Introduction to Abstract and Linear Algebra

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Lecture – 38

More on Eigen Value

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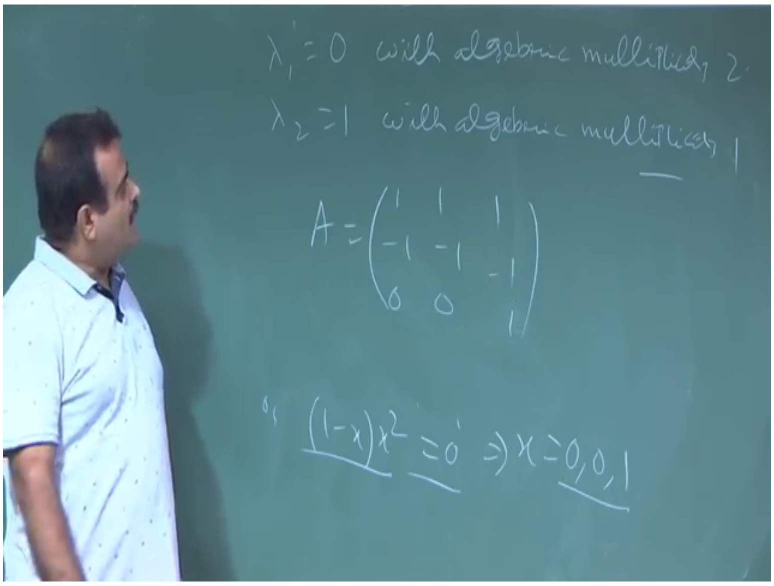

Ok, we are finding the geometric and algebraic multiplicity of a matrix. So, we have seen the matrix is 1 1 1 minus 1 minus 1 minus 1 0 0 0 0 1. And we have seen the roots of this matrix; we got the characteristics equation like this. And if we equate the roots, we are getting the two distinct eigenvalues. So, this 0 is coming twice, so that is why 0 is eigenvalue 0 is having a algebra multiplicity 2. And eigenvalue 1 is having algebra multiplicity 1. Now, we will talk about geometric multiplicity. So, for that we need to find the set of eigenvectors corresponding to this eigenvalue.

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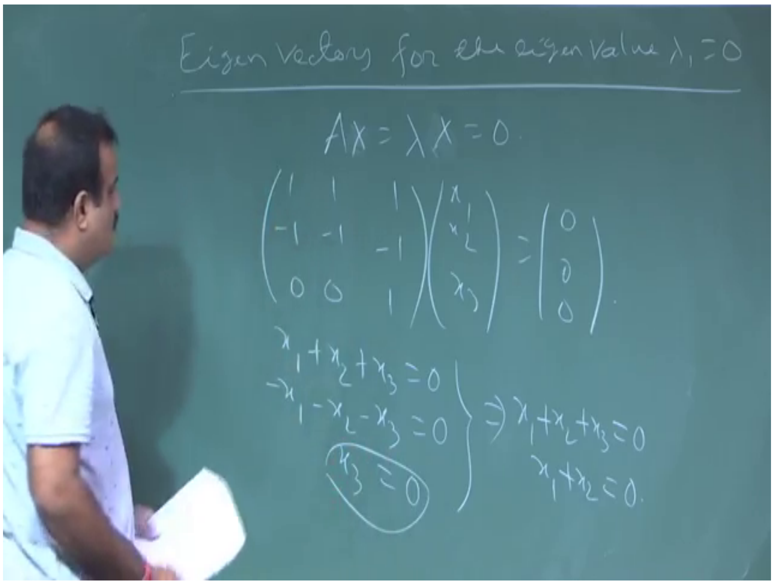

So, let us try with the first one with the eigenvalue 0. And I need to find the eigenvectors the set corresponding to the eigenvalue 0. So, eigenvectors for the eigenvalue lambda 1 equal to 0. So, how to get eigenvectors corresponding to the eigenvalue 0? So, for that we need to get the all the non-zeros so this is X all the non-zero vector X such that this is true. Now, lambda is 0 so that means this is 0. So, we have to find the all the non-zero vector X for which X equal to 0.

So, what is A, A is nothing but 1 1 1, then the minus 1 minus 1 0 minus 1 and 0 0 1, so these into x 1, x 2, x 3, this is the 0 vectors. So, if you simplify this, we will be getting like x the equation likes x 1 plus x 2 plus x 3 n equal to 0, then minus x 1 minus x 2 minus x 3 equal to 0, and then x 3 equal to 0. So, this will give us x 1 plus x 2 equal to 0, and x 1 yeah if you put this to be 0, so we have x 1 plus we have basically two equations, and x 1 plus x 2 equal to 0. So, you have option for x 3, so if you choose say x 3 is 0, we have no option for x 3 sorry.

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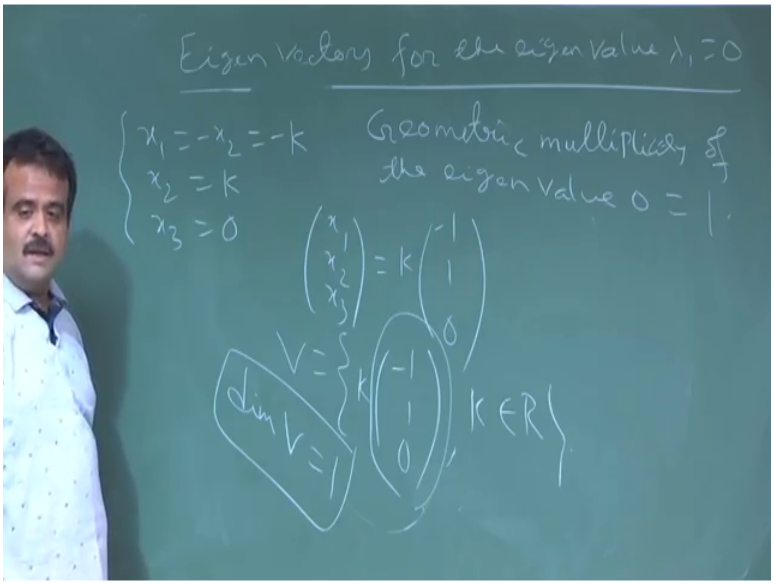

So, now what are the values, now we can choose x 1 to be K or x 2 to be K, so this will give us x 1 equal to minus x 2, which is say minus K, which is x 2 be K, and x 3 is 0. So, these are the, this is the set. Now, this will corresponding to the vector like so x 1 x 2 x 3 if you the write it in a column way, this is nothing but if you take the K common, so minus 1 1 0. So, this is the set of all possible eigenvectors corresponding to the eigenvalue 0.

So, now this is this set along with the 0. So, this set if K could be 0, all these are all real numbers. So, this is the real set. So the, you know this is a vector space, and dimension of this space is called geometric multiplicity of this. So, this is generating by the all vector, so dimension of this is 1. So, the geometric multiplicity of the eigenvalue 0 is 1. Geometric multiplicity of the eigenvalue 0 is equal to 1 ehereas, algebraic multiplicity of 0 is 2, because 0 is coming twice in the root of the characteristics equations.

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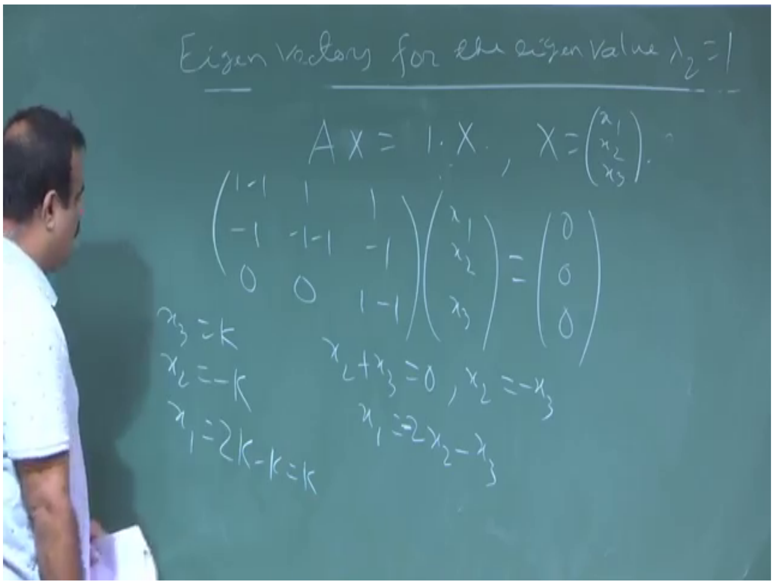

Now, we can similarly we can find the, which is this which is 1, and which is less than 2, which is the algebraic multiplicity of the, this. Now, we try to find the geometric multiplicity of the Eigen value lambda 2, which is nothing but 1. So, to get this we need to solve the homogene yeah will need to solve the equation A X equal to lambda 1 lambda 2 X lambda 2 X. So, this will give us lambda, lambda 2 is 1 we can just write 1 into X. So, if X is X is x 1, x 2, x 3, so this will give us 1 minus 1 1 1, then minus 1 minus 1 minus 1 minus 1, then 0 0 1 minus 1, x 1 x 2 x 3.

This is the corresponding A minus lambda X A minus lambda i 0 0 0. So, this will give us the solution like, so you this is 0, this is 0. So, this will give us solution like x 2 plus x 3 equal to 0, and so this means x 2 equal to minus x 3. And if you solve it, we will be get a x 1 to be 2 x 2 minus x 3. So, if you just choose x 3, x 3 equal to be K, then x 2 will become minus K, and x 1 is becoming 2 of minus yeah so x this is minus; so, 2 K minus K, so this is also K.

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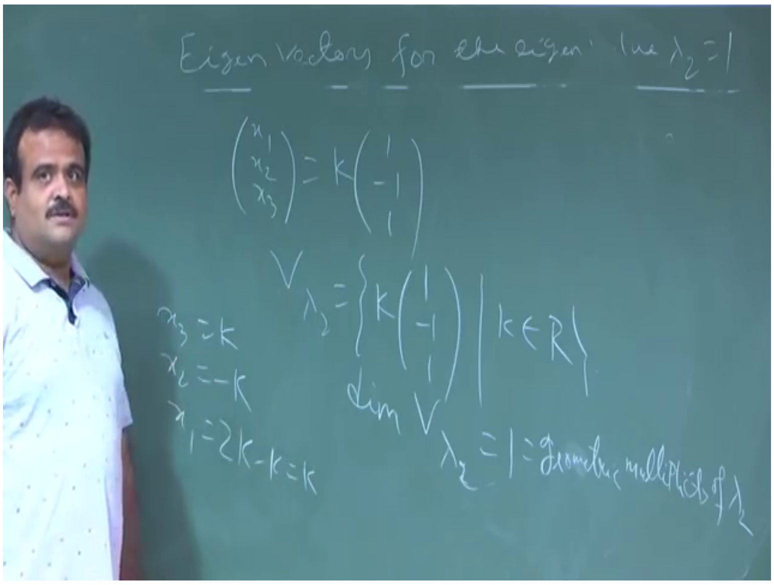

We have this x 1, x 2, x 3 is nothing but of this form. So, x 1 x 2 x 3 we take K common which is 1 minus 1 1. So, this is generating the so this is the corresponding this is the all possible eigenvector. So, every eigenvector is of this form. So, V of lambda 2 is nothing but so the dimension so this vector is generating the group, generating the field.

So, the dimension of lambda 2 is equal to 1, which is same as this is the geometric multiplicity. So, geometric multiplicity is the geometric multiplicity of lambda 2, which is same as algebraic multiplicity here. Geometric multiplicity is less than equal to algebraic less than equal to, so here the equalities occurring here ok. So, these are the geometric and algebraic multiplicity of geometric multiplicity of an eigenvalue ok.

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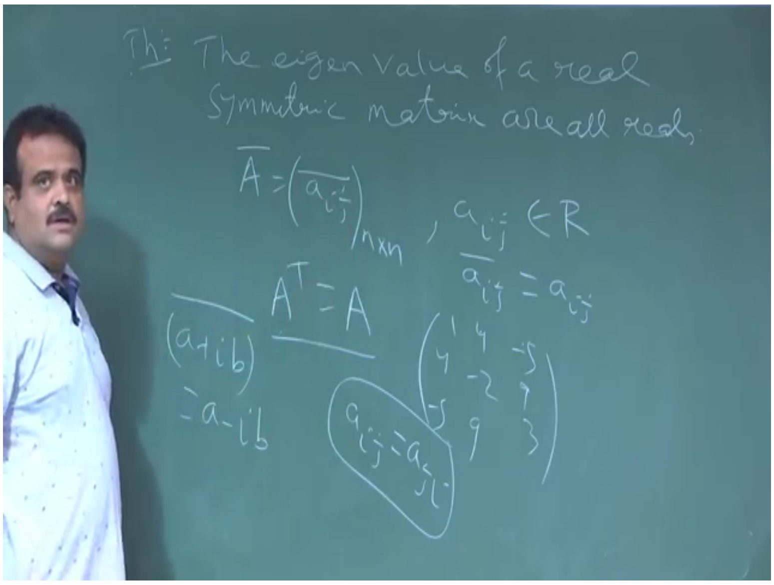

So, now we will talk about some properties of eigenvalues. Say, for example, if we have a real symmetric matrix ok, we if we have a real symmetric matrix, then the eigenvalues are all real. And if we have a skew symmetric matrix, we have we want to see eigenvalues are either 0 or they are completely imaginary. So, let us try first one. So, the theorem is telling the eigenvalue of a real symmetric matrix. We know the form symmetric matrix, symmetric transpose of A is A are all real; so, eigenvalue of all real eigenvalue of a real symmetric matrix.

Suppose, you have a real symmetric matrix a i j n cross n square matrix, where a i j's are real, so that means, if you take the conjugate of this, and that is same as this, this is conjugate. And it is symmetric means transpose if you take the transpose of this, this is same as A. This is an example of symmetric matrix. If we have 1 say minus 2 3, we have diagonal we have nothing to do, but if the value over here is same as value over here. So, if you have a say 4 over here, this has to be 4. So, value over here is same as value over here. If it is minus 5, you have minus 5 over here. If you have a value over here say 9, this has to be 9, so that means, a i j equal to a j i. So, this is the, for all i and j, then the matrix is called symmetric matrix.

So, now this theorem is telling if you have a symmetric matrix, and if the all coefficients are real that means, if the conjugate of a is same as this. So, conjugate means, if you have a complex number, a plus i b a plus i b, then the conjugate of this is defined as a minus i

b. But, if b is 0, then it is called real purely real, then a conjugate is same as this. And if a is 0, it is called purely imaginary, so that is the thing. So, now we have to so conjugate matrix means, we take the conjugate matrix we defined as this where we take the conjugate of each of these numbers.

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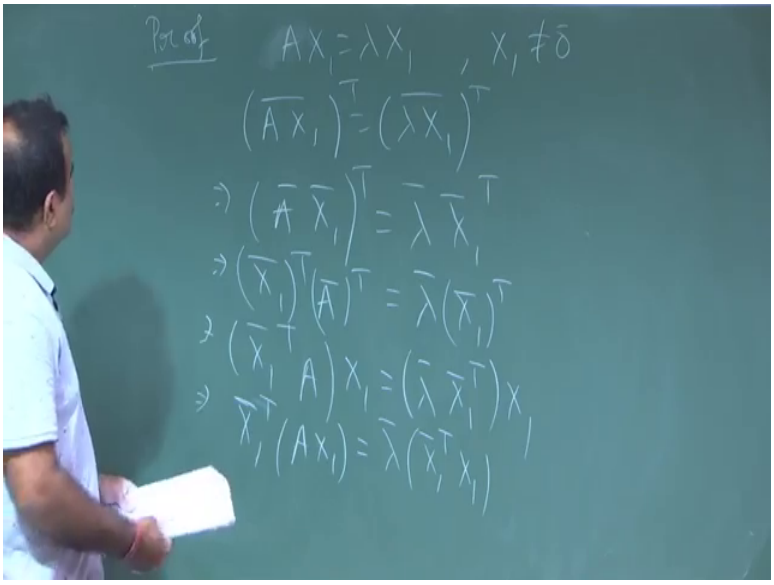

So, you are going to prove this that real symmetric matrix in real eigenvalues only. So, how to prove this? So, for this we taken, so suppose lambda is eigenvalue of the matrix A so that means, A x there is a non-zero vector X such that A X equal to lambda X or we can say X 1. Some non-zero vector X 1 not equal to 0 vector. So, what we do, we take the transpose of this now before transpose yeah, we take the conjugate. So, conjugate must be same, and then we take the transpose. So, if we take the conjugate, then it is nothing but A conjugate X 1 conjugate transpose equal to lambda conjugate X 1 conjugate transpose, then there is a scalar quantity ok.

So, now this will give us so transpose will reverse this, so this is x 1 conjugate transpose, and then lambda bar x 1 conjugate transpose. So, we have to simplify further ok. So, so now this we can write as this X 1 conjugate A X 1 equal to lambda bar, we just taking a both X 1 both side, lambda bar X 1 transpose conjugate X 1. So, if we further simplify this, we take this transpose we have to be carefully this transpose and T bar and transpose. So, this is A X 1, because which is associativity property is there bar X 1 bar transpose X 1 ok. So, let us just so up to this is fine.

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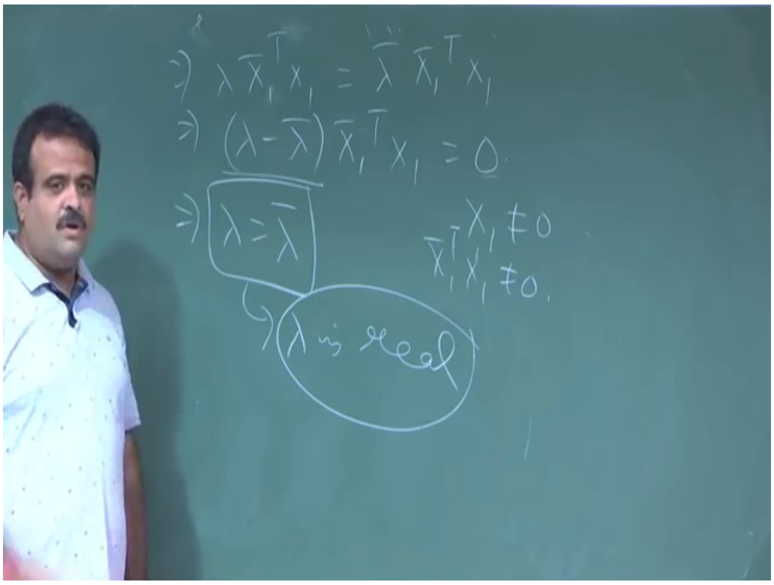

Now, this is again lambda of X 1, so the lambda this is again lambda X 1 bar transpose X 1 equal to lambda bar X 1 bar transpose X 1. Now, this implies lambda equal to lambda bar, because if you take this side, lambda minus lambda bar X 1 transpose X 1 equal to 0. Now, this is X 1 is non-zero, so that that means, X 1 bar transpose X 1 is not equal to 0. So, this means this has to be 0; lambda is equal to lambda bar. So, this means lambda is this means lambda is real purely real, no complex part, no imaginary part in it. So, this is the, this is one of the theorem. If the eigenvalues are if the matrix is symmetric real symmetric matrix, then the corresponding i all the eigenvalues are real.

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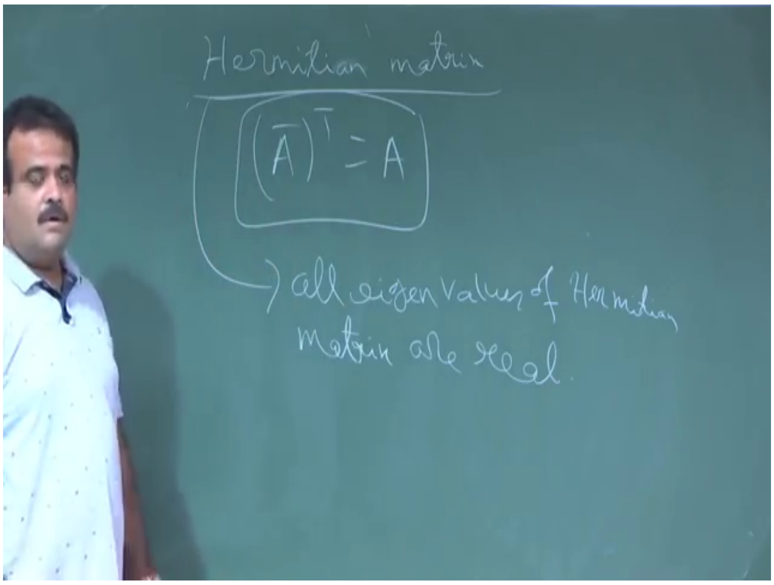

Now, this is true for Hamilton matrix also. So, we will define the Hamilton matrix sorry Hermitian matrix. So, when we say matrix is Hermitian matrix, if A bar transpose transpose equal to A. This is the definition of Hermitian matrix, so then this is called then then the matrix is called Hermitian matrix. Now, the way we proving the last just now same technique will go. And we can say the all the eigenvalues are all the eigenvalues of Hermitian matrix are real all eigenvalues of Hermitian matrix are real ok.

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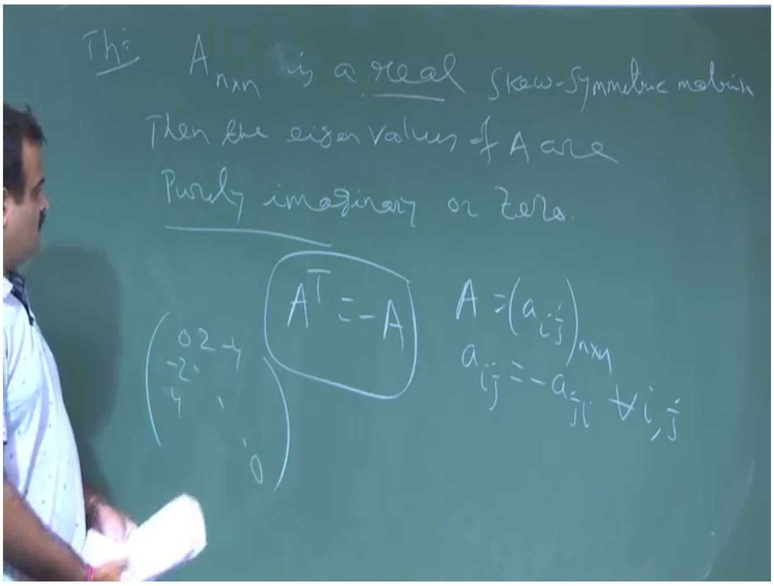

So, now we talk about real skew symmetric matrix. And we will see how their eigenvalue are (Refer Time: 17:38) A is n by n matrix skew symmetric matrix of the coefficient of real; it is real skew symmetric matrix. Skew symmetric matrix means I will I will define that symmetric matrix. Then the eigenvalues, eigenvalues are either purely imaginary or 0 and the eigenvalues of a are purely imaginary means there is no real part. So, like 2 plus 3i it is not purely imaginary. So, this part should not be there, so that is the purely imaginary.

So, eigenvalues are purely imaginary or 0 ok. So, so this is how to so what is skew symmetric matrix, so skew symmetric matrix is A transpose equal to minus A that is the skew symmetric matrix definition that means, if A is a i j n cross n, then a i j must be equal to minus a j i. And this true for all i and j starting from 1 to n, so that is why if the diagonal element, where i is j is equal to i, diagonal event a i i equal to minus a i i, so that means, 2 a i equal to 0. So, a i equal to 0. So, for any skew symmetric matrix the diagonal elements are 0. And off diagonal element if we have a so all the diagonals elements will be 0’s over here, and off diagonal element so the, this is 2 this has to be minus 2. If this is minus 4, this has to be plus 4, so that is the definition of skew symmetric matrix.

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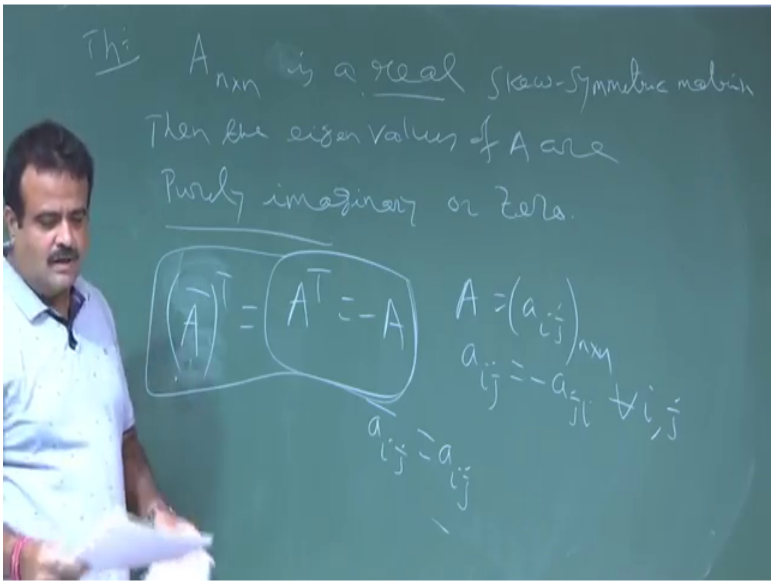

And it is a real skew symmetric matrix the that means, this a i j’s a real number. So, a i j’s conjugate if you take it is a i j so that means, this conjugate of so conjugate of sorry conjugate matrix is same as this. So, this is the definition of real skew symmetric matrix. So, if we have a real skew symmetric matrix minus A, yes if we have a real skew symmetric matrix, then we can show that the eigenvalues are either 0 or 1 sorry eigenvalues are either imaginarily or pure. So, similar way we can argue that.

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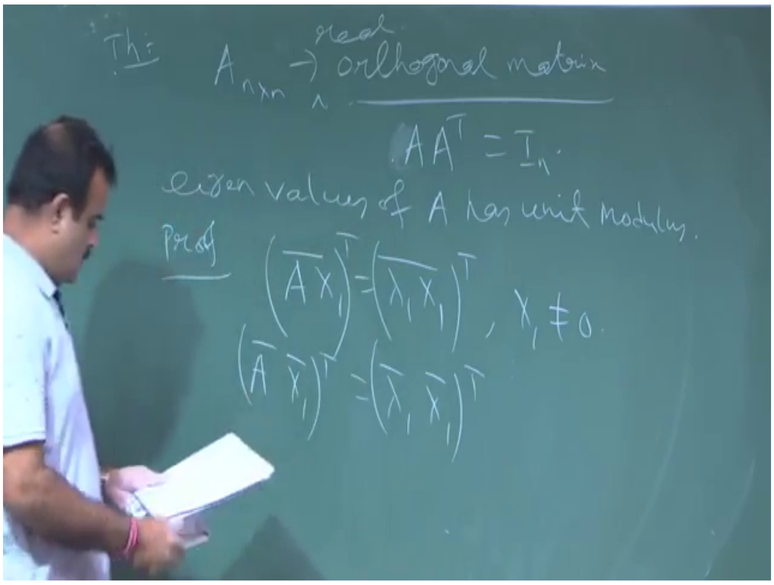

Now, we will talk about orthogonal matrix. So, this prove we are not doing is this will be in the note lecture note. So, now we talk about eigenvalues corresponding to the to a orthogonal matrix. So, so suppose A is orthogonal matrix orthogonal matrix. So, orthogonal matrix means A to A transpose equal to identity, so that means A transpose is an inverse of A. So, if we have a real orthogonal matrix, then the theorem is telling the eigenvalues values of A has unit modulus. Unit modulus means mode of that eigenvalue that means, if lambda is the eigenvalue, it is a complex number maybe mod of this is 1 that is the unit modulus had units mod modulus ok.

So, this we have to proof. So, how to proof this to prove this suppose yeah suppose lambda 1 is eigenvalue so, and corresponding eigenvector is A X 1. So, we have this while lambda, where X 1 is not equal to 0 and lambda 1 this. So, what we do we take the transpose of this, and it is which is the conjugate, then we will take the transpose of this. Then this is nothing but A bar transpose A bar X 1 bar transpose, then this similar thing we did in the last proof. So, it is this is orthogonal matrix real orthogonal matrix. So, we have to use that property also here to achieve that ok.

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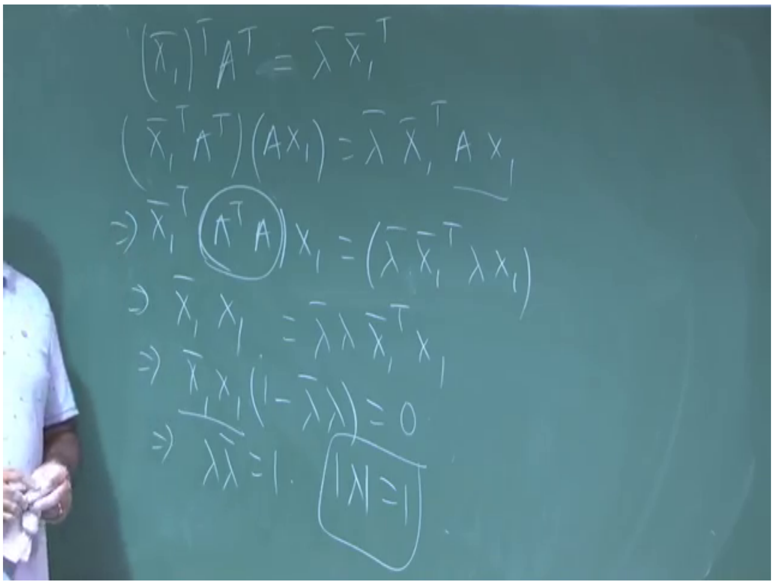

So, A bar transpose so if you take the transpose, it will give us X bar transpose and A bar. A bar is this is a real matrix so this A, so this is nothing but A transpose equal to so X 1 is there, so yeah X 1 is already there equal to lambda bar X 1 bar transpose ok. Now, we multiply both sides by A X 1. So, it is mode of it so we multiply both side by A X 1 transpose this A X 1 equal to lambda bar X 1 transpose A X 1 ok.

So, now we take this together A transpose A X 1 equal to this is A X 1 is nothing but lambda X 1. So, transpose lambda X 1, so this lambda X 1 what we can do yeah. So, this is identity matrix, because this is the property of orthogonal matrix. So, this will give us X 1 transpose X 1, this identity matrix. And this will give us lambda lambda 1 we take this side, so lambda sorry lambda 1 these are lambda or lambda 1 any way lambda is lambda. So, so lambda bar lambda X 1 bar transpose X 1 ok.

So, this will give us X 1 bar X 1 into 1 minus lambda bar lambda equal to 0. Now, this is non-zero quantity. So, this implies lambda bar equal to 1. Now, this implies mod of lambda equal to 1. So, if lambda is the complex number, so modulus is 1. So, it has unit modulus ok, so this has unit modulus. So, we will talk about more on this. In the next class, we will define diagonalization of the matrix. So, those we will we will discuss in the next class.

Thank you.