Introduction to Abstract and Linear Algebra
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Lecture – 37
Geometric Multiplicity

So, we are talking about eigenvector. So, just to recap to I will discuss the Geometric Multiplicity of eigenvector.

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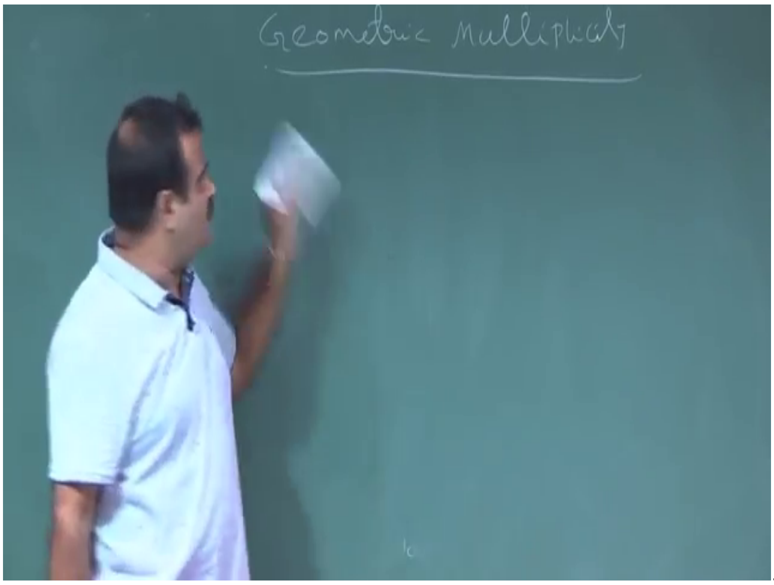

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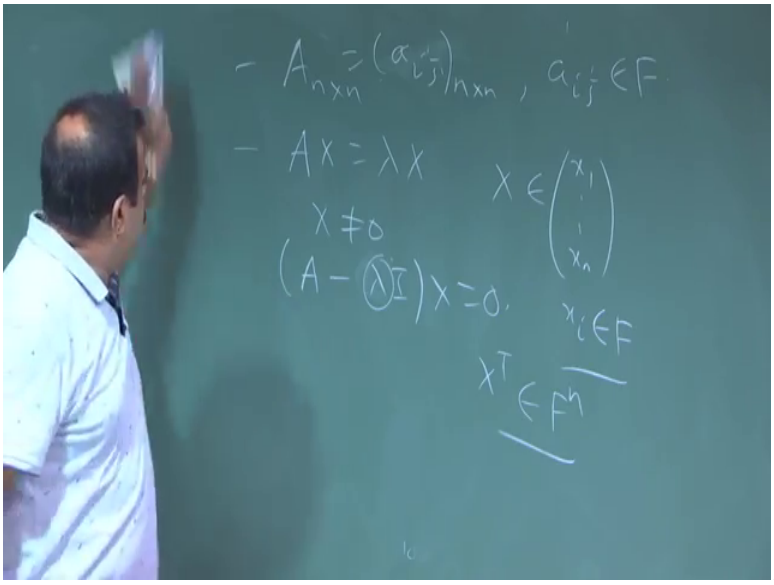

So, just to recap, so, we have a matrix, square matrix n cross n square matrix, then we call a non-zero vector to be eigenvector of this matrix if X is non-zero and this system has a lambda n lambda, this system has a non-zero solution and then that and we have seen also if we have a. So, this is corresponding to a also A eigenvector is corresponding to unique eigenvalue and for a eigenvalue we have a we could have a many eigenvector, but it should be from the same field, where this A i is coming from like if A is A i J n cross n and if the matrix element are coming from the field is then this eigenvector should be coming from the same tuple where each of X i is F.

So, this is. So, basically X T is belongs to F to the power n, but there are, they have they are must be from the same field. So, now, we will take example of a real Eigen real matrix where eigenvector will does not exist because of field. So, let us take an example. So, suppose our field is the real field and we have a matrix like this 0 minus 1 1 0.

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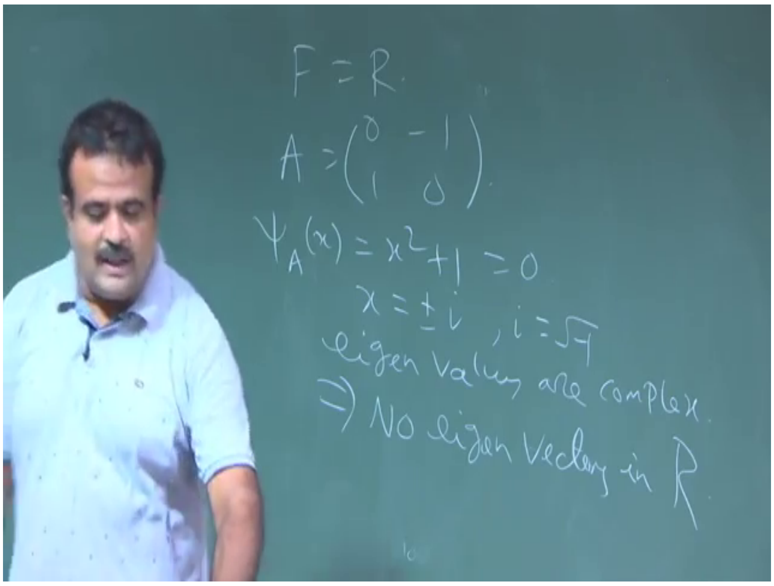

So, what is the characteristics polynomial? Characteristics polynomial is x square plus 1, so, characteristic equation is this. So, we have seen this example in the last class. So, we have two eigenvalues this, this is a complex number. So, this eigenvalues are complex, so, the so; eigenvalues are comple;x so that means, this imply there is no eigenvector exist in the real field, because of underlined filed is real, but if you allowed to have the complex field then we will come to that.

So, no eigenvectors in the real in the field. So, no eigenvector exist in that case, but if we allow our field to be the complex filed, then we can find the eigenvector corresponding to this eigenvalues.

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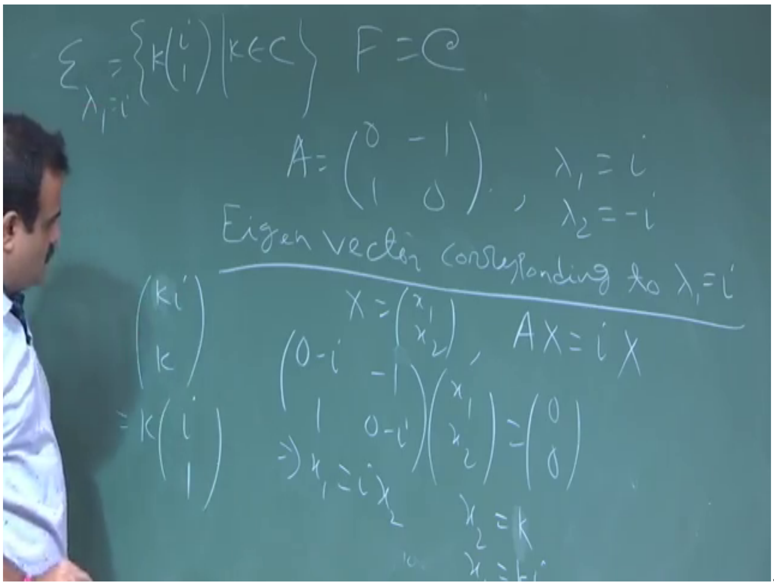

So, let us take this matrix again and now our field is a complex filed set of all complex number. And we have two Eigen values, lambda 1 is equal to i, lambda 2 is equal to minus i plus minus i 2 eigenvalues. So, we want to find the eigenvector corresponding to these two. So, eigenvector corresponding to lambda 1, this is i. How to find? This is a set of all non zero vector such that set of all non-zero vector now, they are coming from complex such that A X equal to lambda 1. So, lambda 1 is i X. So, this if we simplify, we will get the homogeneous equation like this, 0 minus i minus 1 1 0 minus i x 1 x 2 equal to 0 0.

So, that will give us e on equal to i x 2. So, that; that means, if we took x 2 to be K, any complex number then x 1 is x 1 is, K i so; that means, eigenvector of this form. So, 1 is K, now x 1 is K i and x 2 is K, so, this is of the form i 1, ok. This set if you vary K, K is a complex number; so, this is the this set is called the eigenvector, we can corresponding to the eigenvalue, this is the set of all vector like this where K is a complex number, ok. So, this is the eigenvector corresponding to this.

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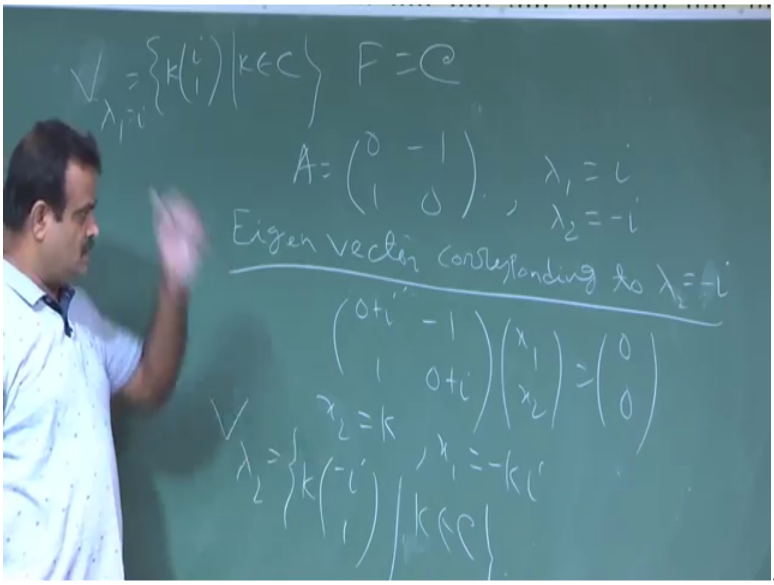

Similarly, we can find the eigenvector corresponding to other eigenvalue. So, corresponding to lambda 2, which is minus i, if we do that we will get the equation like this 0 plus i minus 1 1 0 plus i x 1 x 2 equal to 0 0. So, this will give us something like x 2 is equal to K and x 1 equal to minus K i. So, this is basically the set, so, this is eigenvector. We can write V actually so, this is a set of K into minus i 1, here coming from the complex field. So, this is the eigenvector. So, these are the eigenvectors corresponding to this, ok.

So, now we will get some result on the eigenvector. So, if you have two eigenvector which is corresponding to two distinct values then they are independent. So, this is the theorem.

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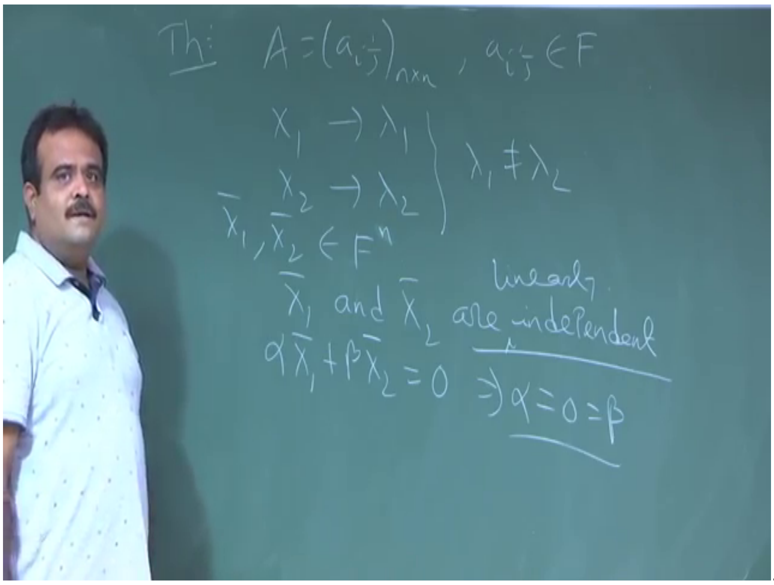

Again, suppose A is a square matrix, A are coming from the field, A are coming from the field a i J, if, and I suppose we have two eigenvector X 1 corresponding to lambda 1 and X 2 corresponding to lambda 2, we have two eigenvectors, they are corresponding to two distinct eigenvalues such that lambda 1 is not equal to lambda 2.

And this X is are coming from X 1 and X 2 are coming from the n tuple of the field where elements are from the fields this n tuple, ok. So, now, our claim is, these two are independent X 1 and X 2 are linearly independent set of vector, independent linearly independent, linearly independent over this. So, how to show that? To show that we need to take a, a linear combination of this and we have to show, how to show two vectors are linearly independent? So, we have to take alpha X 1 bar plus beta X 2 bar and if we are take it to 0, if this implies alpha equal to 0 equal to beta, then this is the linearly independent set of vectors, ok. You can take it column wise also does not matter.

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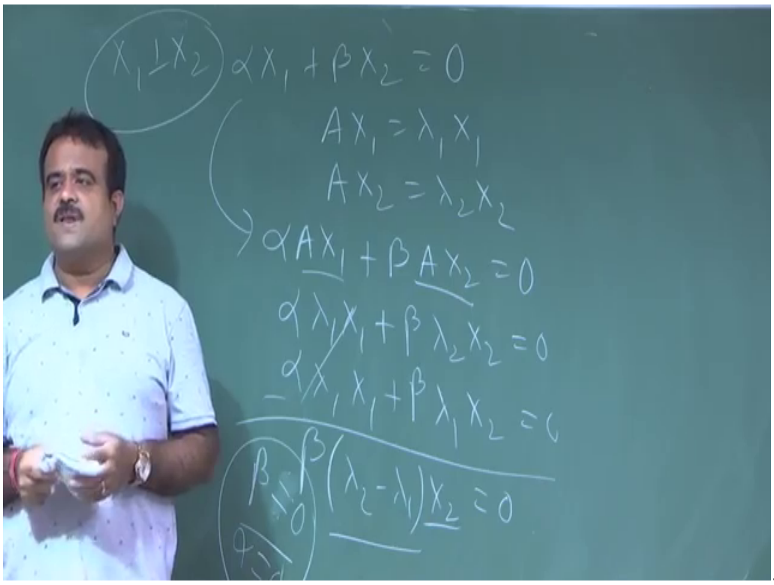

So, now, we take now to prove this linearly independent set of vector what we do, we just apply. So, we have in column wise if we take alpha X 1 plus beta X 2, we are taking this is 0, this is a column vector. Now, we know that A X 1 equal to lambda 1 X 1, A X 2 equal to lambda 2 X 2. So, now, if we apply A on this so, A lambda X 1 plus B sorry, alpha A X 1 plus beta A X 2, this is 0 vector. Now, so, now, this is one equations. Now, if we just alpha A beta Y, now alpha this is give us what this is the alpha lambda 1 X 1 plus this is A X 2 is lambda 2 beta lambda 2 X 2 equal to 0.

Now, if you multiply this with lambda 1, then we get alpha lambda 1 X Y plus beta, we are multiplying this with lambda 1, lambda 1 X 2 equal to 0. So, if you subtract this, this is cancel out, this is beta into lambda 2 minus lambda 1 X 2 equal to 0, but here none of them are 0, X 2 is non zero vector and these are distinct we assume, so, this gives beta equal to 0. Similarly, if you multiply this with lambda 2 and if you subtract will get sorry beta equal to 0, we will get alpha equal to 0 so; that means, these two are set are linearly independent X 1 and X 2 are linearly independent set of vectors. So, any two vector which are corresponding to the distinct eigenvalues, they are independent.

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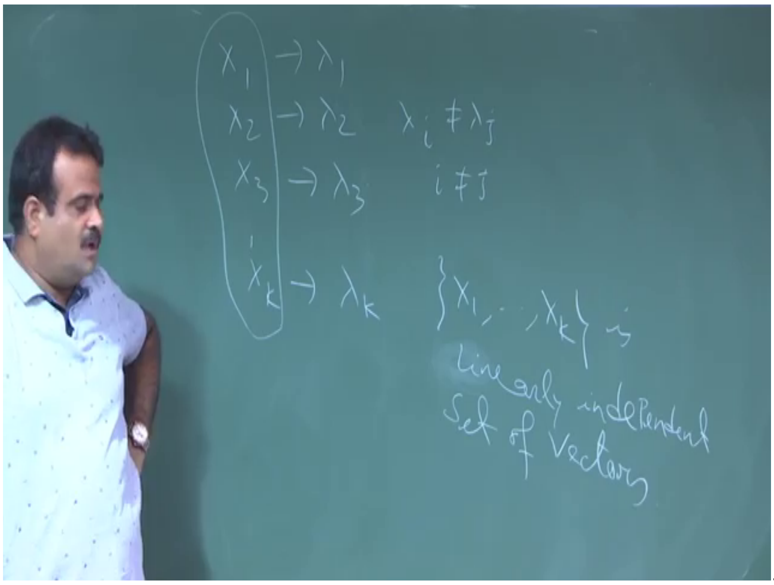

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So, this result hold for more than two also. So, if we have three eigenvectors corresponding to three distinct eigenvalues. So, if we have X 1 corresponding to lambda 1, X 2 corresponding to lambda 2, X 3 corresponding to lambda 3, if they are more also X K corresponding to lambda KS.

And if they are distinct, if lambda k are distinct if lambda is not equal to lambda J, i not equal to J; that means, this say this is corresponding to distinct eigenvalues, then this set is a linearly independent set of vectors. Then this set is a, is linearly independent set of vectors. We prove for two, we can extend it for more than two by the help of even for the induction inductive prove we can do, ok. Now, this is one observation and now we are going to define the geometric multiplicity of a eigenvalue for that we need to bring the vector space subspace. So, let us have the, ok.

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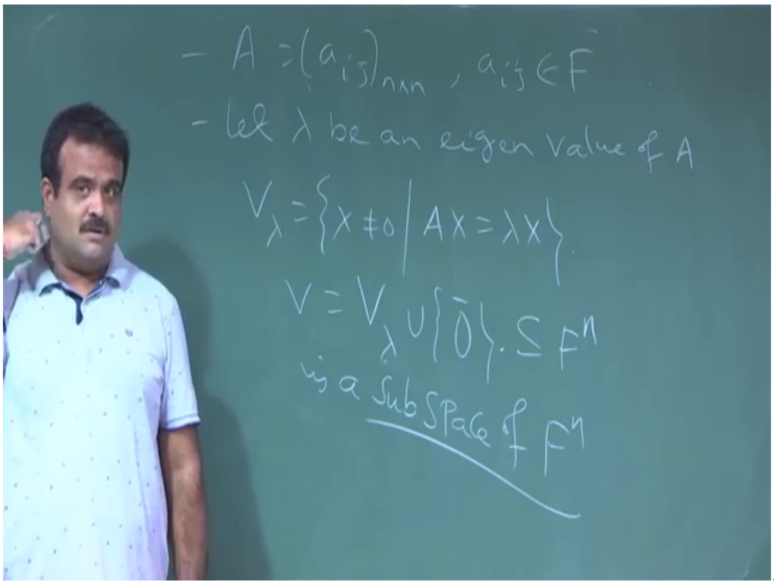

Now, A is an A is n cross n matrix coming from the field F. Let lambda be an eigen value of A, ok. So, it has many eigenvectors. So, that eigenvector we take it as this. So, this is the set of all non zero vectors, these are coming from the same field such that A X equal to lambda X, this is infinite set we have seen. Now, if we add this with if we take this vector V which is if we add the 0 vectors, because 0 is; obviously, has a will satisfy this. Then this is a, I mean if you take the transpose of this is a this is a subspace, this is a vector space and vector space this is a subset of V to the power n, I mean transpose of that V to af, af to the power n, n tuples, ok. So, and we know this is a vector space now this is subset.

Now, when we can say a subset is again from a vector space we have some measures in subset condition we will come to that. So, we have to show this is a, this V along with this is a subspace of this, ok. And the dimension of that subspace is nothing, but the geometric multiplicity of this lambda. So, so, first of all you have to show this is subspace. So, how to show this is subspace, ok. So, to show subspace we need to take two vector from this.

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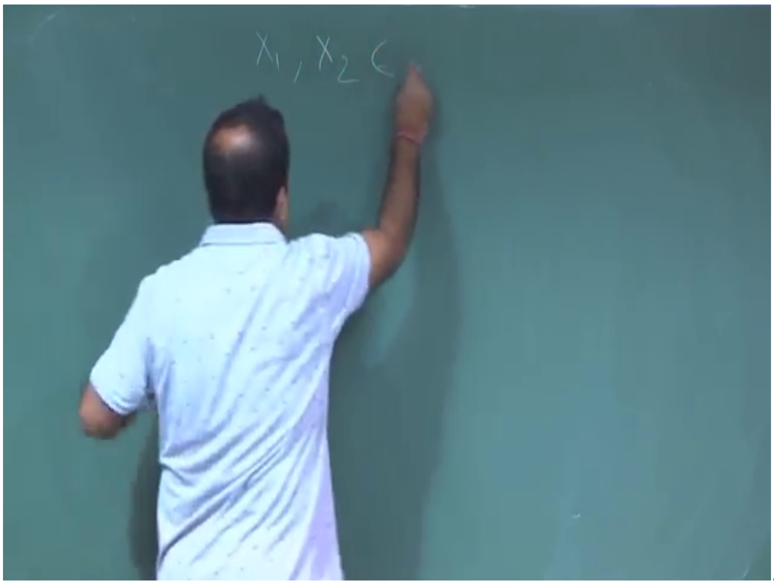

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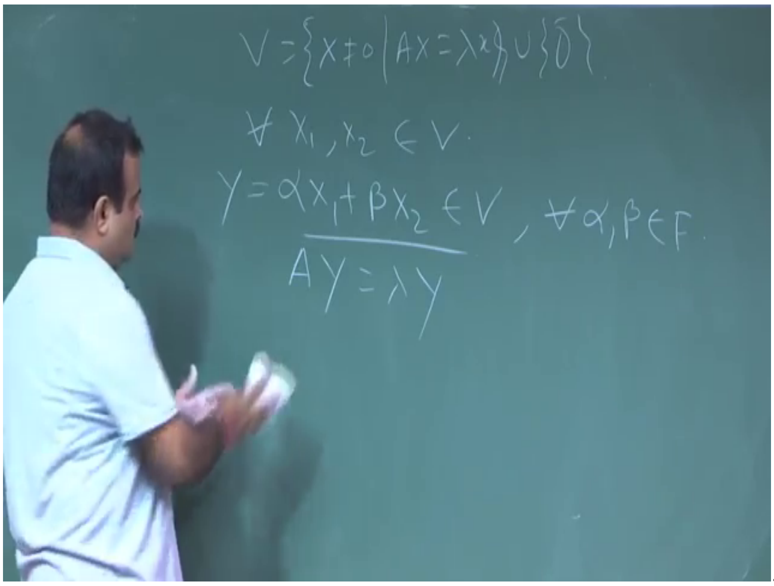

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So, let us have a vector space again. So, this is nothing, but the set X, we can take the transpose if you are comfortable with the row or column vector, but anyway these are the notation. So, X such that X is non zero such that A X equal to lambda X and we include the 0 vector you know 0 is obviously, a solution of this.

Now, if you take X 1 and X 2 from this, two vector X 1 and X 2 from this and if we can solve alpha X 1 plus beta X 2 also belongs to this. And if this is true for all such X 1 X 2 and for all the scalar this. Then this, this was a if you remember the vector space subspace there we prove that this is a condition necessary in sufficient condition to a subset to be become a any vectorspace under the same two operations. Anyway, so, we have to show this. So, how to show this? So, to get this we have to do A so, this is a this is Y. So, we have to show Y is belongs to this. For that we need to show A Y is equal to lambda Y, if we can show that then Y belongs to Y is again a vector space i, i in vector. So, what is A Y? A into alpha X 1 plus beta X 2.

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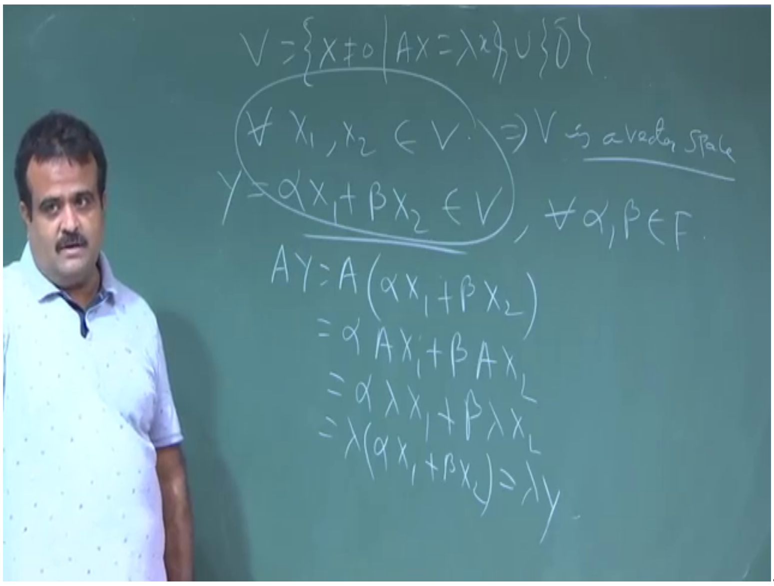

So, this which I have write alpha A X 1 alpha beta plus scalar. Now, this, this two are eigenvector corresponding to eigenvalue lambda. So, this is nothing, but lambda X 1 plus beta, lambda X 2. So, we take lambda common. So, this is alpha X 1 plus beta X 2.

So, this is lambda Y. So, A lambda is equal to lambda Y so; that means, Y is again A Y is either 0 or it is again a eigenvector of corresponding to lambda. So, this result is true. This implies that V is a vector space, which is a subspace this is a subset of the F to the power n. And the dimension of this vector space is defined as geometric multiplicity of lambda. The dimension must be less than n, less than equal to n, because n is the n is the dimension of the F n and it is a subspace. So, dimension has to be less than n. So, that is the geometric multiplicity of the eigenvalue lambda.

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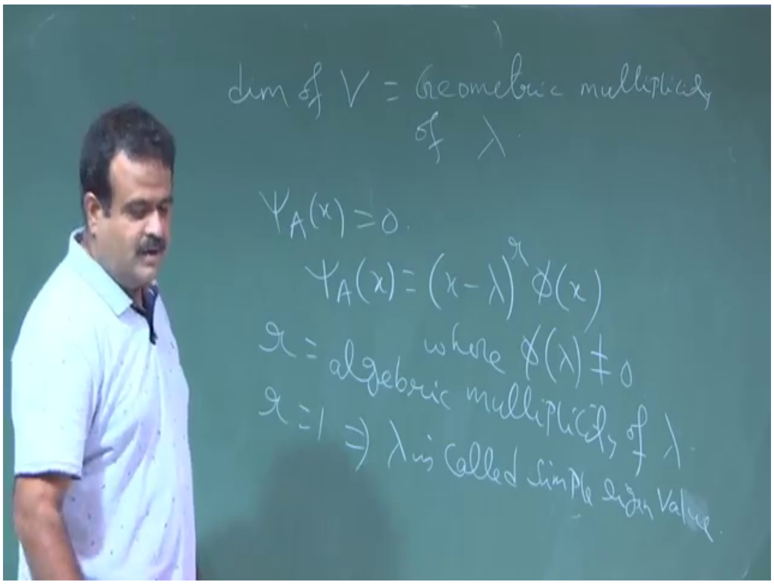

So, the dimension of V is called geometric multiplicity of lambda. We know the algebraic multiplicity, algebraic multiplicity is the number of times lambda is coming in the roots of this polynomial characteristic equation. So, that was the algebraic multiplication. What was the algebraic? If this is the; if this was the characteristics equation then if lambda is eigenvalue and lambda will be the root of this equation. So, if it is coming, if the root is of multiplicity r; so that means, lambda coming r times in the root. So, then we must write x minus lambda to the power r then some other polynomial where phi of lambda is not equal to 0. Because r is then r is called algebraic multiplicity; r is called the algebraic multiplicity of lambda, ok.

And this dimension of this vector space is called geometric multiplicity of lambda. So, we have a result that is called if r is 1; that means, if the lambda if the r is the lambda is the root with multiplicity 1 then lambda is called simple eigenvalue, ok and we have a result for geometrical algebraic multiplicity. So, that is telling us that algebraic multiplicity geometrical multiplicity is less than equal to algebraic multiplicity. We will get the geometric algebraic multiplicity through an example. So, this theorem is telling.

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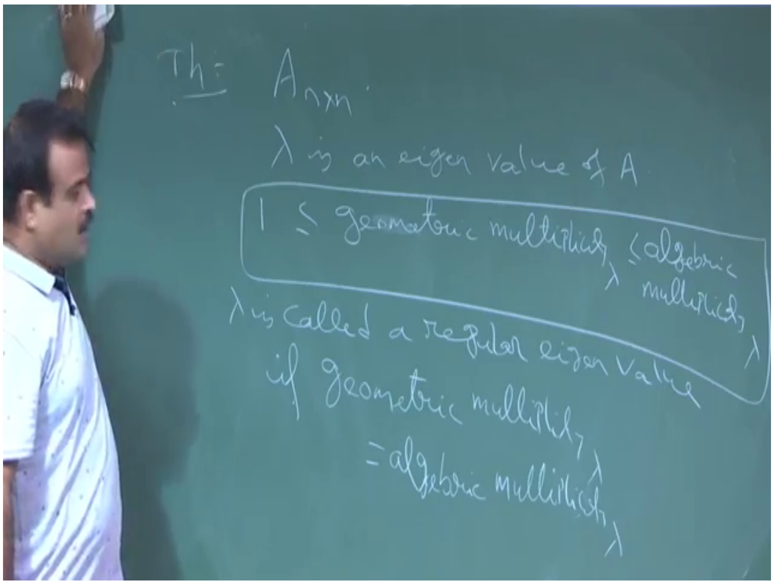

So, suppose A is an n by n matrix, a lambda is an eigenvalue value of A any eigenvalue, then we have this result; 1 less than geometric multiplicity of lambda, multiplicity of lambda is less than algebraic multiplicity, ok. I am not going to prove this proof is there in a text book. So, geometric multiplicity is always less than equal to less than equal to these are all less than equal to algebraic multiplicity. And lambda is called a regular eigenvalue, if these two are same; lambda is called a regular eigenvalue, if geometric multiplicity of lambda is same as algebraic multiplicity of lambda.

If the geometric multiplicity of lambda is equal to algebraic multiplicity lambda then it is called a then it is called a, it is called a regular eigenvalue, ok. We will take an example and we will find this both the multiplicity for a given matrix A. So, let us take an example.

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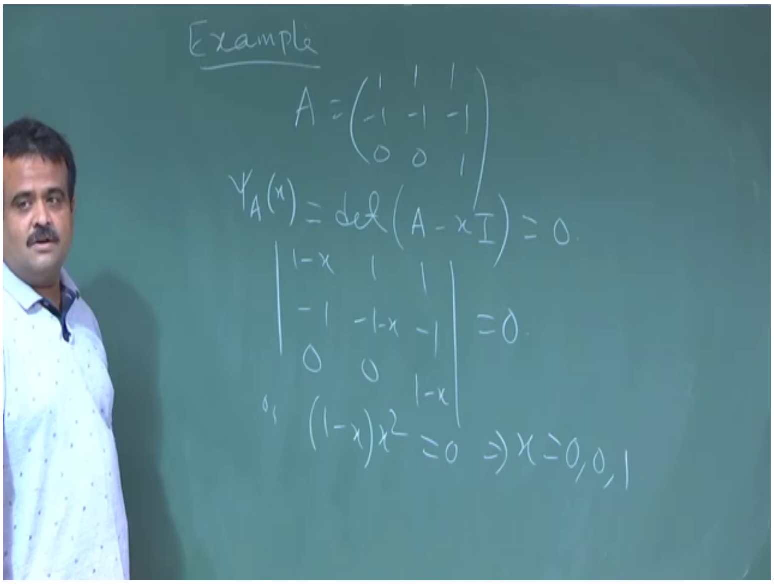

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So, we take a matrix over the real 1 1 1 minus 1 minus 1 minus 1 0 0 1. So, we take this matrix. Now, we have to find the eigenvectors of eigenvalues of this matrix. So, for that we need to construct the corresponding characteristics polynomial and characteristics equation. So, characteristic polynomial is nothing but A minus x I. So, determinant characteristic giving us this.

So, this is nothing but 1 minus x 1 1 minus 1 minus 1 minus x then this is minus 1 and 0 0 1 minus x. So, determinant is 0. So, if you solve this, it will give us the if you calculate this determinant 1 minus x into x square equal to 0. So, this is giving us the solutions x equal to 0 0 1. So, it has two distinct eigenvalues 0 and 1 and 0 is coming twice; so, multiple this root are 0 and 1. So, the roots multiplicity of 0 is 2. So, that is the algebraic multiplicity and geometric and the algebraic multiplicity of 1 is 1, ok.

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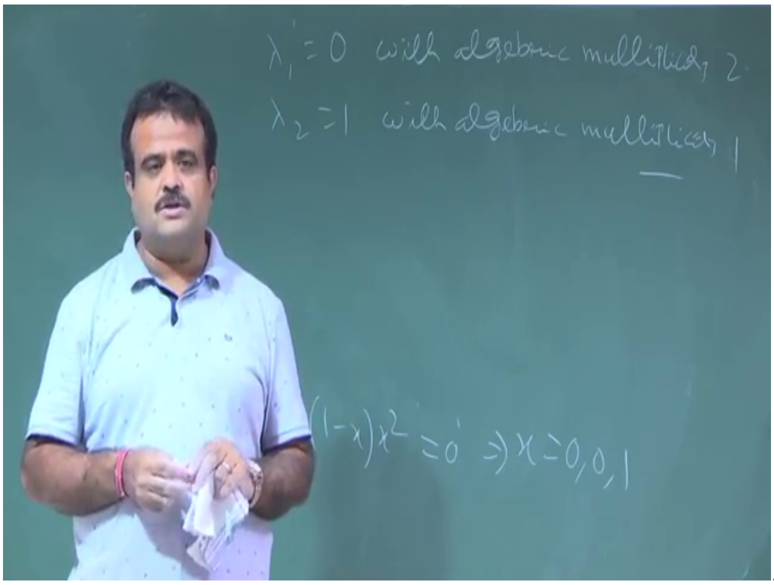

So, that is the, so, we have two eigenvalue, two distinct eigenvalue lambda 1 is equal to 0 lambda 1 0 with algebraic multiplicity 2 and lambda 2 equal to 1 with algebraic multiplicity 1, ok. So, these are the algebraic multiplicity. So, in the next class, we will discuss we will find the geometric multiplicity of this eigenvalue. So, we have two distinct eigenvalues 0 and 1. So, we will discuss the geometric multiplicity of these two eigenvalues. And for that we need to take the set of the vectors, the eigenvectors corresponding this to this eigenvalue that we will discuss in the next class.

Thank you.