Introduction to Abstract and Linear Algebra
Prof. Sourav Mukhopadhyay
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture - 35
Eigen Value of a Matrix

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So, we will talk about Eigen Value eigenvector of a Matrix. So, for that we need to just define first what do you mean by characteristics equations of the matrix characteristics equation of a matrix ok. Suppose, we have matrix A which is a i j this is a square matrix n cross n matrix, A has to be a square matrix. We have a square matrix n cross n cross square matrix. Now, if we can see and these are coming from F in and this could be real field, but in general this is a coming for the field F. Now, if you consider the determinant of this. So, this is det of A coma x I n this is a n cross n identity matrix.

And if you make it to be 0, this x is a in determinant I mean the variable. So, then this will give us a n degree polynomial and that polynomial is called construct this polynomial and the F equal to 0 that will give us a that will that is called the characteristic equations. Now, let us take a what do you mean by this determinant.

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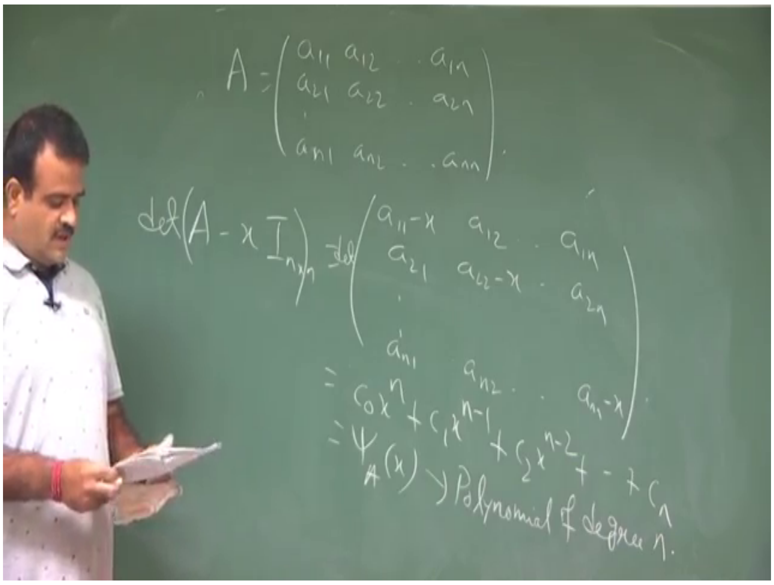

So, we have a matrix A a 11 a 12 a 1n a 21 a 22 a 2n a n1 a n2 a nn. Now, how to get this value his determinant? This matrix is A minus x of I n is nothing but identity matrix of order n cross n, this is nothing but a 1 1 minus x remaining will be same, only the diagonal element will change a 2 1 a 2 2 minus x a 2 n dot dot dot a n 1 a n 2 a n n minus x. This is the corresponding matrix.

Now if you take the determinant of this matrix, determinant of this that will give as a that will give as a n degree polynomial and that polynomial is we are denoting by psi A of x. This is nothing but this is n degree polynomial. This is something C 0 x to the power n plus C 0 is minus 1 to the power n C 1 x to the power n minus 1 plus C 2 x to the power n minus two dot dot dot plus C n ok. So, this is a polynomial, this is denoted by psi A of x. This is a this is a polynomial of degree of degree n. This is a polynomial of degree n ok and where coefficients are coming from the principal minors.

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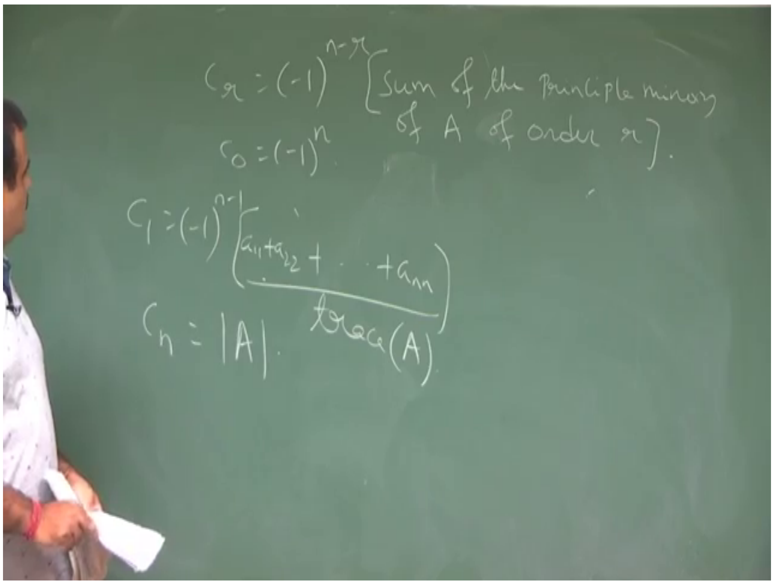

So, where C r, C r is, so c r is nothing but minus 1 to the power n minus r sum of the principal minors of the principal minors of A, principal minors means we have this these are the principal minors the diagonal the diagonal elements sum of the principle minors of A of order r minors of A of order r ok. So, yeah, so this is the thing.

So, what is C 1? So, C 0 is basically minus 1 to the power n. And what is C 1? C 1, if you calculate C 1 is minus 1 to the power. So, as to r is equal to 1. So, this is n minus 1. And principal minor of order 1, order 1 means the diagonal elements, so a 1 1 plus a 2 2 just remove this. So, a 1 1 plus a 2 2 dash a n n, these are the trace, this is called trace of the matrix. The sum of diagonal elements is called trace of a matrix. So, this is the this and what is C n? C n is this will become 0, and principal minor of order n that is the that itself is the determinant of A, det of A. So, this is the this thing and so the is so this is the coefficient of the characteristic polynomial.

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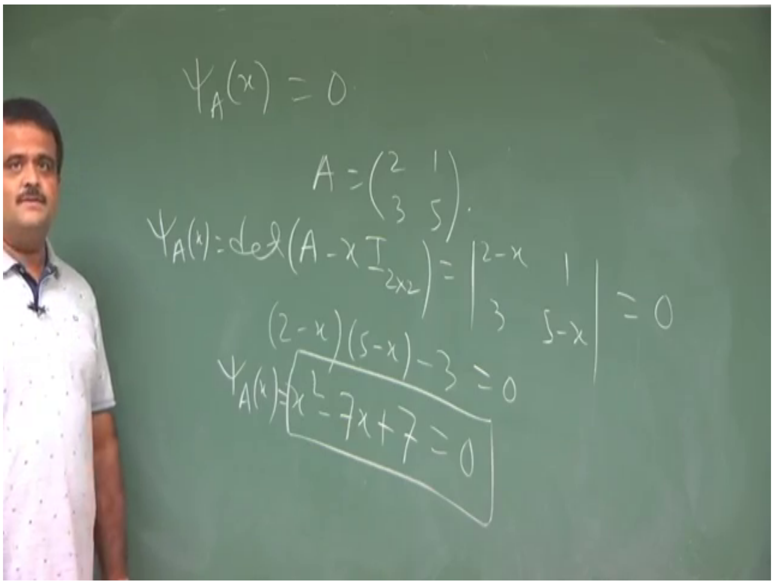

Now, if you make the characteristic polynomial to be 0 that is called characteristics equations. So, psi of A x is 0 this is called characteristics equations ok. We will take an example, how we get this characteristic equation. So, this is the n degree polynomial equations. So, we will take an example, simple example. Say we take A is a 2 by 2 matrix; this should be a square matrix 3, 5. Now, we want to find the corresponding characteristic equation for this.

So, how to get that? So, we have to get the determinant of A minus x 2 cross 2. So, this is nothing but determinant of 2 minus x 1 3 5 minus x. So, this is this we want to be equate to 0, so that will give as a psi of A x. Now, what is this, this will give us. So, this multiplied by this. So, 2 x into 5 x minus 3 equal to 0 therefore, simplify this we are getting x square minus 7 x plus 7. So, this is the corresponding polynomial, and this is 0. So, this is our characteristics equation corresponding to a this matrix ok.

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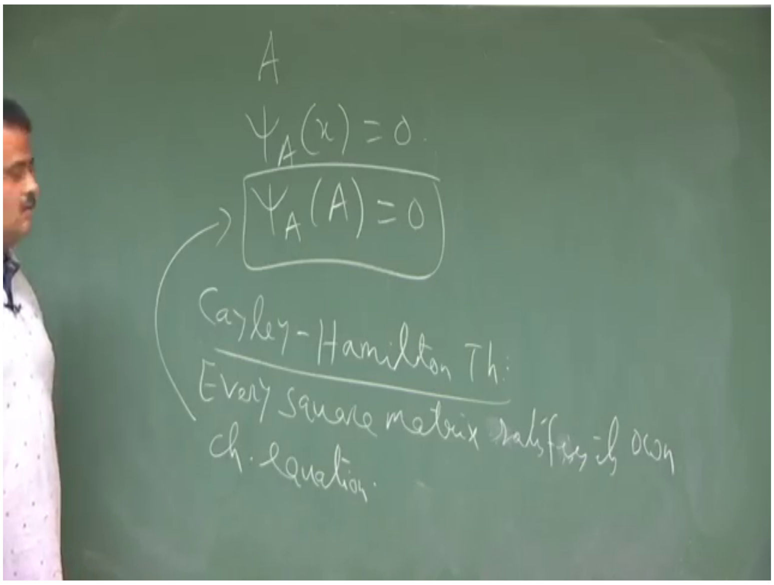

Now, Cayley-Hamilton theorem is there, so that is telling every matrix satisfies its characteristics polynomial that means, you have given a matrix which have corresponding characteristics n equation. So, it is telling this is polynomial in matrix ok. So, this is polynomial in matrix. So, this is the, this is called a Cayley-Hamilton theorem, c l y l e y Cayley-Hamilton Hamilton theorem. So, what it is telling, every square matrix square matrix satisfy its own characteristics equations every square matrix satisfy satisfies its own characteristics equation. So, that means, this psi A of A is one 0 ok.

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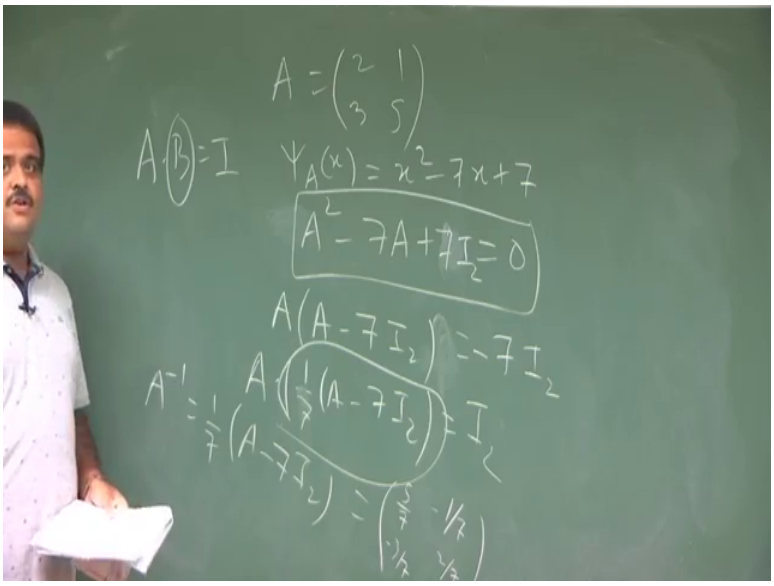

So, for that example what we have. So, if we have that matrix like this just now we have seen 2 1 3 5. And the corresponding characteristics equation you got characteristics polynomial you got is this x cube minus 7 x plus 7. So, now, the corresponding, so now, this Cayley-Hamilton's theorem is telling, so A cube A square sorry x square 7 A plus 7, this has to be 0. So, this is the Cayley-Hamilton theorem. We can verify this if we get, if we calculate a square, then if we calculate this value, we can easily verify this, but proof is there we are not going to prove it. So, this is the Cayley-Hamilton theory every square matrix satisfy its own characteristics equations. So, this is the meaning of this ok.

Now, if we have this, sometimes it will help it it help us to get the some power of a matrix. So, suppose an inverse of a matrix, suppose we have this. Now, from here how we can get the inverse of this matrix? So, we can just do like this, we can take A common A minus 7 it is the identity matrix of order two we can say just two this is equal to. So, we take this two and this is equal to minus oh sorry sorry this is 7 sorry this is 7 into I 2, this is 7. So, if we replace by this matrix, we have to get the identity matrix yes.

So, this is minus 7 I 2 this is the way. So, that means, if you take this A into 1 by 7 A minus 7 I 2 this is giving us the identity. And we know A into B if it is matrix, then B is the inverse of A, so that means, this is the inverse of A. So, A inverse is nothing but 1 by 7 sorry this is 7 into A minus 7 I 2. So, if you calculate it you will get this matrix like this 5 by 7 minus 1 by 7 minus 3 by 7 and 2 by 7. So, this is the inverse of this matrix. So, this is the way characteristics the Cayley-Hamilton theorem help us to find the inverse by this way. So, it has another application, we will talk about that to get the power of some matrix. So, we have given a matrix S, how to find A to the power 50 or A to the power 100.

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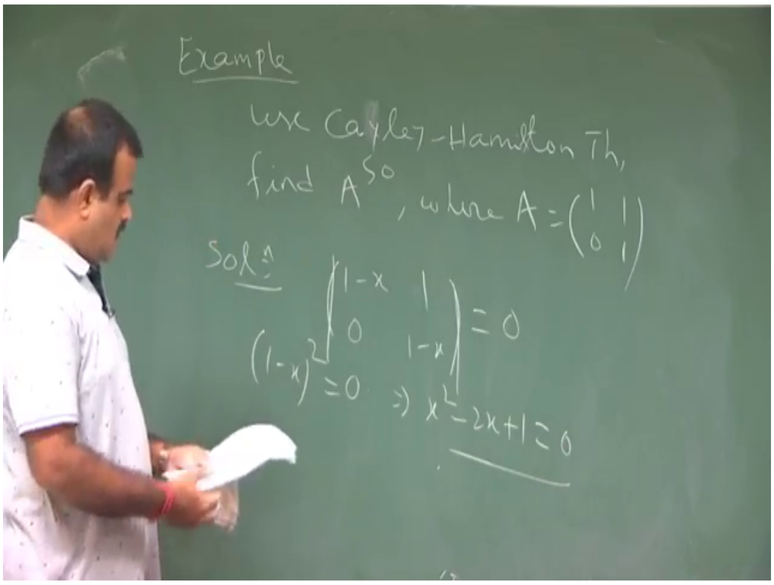

So, this type of example; so, use Cayley-Hamilton theorem sorry Cayley-Hamilton. Find A to the power 50, where A is A is some matrix say 1 1 0 1; it is not identity matrix it is the identity matrix of the 50 is the A is the identity matrix ok. So, how to get the, so how to get this characteristics polynomial? So, characteristics polynomial 1 minus x 1 0 1 minus x this is determinant of this is 0. So, this is giving us 1 minus x square is 0 ok. So, this is telling us this is the x square minus 2 x plus 1 is equal to 0.

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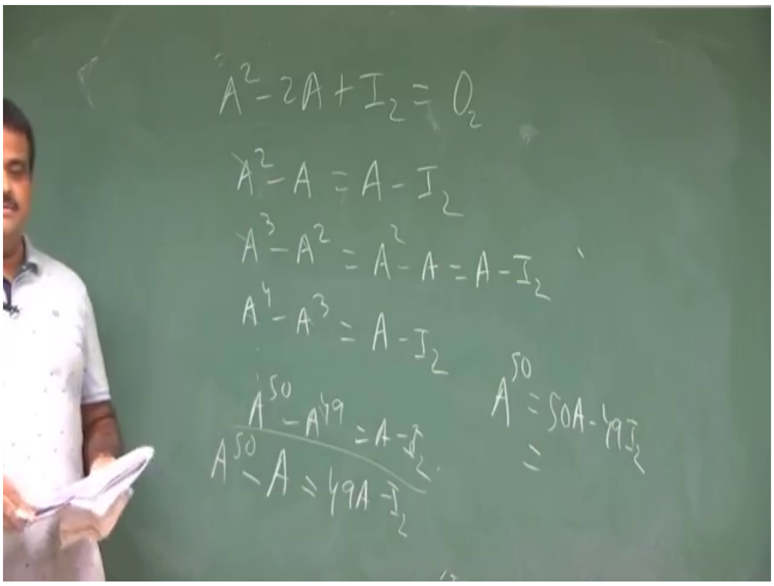

So, the by Cayley-Hamilton theorem what we have. So, A must satisfy this. So, we must have A square minus 2 A plus identity is equal to 0 matrix. This is the 0 matrix 0 0 0 0, this is the identity matrix. So, you simply write 0 matrix 2 by 2 zero matrix ok. So, we have this. Now, this we can write as this we can write as A square minus A is equal to A minus I 2 ok.

So, now what you do we just multiply keep on multiply A A with this. So, if you multiply A A with this one A cube minus A square is equal to A, A square minus A, so that is nothing but A square minus A, so that is nothing but A minus I 2 like this. So, A to the power 4 minus A cube. Again you multiply with this is same as a minus I 2 like this. If we continue like this A to the power 50 minus A to the power 49 is A minus I 2 same way, we are doing. So, now, we are going to subtract this two, this all. So, if we added up, we will get A to the power 50 minus A is equal to how many times 49 times 49 A minus I 2.

So, from here we can just get A to the power 50 is equal to 50 times A minus 49 I 2. So, this we can easily calculate if we know A. So, this is the way the Cayley-Hamilton theorem will help us to find the some powering of matrix like this. So, this we can have some example in the lecture note. So, now we will define what do we mean by Eigen value of a matrix.

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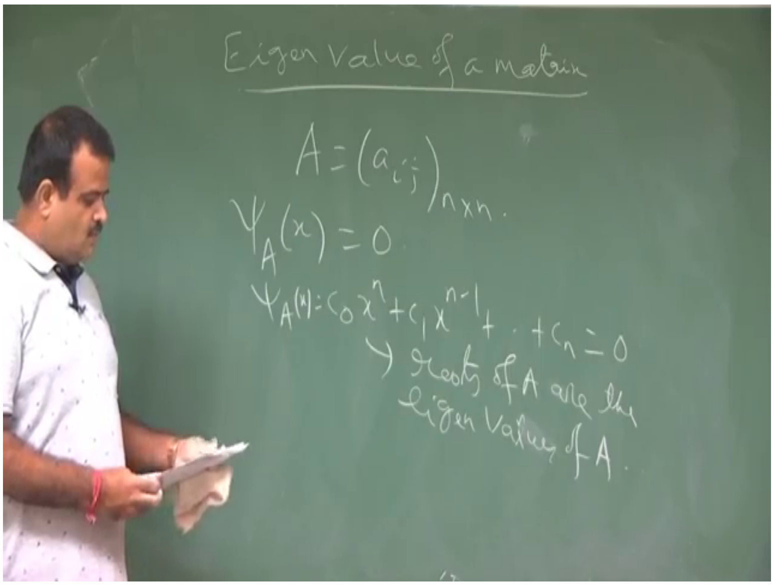

Ok, so we have a given matrix a i j this is square matrix. Now, we have the corresponding characteristics equation, this is the n degree polynomial equations. Now, the solution the solution of this is give us the, this is a n dv equation. So, it has n values. So, these are each values are giving us the, this lambda Eigen roots of this equation. So, this is the n degree polynomial. So, this is C 0 x to the power n plus C 1 x to the power n minus 1 dot dot dot plus C n equal to 0. So, this is the n degree equations. So, it has n roots, maybe some roots are equals. So, multiplicity is there, but this is our psi A of x ok. Now, the roots of these are basically the Eigen values of this matrix roots of A is the are the Eigen values of values of A ok. Now, suppose lambda is a lambda is a Eigen lambda is a root with multiplicity r.

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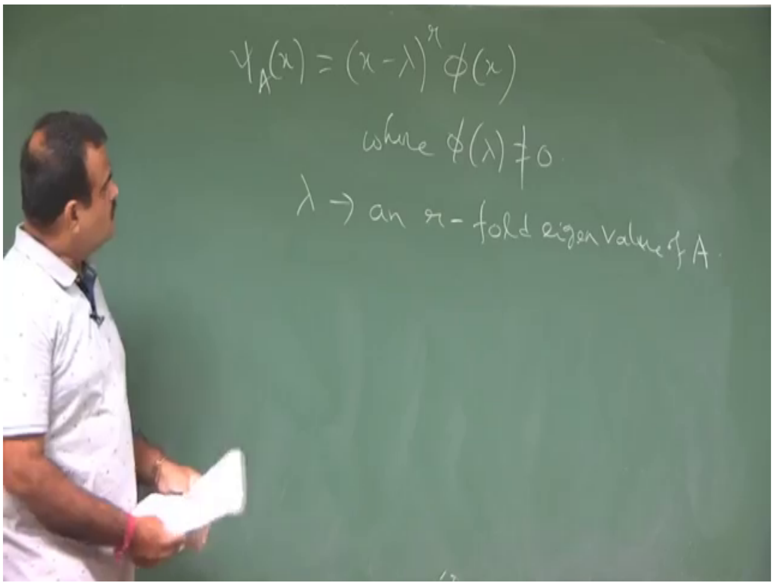

So that means, what; that means, psi A of x will be written as x minus lambda to the power r some psi, where is not equal to 0; if it is equal to 0, then again lambda will be another root ok. So, this means what, so if we have a, so we know a n degree polynomial having n roots and we know some formula of the roots some of the roots are product of the roots what is this. So, this we know. So, so if we have a polynomial equation like this is the polynomial equation of degree n, now if you had a if you had a if it has a root of alpha 1, alpha 2, alpha n, so this will be written as some x minus alpha i lambda 1 x minus lambda 2 like this x minus lambda n with some constant ok.

So, this is the so now, if lambda is a root which is coming r times. So, if the r roots lambda is a root, it is coming r times and no more times this is the indication then lambda is a r fold Eigen value of A, this is called multiplicity r fold Eigen value of A. Any root of this is called Eigen value. And but if the lambda is a having the characteristics having the multiplicity r of the corresponding this characteristic equation, then it is called r fold Eigen value of A ok.

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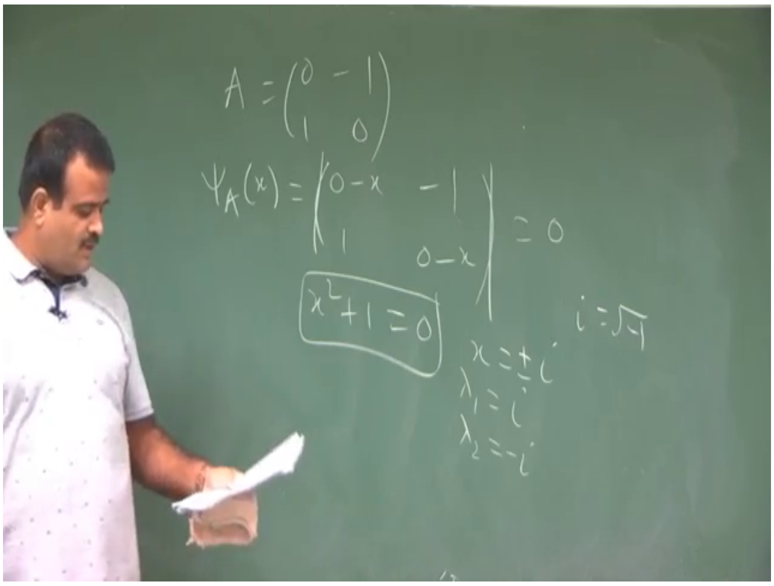

So, now, let us take the example how we can find the Eigen values of a matrix. So, suppose we have a matrix like this 0 minus 1 1 0, this is a real matrix the elements are coming from real number. So, what is the corresponding characteristic equation, so this is 0 minus x minus 1 1 0 minus x. So, determinant of this that equal to 0. This will give us the so this is nothing but x square plus 1. So, x square plus 1 equal to 0, this is the characteristic equation corresponding to this polynomial.

Now, what are the roots of this polynomial, this equation, roots are x equal to plus minus i, i is the minus 1 to the power. So, this is the complex number. So, here the for there the, these are the here these are the Eigen value. So, lambda 1 equal to i, lambda 2 equal to minus i these are not real. So, it is not necessary that every if you have a real polynomial, if you have a real matrix or the field I mean the, these are coming from field here, it is real field, then the Eigen value will be from the same field that is not necessary. Here the field is the real numbers, but the Eigen values are the complex number. So, this is one example where the Eigen values may not be belongs to the same field ok. So, but if the A is a symmetric matrix, then it is coming from the same field, so that is the.

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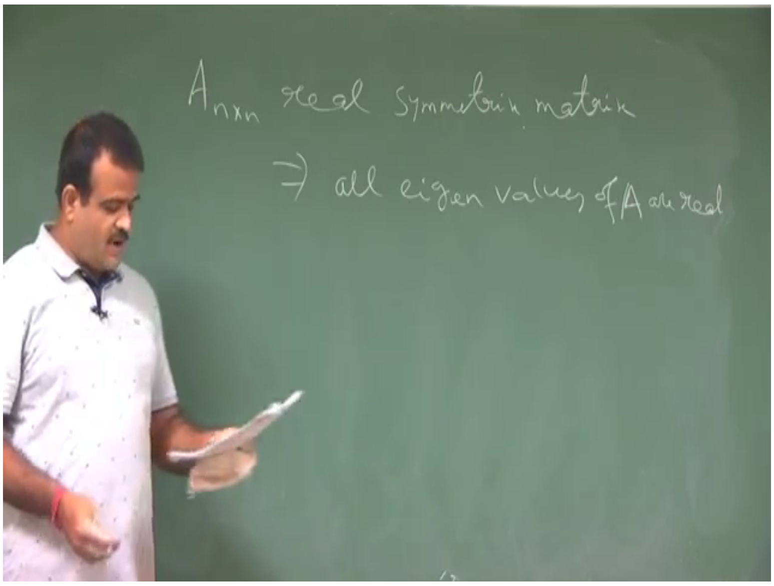

So, if a is a symmetric matrix, suppose A is n by n real symmetric matrix, then all Eigen values are real. Real symmetric matrix means a i j is equal to, so diagonal elements are there. So, what about the elements of array a 1 2 same will be here a 1 2. So, a i j is equal to a j i and this is true for all i and j then that is called symmetric matrix. And q symmetric is if it is minus. So, for q symmetric matrix the diagonal elements are 0 ok. So, this is a symmetric matrix then the all Eigen values are real values of a are real, so that coming from same field real numbers all Eigen values are real ok. Now, if A is q symmetric matrix, then they are purely imaginary or 0 anyway this proof I am not going. So, now, we will discuss the product of the eigen values and some the properties of this Eigen values.

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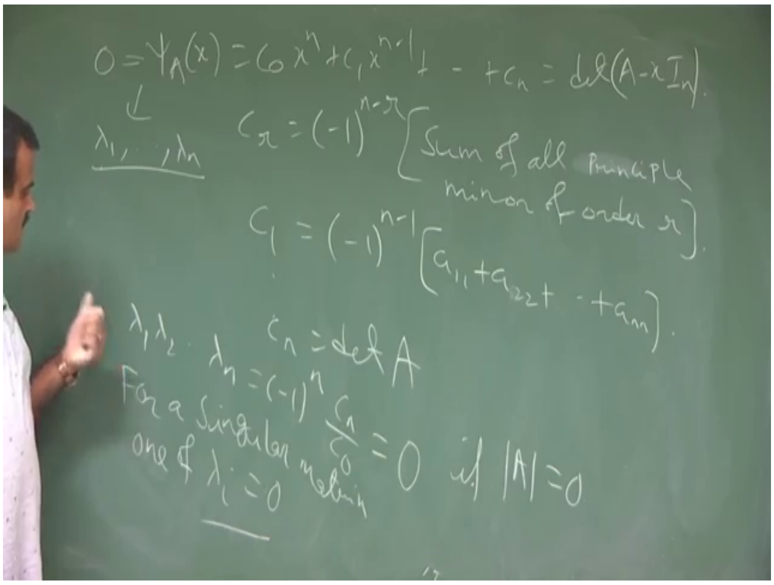

So, we know the characteristics equations of where this Eigen values are coming this is C 0 x to the power n plus C 1 x to the power n minus 1 C n. This is nothing but determinant of a minus x i n. And this we are this is the this what is C i, C r we know, C r is minus 1 to the power n minus r this is the sum of all principle minor principle minor principle minor of order r ok. So, and we have seen C 1, C 1 is nothing but minus 1 to the power n minus 1 trace of the matrix because principle minor of order 1 is the trace of the matrix a 11 a 22 a nn. And what is C n? C n is the determinant of A that is the principle minor of order A ok.

Now, if A is non singular, now we know the, this so, product of the Eigen values we know. So, if the Eigen values if we equate these two 0, then this will be so this will give us the Eigen values there are n, this is the n degree n n degree polynomial. So, it has n roots either real so it could be real or imaginary, but it has n roots. So, these are the roots. So, now the products of the Eigen values like lambda 1, lambda 2 we know the products of the roots is products of the roots is minus 1 to the power n C n by C 0, this is the formula we know.

Now, C n is the determinant of A. Now, if A is non singular then the determinant is 0 I sorry if A is singular then determinant is 0; if determinant is 0 that means, one of these Eigen value has to be 0 that means, one of these so if for a non singular matrix for a sorry for a singular matrix for a singular matrix one of the Eigen values lambda i should be 0, because this product is 0 product is 0 means one of this lambdas has to be 0. So, this is all observation.

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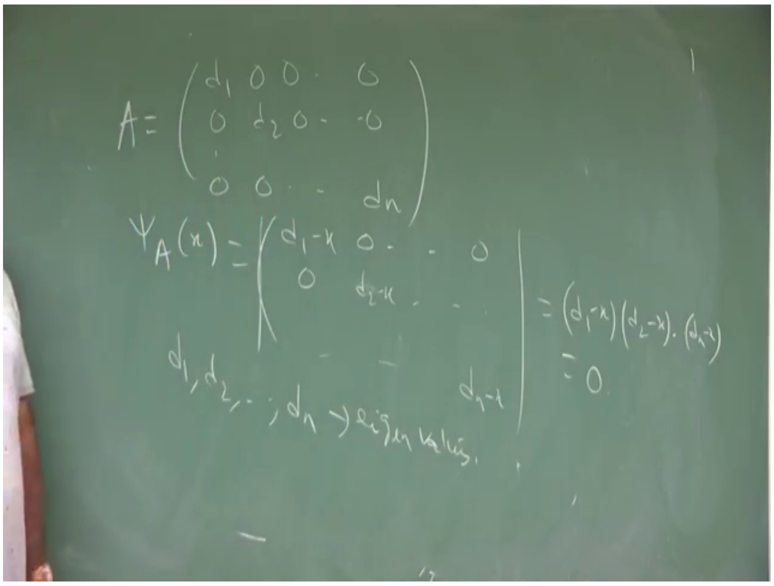

Now, for diagonal matrix if you have a diagonal matrix then, what are the Eigen values? Now, if you have a diagonal matrix say d 1 0 0 0 0 d 2 0 0 0 0 0 d n, suppose this is our A, these are the diagonal element. Now, what is the lambda A x this is nothing but d 1 x 0 0 0 0 determinant of this d 2 x so d n x. Now, the corresponding polynomial is d 1 x d 2 x d n x. And this we want to become zero to get the equation, so that the so d 1 d 2 d n are will be the Eigen values. So, for the diagonal element the Eigen values are the nothing but the, that for the diagonal matrix diagonal elements are nothing but the diagonal the one zero diagonal element of that matrix ok. So, yeah we will talk about more on Eigen values and Eigen vectors in the next class.

Thank you.