are orthogonal.

So, if we have a real symmetric matrix, then we know then we know that the eigenvalues are all real. Then if you take two distinct eigenvalues, then the corresponding eigenvectors are orthogonal. So, this is the, this theorem is telling that. Now, (Refer Time: 06:24) I have more result on the eigenvalue say if we have a orthogonal matrix real orthogonal matrix, then if lambda is eigenvalue, then we will show that 1 by lambda will be also a eigenvalue of A, so that is another theorem.

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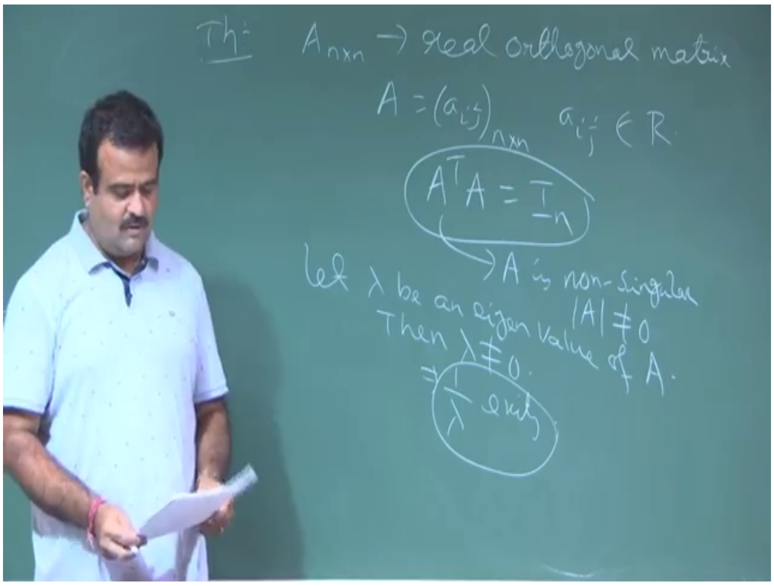


So, A is an the square matrix A is an real orthogonal matrix real orthogonal matrix, so orthogonal so that means, real means the all elements are coming from real field, so here our (Refer Time: 07:20) field is the real number. An orthogonal means a A transpose equal to A transpose A is equal to identity element. So, this is the orthogonal property of a matrix. Now, this is given. Now, since it is orthogonal, so that means, it is a non-singular matrix, this implies A is non-singular that means, determinant of A is not equal to 0. Now, once det A is not 0, then we know that all eigenvalues are non-zero.

So, so if we take a eigenvalue, let lambda be an eigenvalue of A, then lambda is non-zero, lambda is not equal to 0. Because, if lambda is becoming 0, then we know that product of the eigenvalue is det A is determinant of A of the form, so then the determinant of A must be 0. So, this result we have seen. So, then for any eigenvalue is not equal to 0. So, once it is non-zero, then we can think for 1 by that means, 1 by lambda exist in the real field ok. And we are going to show these lambda is a eigenvalue of this, then 1 by lambda is also an eigenvalue of A, so that is this theorem is telling all about. So, we have to prove this.

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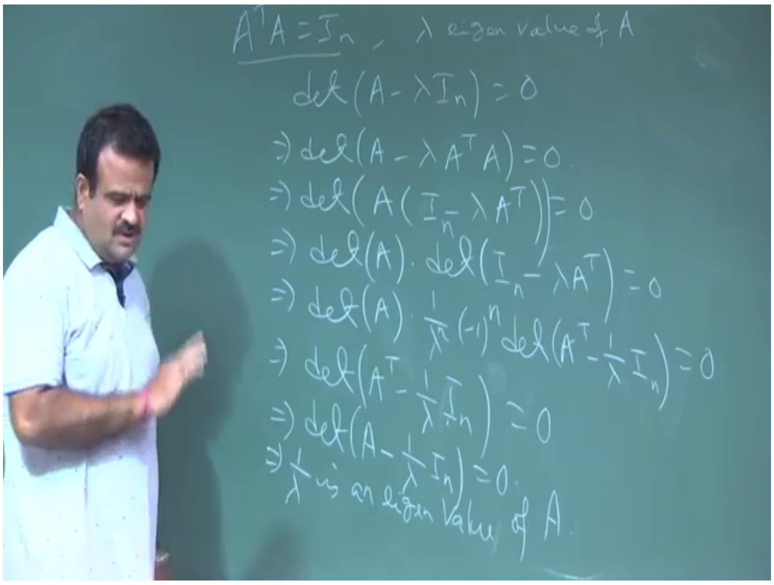


So, how to prove this? So, let us just so, a a A is a real orthogonal matrix, and lambda is an eigenvalue of A, then we want to show that 1 by lambda is also an eigenvalue of A. So, how to show this? So, since lambda is an eigenvalue that means, the characteristic equations will satisfy the root one of the root of the characteristic equation is lambda, so that means, lambda must satisfy, this det of A minus lambda I n is equal to 0.

Now, A is orthogonal, we can use this property over here. So, this implies det of A minus lambda of A transpose A equal to 0 ok. Now, we can take the determinant of det of A, we can take common, so this is I minus lambda of lambda of A transpose is equal to 0. Now, this we can take the determinant of A into determinant of I n lambda of A transpose equal to 0 ok.

Now, we want to take this side, so this minus 1 to the power n will come, and we want to take 1 by lambda. So, this will give us det of A into 1 by lambda to the power n minus 1 to the power n det of A transpose A transpose minus 1 by lambda I, this is to be 0. Now, this implies these are all non-zero, so this implies det of A transpose minus 1 by lambda I n this is to be 0.

Now, we can take the transpose of this that is also 0. So, transpose of this is also 0. So, this will give us so, this means this implies det of A minus if we take the transpose on that 1 by this is to be 0 ok. So, this implies so, this is this imply 1 by lambda is satisfying the characteristics equation, this implies 1 by lambda is an eigenvalue of A. So, this is the proof.

If lambda is an eigenvalue of A for a orthogonal real orthogonal matrix, then 1 by lambda is also an eigenvalue of A ok. so this is these are some more properties on this, we can have more properties in the lecture note. So, now we will move to the definition of a diagonalizable of a matrix, when we say matrix is diagonalizable. So, for that, we need to define the concept of singular matrix. So, when we say two square matrix of same size n cross n are similar, so that is the definition.

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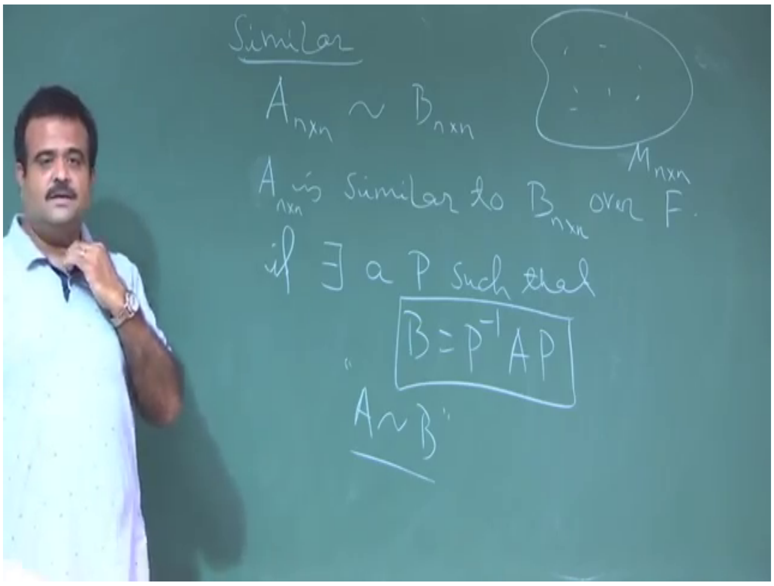


So, similar, so we say two matrix, this is n cross n is similar to B matrix, this is also n cross n. We have two square matrix, this symbol we can use for similar is not kind it is not necessarily row reduced or column reduces echelon form, but similar has a definition. So, A is a A is the matrix A is similar to a matrix B, which is of same size over the same field that is important over the same field. If there exist a non-singular matrix over the same field, such that such that B is of B is equal to P inverse A P. This is the definition, B is equal to A P inverse A P, P is a non-singular matrix, so p inverse will exist. So, then we say A is similar to B. This is the definition of this relation this is a relation between the matrix square matrix.

If we have if we have set of if we considered the set of all possible square matrix of same size n cross n, this is the matrix set of matrix of size m cross n, then we take any two matrix from this set, and we say they are related with this relation similar if and only if yeah if and only, one can obtained from other by just doing this operation P inverse A p ok. Now, if A similar to B, then B is also similar to A that is this relation is an and also A is similar to A, because in that case P is the identity matrix. So, this is a, this relation is a reflexive, because A is similar to A, and this relation is a symmetric.

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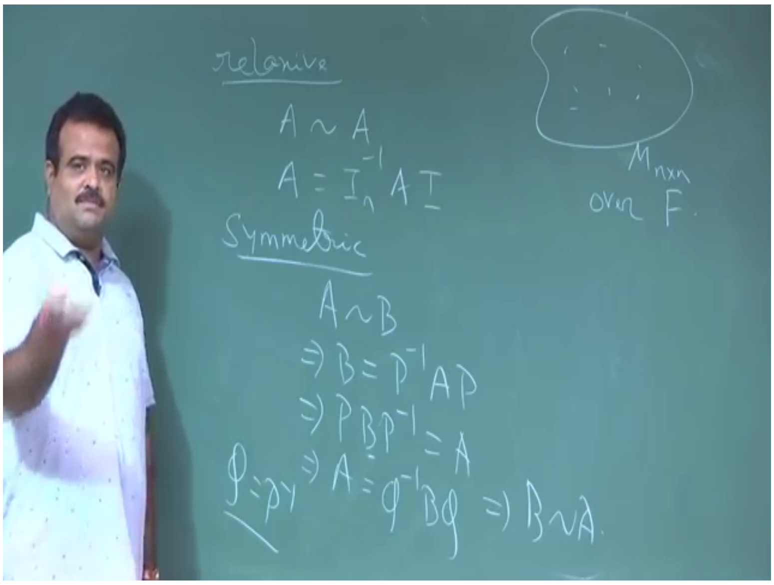


Because, if because if A is similar to B, so reflex, this is symmetric. The reflexive is the reflexive means A is similar to A, because we can write A to B identity matrix inverse A identity matrix, and this is our P ok. Now, symmetric, symmetric means symmetric means if A is similar to B, then we need to show B is similar to A. So, A is similar to B implies B should be written as P inverse A P. And from here we can write, we can write this as we can P B P inverse, so if you take this is to be A.

Now, if you take then A this implies A is written as Q inverse B Q, where Q is P inverse; P is non-singular, so Q is also non-singular. So, this implies B is similar to A. So, A is if A is similar to B, then B is similar to A. This is the symmetric property of the of this relation. Now, we will talk about their eigenvalue. Suppose, two matrices they are similar, then what about their eigenvalues, we have seen this result.

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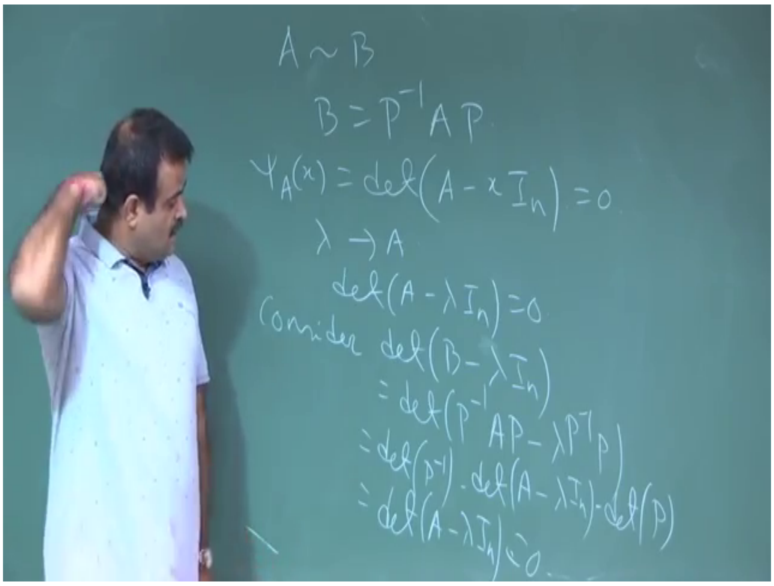


So, suppose we have two matrix A and B, which is similar, so the that means, either one of them can be written as from P inverse A P. Now, we have seen they have the same characteristics polynomial characteristics equation, so that will give us the same eigenvalue, so they have same eigenvalue. So, how to check that? So, characteristics equation of A is nothing but determinant of A minus x I n equal to 0 or if lambda is eigenvalue of A, then we have determinant of A minus lambda I n equal to 0. Then we have to show the lambda is also an eigenvalue of B.

So, for that, we can just take this consider determinant of B minus lambda I n. So, if we can show this determinant is 0, then lambda is also an eigenvalue of B. So, this determinant we can just write det of B is nothing but P inverse A P, and this lambda I n we can write P inverse P. So, this is nothing but det of P inverse into det of A minus lambda of I n into det of P. And the det of P, P inverse will get give us the one, so this is nothing but det of A minus lambda of I n. So, lambda as so, since this is 0, this is also 0, in fact we have the same characteristic equations.

This result we have seen earlier, so the that means, if lambda is the eigenvalue of A, then lambda will be an eigenvalue of B. So, they have same set of same set of eigenvalues A and B, if they are similar ok. But, the converse is not true that means, if the ,we have two matrix, which having same set of eigenvalues, they may not be similar ok. We have to we have to justify that by taking an example converse is not true.

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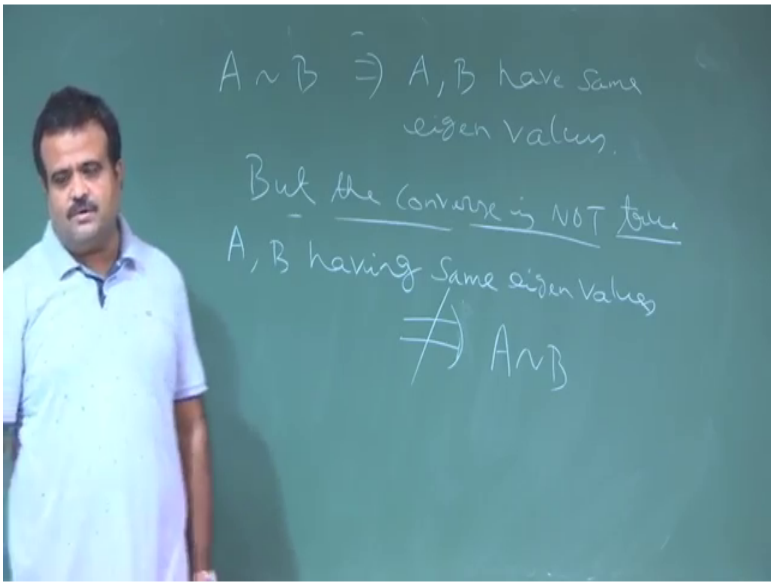


So, so result this result is if A, B are similar, this implies A, B have same eigenvalues, this just now we have seen, but the converse is not true converse is not true, so that means, if we have a two matrix, so we have to take a counterexample. We have to take two matrix which are having same eigenvalue that means, if you have two matrix having same eigenvalue eigenvalues, it does not mean that this does not imply that they are similar, this does not imply that they are similar. So, how to prove this to I mean this is a counter example, we have to show we have to get take a counter example on this. So, let us have two matrix A, B, where the eigenvalues are same, but they are not similar.

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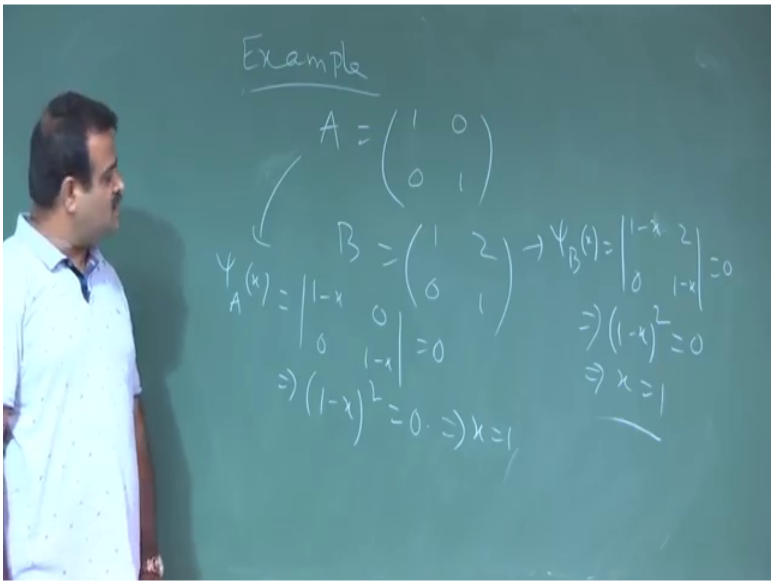


So, I just take two one example on (Refer Time: 21:35). So, suppose A is identity matrix 1 0 0 1, and B is 1 2 0 1 ok. So, what is the characteristic how to get the eigenvalue of this I have to get the eigenvalue of this, we have to take the characteristic equations, which is determinant of 1 minus x 0 0 1 minus x this is to be 0. So, this gives us 1 minus x square is equal to 0, so that means, x is x is 1. So, x is 1 with multiplicity two algebraic multiplicity.

So, we have a eigenvalue 1 with now, now how to get the eigenvalue of B, so to get the eigenvalue of B, we need to find the again we need to find the characteristics polynomial of B, so lambda B x, which is 1 minus lambda 2 0 1 minus sorry 1 x. So, this is the equation. So, this will give us the same equation 1 minus x square equal to 1. So, they have the same characteristic equations, and they have the same eigenvalues. Now, the question is the A, B are similar. So, to be similar, we need to have a square matrix like yeah we need to have a non-singular matrix P, such that B should be written as some A P transpose A B.

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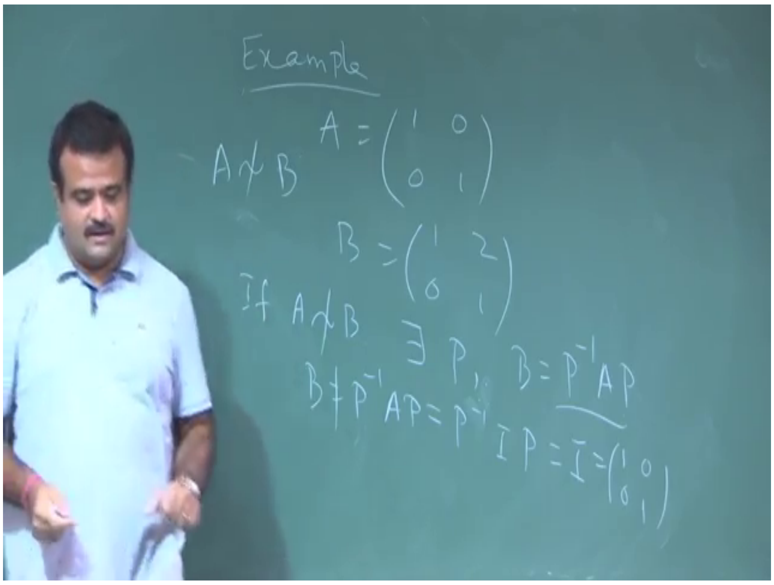


So, we need to have a; so, if A b are similar, this implies then there has to exist a square matrix non-singular matrix P, such that one of them should be written as either B or A, one of them should be written as P inverse A P, where P is a non-singular matrix. Now, we have to show that we have to check whether such a matrix P non-singular matrix exist. Now, what is this quantity? So the, that means, B should be written as P inverse A P. Now, A is identity and P is non-singular, so this is nothing but P inverse identity P, so this is nothing but identity again, so this is 1 0 0 1. But, B is not 1 0 0 1, B is 1 2 2 0, so the B this is not so the, that means, A is not similar to B. So, this is the one example, where two matrix is having same eigenvalue, but they are not similar ok. Now, we will define the diagonalizable of a matrix.

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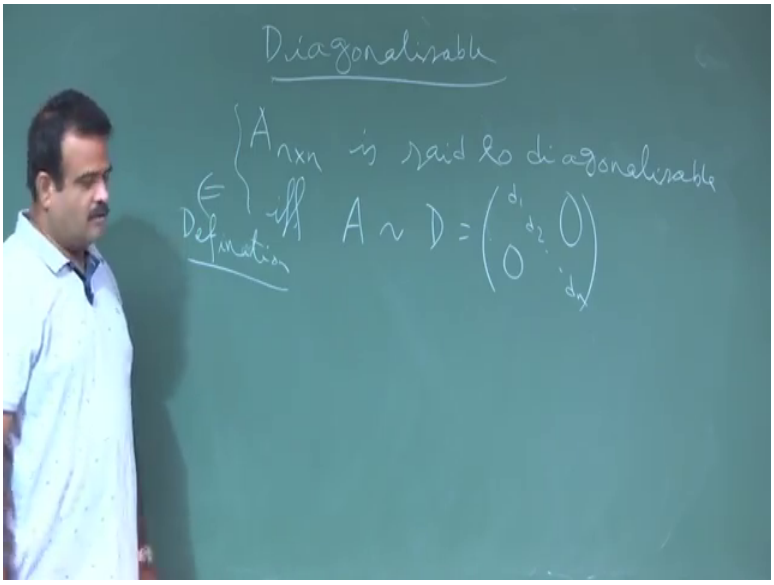


So, we say diagonalizable. So, suppose we have a square matrix A of size any size n by n, now we say this is diagonalizable, if it is similar to a diagonal matrix that is the definition. So, A is A is said to be diagonalizable, if and only if and only if A is similar to a diagonal matrix D, which is a diagonal matrix like we have some d 1, d 2 diagonal elements are non-zero, others elements are 0. So, then we say A is diagonalizable. So, if it is similar to a diagonal matrix, then we call A is diagonalizable. This is the definition ok. So, similar we know (Refer Time: 26:41) ok. So, we will we will just continue this in the next class.

Thank you.