Introduction to Abstract and Linear Algebra
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Lecture - 34
Row Rank and Column Rank

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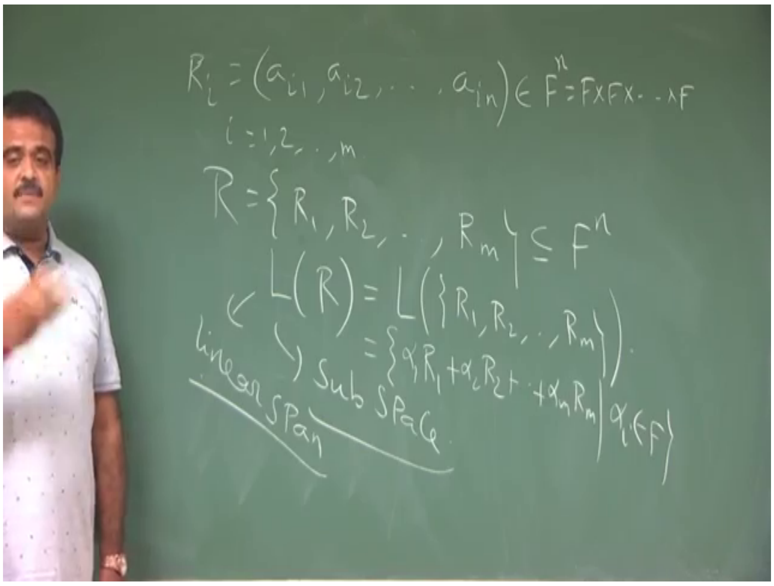

So, we are talking about rank of a matrix. So, today we will discuss the row what do we mean by row rank and column rank of a matrix. So, suppose we have a matrix A, which is a m by n matrix, so m row n columns. And these elements are coming from a field, we can take F to be a real field also. But, in general this is a field where F is a field with two operation any field, we can talk about over any field.

So, now so A is a m by n matrix. So, it has m rows, this is the first row, a 2 2 this is the way we, so a m 1 a m 2 a m n. This is the m by n matrix, where the elements are coming from the field. So, this one is our first row, so this is our R 1, this is our second row R 2 like this dot dot dot this is our there are m rows R m.

And the column if you consider, this is our 1st column C 1, this is our 2nd column C 2 like this, this is our nth column C n. So, these are the rows and columns of the matrix. So, each of this row is a n tuple. So, it is a it is a basically a element from F to the power n, because our field is n. So, the ith row is a i 1, a i 2, a i n. This is the ith row i is varying from so i is varying from 1 to m, and this is a n tuple. So, this is a element from F to the power n, this is nothing but Cartesian product n times.

So, F cross F is nothing but the set x, y, where x, y are both coming from F, this is the Cartesian product. Two times if it is n tuple, so this is coming from F to the power n. So, if F is R this is coming from R to the power n, this is a n tuple. And similarly, this is the row this is a ith row, now we have and similarly a column we will we will come to the column. Now, if you take this collection of rows, say this set R, which is R 1, R 2 there are total m rows R m. So, this is a subset of; this is a subset of each of this element is a coming from F to the power n. So, this is subset of F to the power n.

Now, if you consider the space which is generating by this vectors this is a n-dimensional vectors. So, if you consider the space, which is generating by these vectors so that is called linear span of this. So, this is nothing but linear span of this R 1, R 2, R n. So, this is the all possible linear combination this alpha 1 R 1 plus alpha 2 R 2 dot dot dot alpha m R m, where alpha is are coming from a field all possible alpha i. So, we are taking the linear combination of this, we are taking the linear span of this, so this is a subspace and we know this is a sub space, we have shown this. We know this is a this will again form a vector space, this linear span this is called linear span this is called linear span. And we know this is a subspace, we can prove that.

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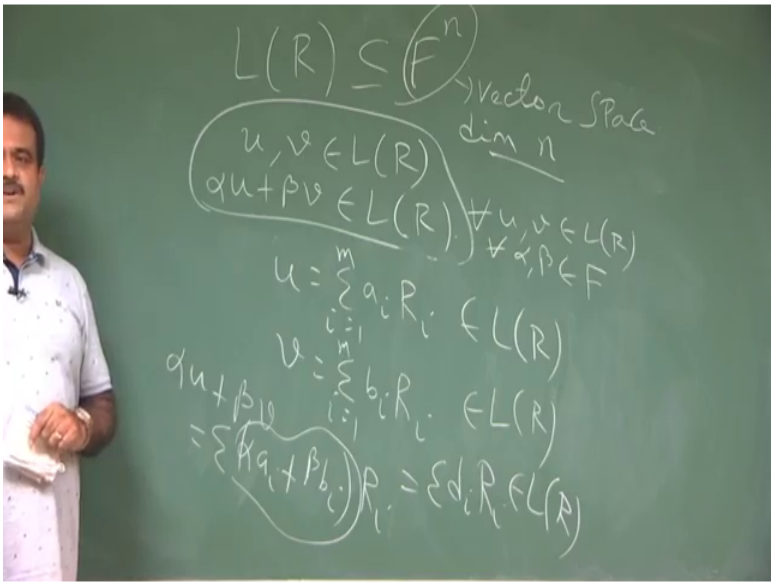

To show this is a subspace, what we can do we have to so this is a subset of F to the power n. Now, F to the power n is a vector space n dimensional vector space vector space with dimension n. Now, this is a subset. Now, when we how to prove a subset is again a vector space, then it is called sub space. So, to show this is a again a vector space what we need to show, we need to take two vector two element from here, and we need to take the so suppose we take two vector from here u and v say u and v from this. And what we need to show, we need to show alpha u plus beta v is also belongs to this. This is a necessary and sufficient condition to become a subspace to be a, subset to be a subspace.

Now, this easily we can show this, because if you take u from this, so u is nothing but a linear combination of so if u is a linear span of this, so u will be written as summation of some a i R i for i is equal to 1 to m. And v will be also written as summation of b i R i, i is equal to 1 to m. Now, again then alpha u plus because these are coming from L of R.

Now, alpha u plus beta v, this is again can be written as summation of alpha a i plus beta b i's then R i. So, this is some d i's so summation of some d i R i, so this is also belongs to so this is a linear combination of the rows of the so this is this is for all u, v belongs to L R. And for all alpha beta belongs to the scalar, so this is a vector space. And this space is called the row space. So, this will again form a vector space, and this vector space is called row space.

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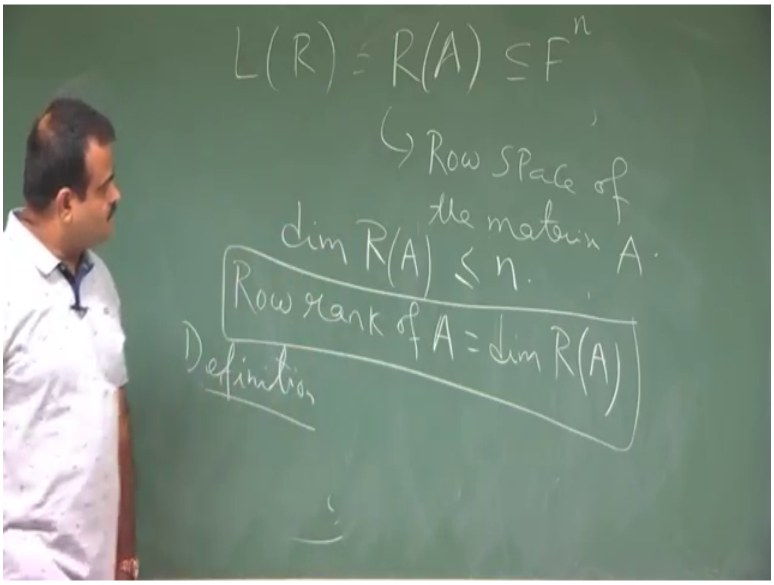

This is called this is R of A this is called row space of the matrix of the matrix A. And since it is a vector space, it has a dimension. And the dimension is it is a row, it is a subspace. So, dimension of l of dimension of this R of A is called it is it must be so this is a subspace, so this is a subset of so dimension must be less than equal to n, it could be at most n. So, now the dimension of this is called row rank of the matrix A. Row rank of A is defined as the dimension of R of A. This is the definition of row rank. This is the definition we defined the row rank as this. Row rank is the dimension of the subspace this R of A.

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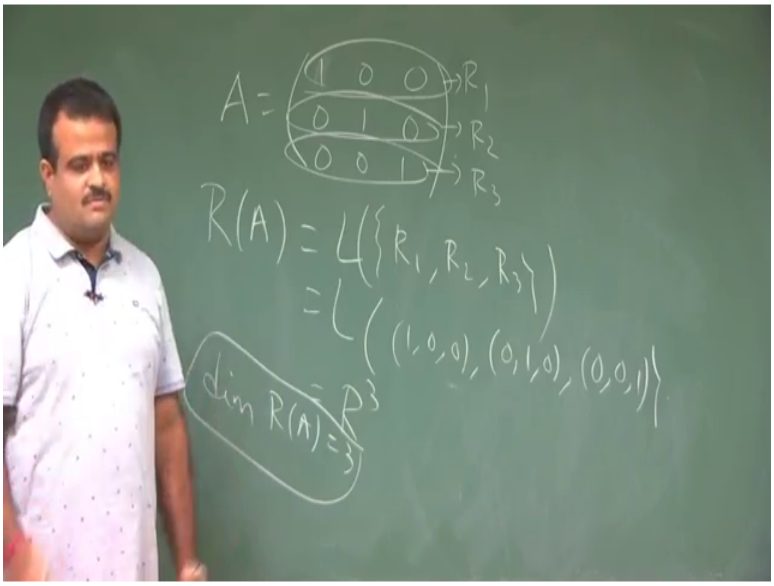

We can take an example. Suppose, we have a matrix like this, say 2 2 0 or simple matrix 1 0 1 0 1 0 1 0 0 0 3 by 3 matrix, the identity matrix (Refer Time: 10:14) So, this is the now what are the rows these are the rows, this is R 1, this is R 2, this is R 3. So, now what is the row space, row space of this is a the linear combination of the linear span of the this R 1, R 2, R 3.

Now, here R 1, R 2, R 3 are the standard vector, so they are 1, 0, 0; 0, 1, 0; 0, 0, 1. Now, this is this we know this is are all independent vectors, and this will form R 3 the real number field R 3. So, the dimension of this is 3, but our matrix is not always like easy matrix. We will take some example, where we can show how the (Refer Time: 11:29) the how to get the dimension of the this how to get the so this is a linear we know this is a vector space. So, we need to find the basis of this vector space. Now, since this are these are all three linearly independent vectors, so this will form a basis. But, in in for general matrix these rows may not be linearly independent, we may have dependent vector in the rows, then you have to get the basis of that. So, how to get the basis of that we will take the example.

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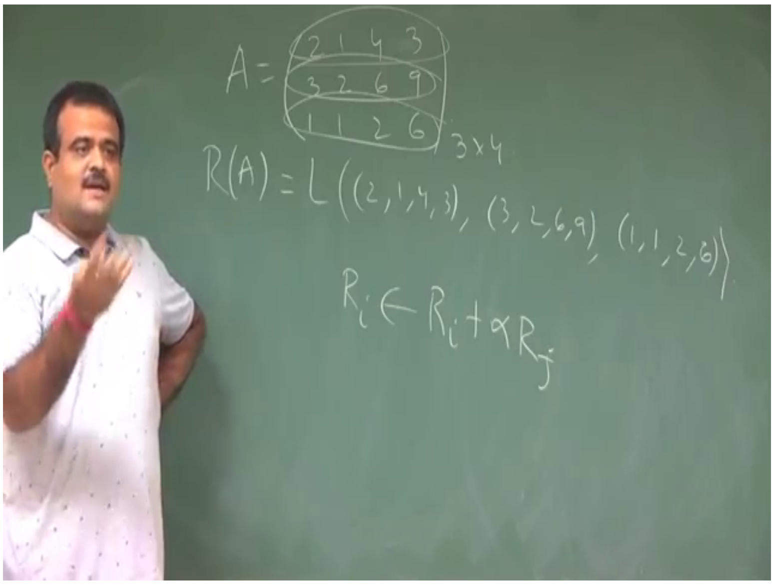

So, let us take a another example, which is not trivial like this one. So, suppose we take a matrix like this, 2 1 4 3 3 2 6 9 1 1 2 6. So, this is a 3 by 4 matrix. So, who are the rows, these are the rows. So, we take so what is the row space, row space is nothing but the linear combination of these rows 2, 1, 4, 3, 6, 9 this vectors 6.

Now, we have to check whether these vectors are can form a basis or not. If it is forming a basis, then the dimension of this the span is linear span is 3. So, to check that what we do, we will apply the elementary row operation on this. And we will convert into row echelon form. And once we convert into row echelon form, then it will give us 0, 1s entities in that corresponding row equivalence matrix. And that is basically we are when we are applying the row operation, we are basically doing the again we are doing the linear combination of the rows.

Because, if we replace one row by the by subtracting say R j, R j is replacing by R j plus some alpha R i plus some alpha R j. So, this is nothing but the linear combination of the rows. And exchanging two rows is just nothing and applying the constant on the row is again a linear combination of the row. So, basically we will apply the row operation, elementary row operations on this matrix to reduce this to a standard form echelon row reduced echelon form, and to get the basis of that subspace, so we will do that.

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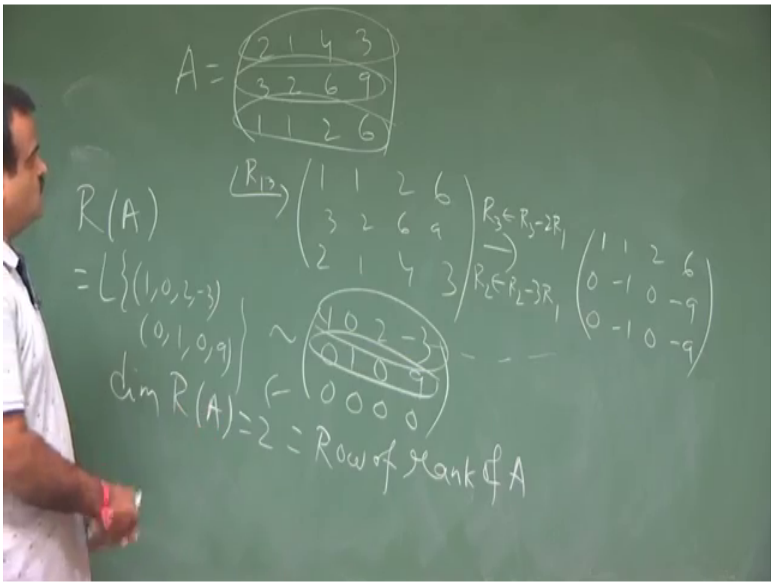

So, we will let us take this matrix. So, we will first apply say we will apply the row operation. So, we exchange this two row R 1 3. So, this will become 1 1 2 6, and this will come here, and this will remain same. Then we apply some row operation R 3, we multiply these with two and subtract with this. So, R 3 is replacing R 3 minus 2 R 1. And again this one we will do, so we want to make this to 0. So, R 2 is replacing R 2 minus 3 R 1.

So, if you do so our matrix will become like this, so 1 so the first row will remain unchanged, the 2nd row we are multiplying 3 with this, we are subtracting with this, so this will become 0, this is 3 so minus 1, so 0 minus 1 0 minus 9, similarly 0 minus 1 0 minus 9. So, now these two are identical. So, we can just add these two, and we get so finally this will if you keep on applying this, so this will be the row reduced echelon form is 1 0 2 minus 3 0 1 0 9 0 0 0 0. So, this is our row reduced echelon form.

And this will give us the, this will give us the basis of this vector space. So, this is these are the element in the basis. So, this L of R L of I mean the row space of row space of A is nothing but linear combination these vectors so, 1, 0, 2, minus 3. Only two vectors is there 1 0 vectors, and they are independent. So, dimension of R of A is 2, this is the row rank of A, this is the row rank of A. So, this way we will get the, this way we will get the dimension of the victor corresponding vector space.

Some so similarly, we will check that, and this is the row reduced echelon form standard form. And this operation we are doing all the linear combinations of the rows, so that is. So, finally we got the two linearly independent vectors in the basis this is the basis. This set will form the basis, and number of element in the basis is called dimension, so dimension is two. So, this is the way we find the dimension of a row space.

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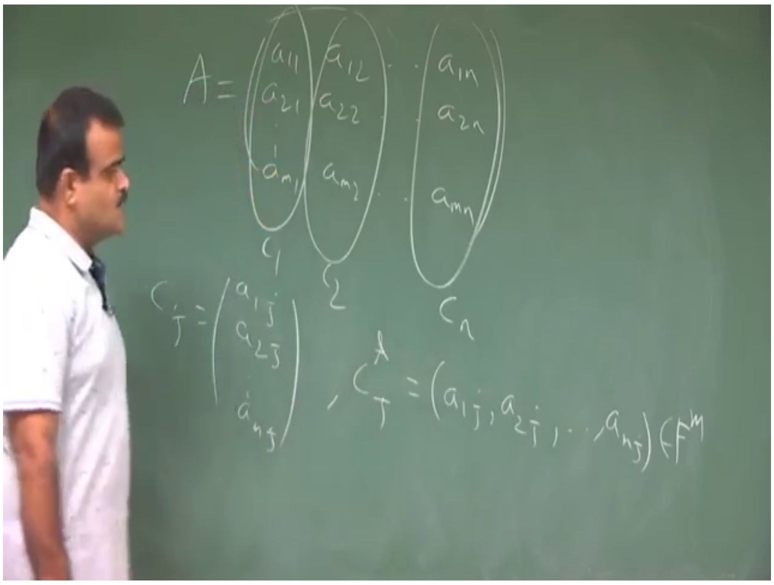

Now, we define column space of a matrix and column rank of a matrix. So, again suppose we have a matrix like this a 2 1 a 2 2 a 2 n dot dot dot a m 1 a m 2 a m n. Now, these are the column, this is 1st column, this is 2nd column, we have there are n columns. And these are so C i is basically C j jth column is a 1 a 1 j a 2 j up to a m j, this is jth column. So, these we want to show in the in the Cartesian product form I mean n this is m tuple. So, to do that we need to take the transpose of this, because the way defined the Cartesian product.

So, these are all coming from F, so F cross F we define like this x comma y form, it is not y comma x form, but both are equivalence. So but to get this form here, we need to take the transpose of that. So, this is just a convention notational convention. So, if you take the transpose of this, it will become this column vector will become the row vector. And then we can talk about the tuple, which is a Cartesian product of F a 1 j, a 2 j, a m j. And this is this is now we can write F to the power m, so this is F to the power m.

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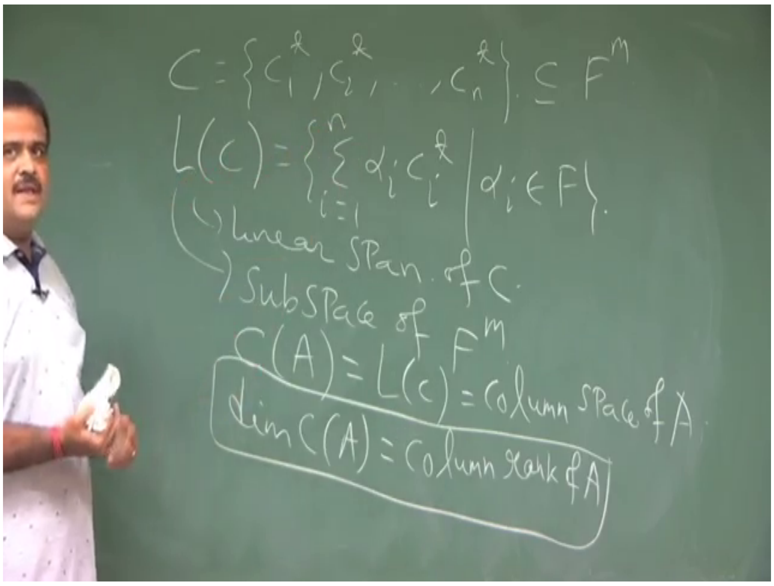

Now, if you take this collection of this column vector and then if you take the linear combination linear span of this that, we will give us a set which is suppose C, C is the collection of this C 1 t, C 2 t, C how many columns are there, there are n, there are n there are n columns C n t. And each of these columns are coming from F to the power this.

Now, if you take the linear span of this, each of this is a coming from F to the power m. If you take the linear span of this, so that is nothing but the linear combination of say alpha i C i t, i is equal to 1 to n all possible combinations linear span linear span of C. And we know the linear span forms a subspace of the corresponding vector space, and that subspace is called column space. So, this will form a subspace of F to the power m, F could be R also. If we are talking about real field, then this F is nothing but R, so then this is a subspace of R to the power n.

So, now this is this is a subspace, now this is called column space this is called column space, and this is denoted by C of A column space of A. And since it since this is a vector space, it has a dimension; dimension is the number of element in the basis, so that dimension is called column rank of this. So, dimension of C of A is called column rank of A, this is the definition of the column rank. Now, this is the subspace for F to the power n, this is a m tuple over the field. So, definitely column rank will be less than n.

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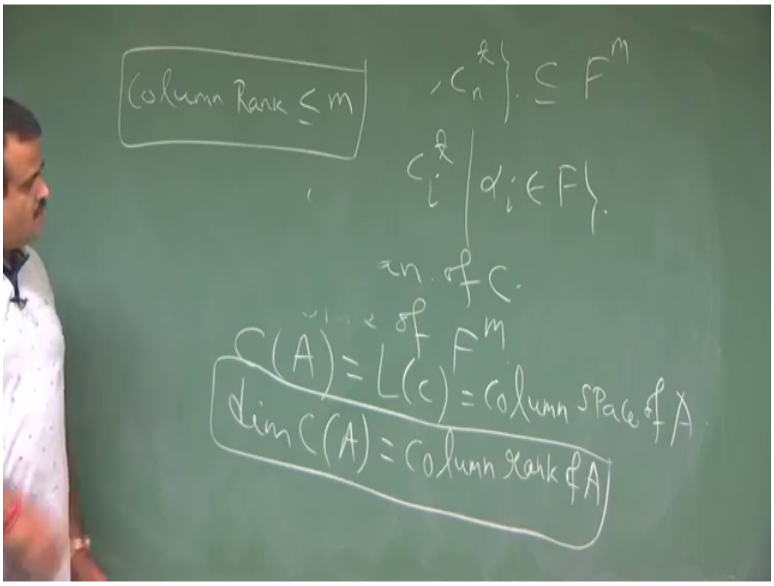

So, definitely column rank will be less than equal to m, this is all observation. So, similarly how we can find the column rank by the similar way, we can just get the column vector we will we will take that example. So, let us take the same example, where we found the row rank of that matrix.

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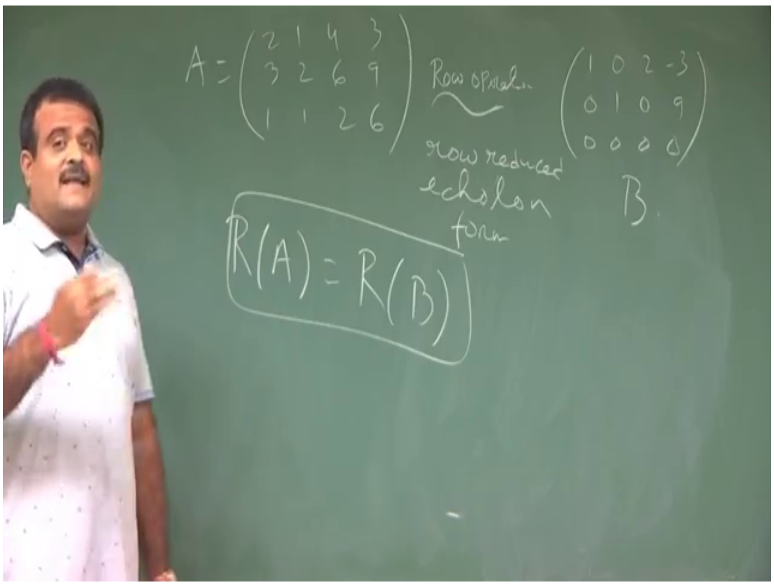

So, let us take this example the same matrix 2 1 4 3 3 2 6 9 1 1 2 6. So, for finding row rank what we are doing, we are applying the row operation this is the row equivalence form. We are applying the series of row operations, and we transfer into a what is called row reduced echelon form row reduced echelon form, this is a normal standard form. And this we know we have seen this is becoming this became of the form like this 1 0 2 minus 3 1 0 1 0 9 all are 0, this is the normal form.

So, the rank of this so we know the rank of a matrix we will come to that theorem. So, rank of a matrix is same as row rank and equal to same as column rank. But, from here to get these what we did, we did just the row operation on this rows, so that means, this is still again the linear combination of this rows, so that means linear combination of so if we denote this matrix as B, then the row space of A is same as row space of B, because the matrix B we got by applying the row operation on the rows of A. Row operation is nothing but we are just doing the linear combination of this, so their linear span is same and the rank of this matrix, because these two are only the independent vector in the space. So, the rank of this matrix is two rank of the space.

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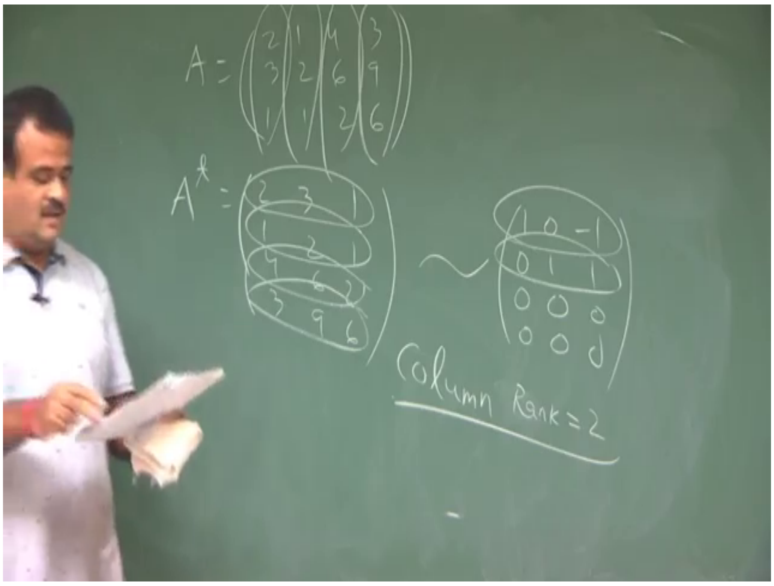

So, now this is the row, now to get the column what we do, so this is a 1st column, 2nd column like this. So, for column we have to take the transpose, so instead of doing that we have to take the, we can take the transpose of this matrix, then the rows the corresponding rows will become the columns. So, if you take the transpose, this is becoming this 1 2 3 4 6 2 then 3 9 6.

And now, these are columns, these are the columns I mean transpose of the columns. So, this will form the linear combination of these vectors will form the column space. Now, again to get the column rank of this matrix what we do, we have to apply the again the elementary operations. So, if you apply elementary operation on this, we will be getting like this is the again the row reduced echelon form, we will get 0 1 1 0 0 0 0 0 0, so that means, the column space of this matrix is same as row space of this matrix is same as row space of this matrix. And the number of independent vector here is 2, so the column rank is, column rank is equal to 2. So, we have seen the row rank also 2. So, this is in general; in general this is true that row rank and the column rank are same. So, we will just state those result we are not going to prove it, we will supply proof in the lecture note.

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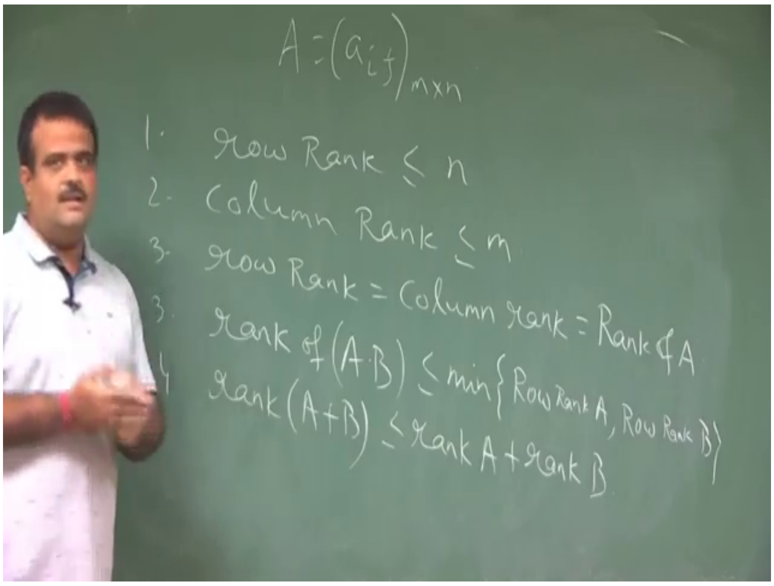

So, the theorems are you have a matrix like this, now the first result is row rank this we know, row rank is less than equal to n. Column rank is less than equal to m this we have seen. And row rank is same as column rank row rank is same as column rank, which is same as rank of the matrix. We know the rank of the matrix by taking the minors, we have seen in the last class that the determinant rank that is called determinant rank. The minors the if the rank of a matrix is R means, we have a R order minor non-zero minor.

Whereas, if you take upper than R order, R plus 1 order R plus 2 order all the minors are zero. So, this is the rank of A, this is this is called determinant rank. This we can also prove that.

And then the product of the matrix; so, rank of A B if we have two matrix A B, they are product multiplication is less than equal to minimum of rank of both. Actually, rank of A B is same as rank of row rank of A B. And row rank of A B can be shown as less than equal to minimum of row rank of A comma row rank of B, because this is the just the linear combination of the rows. And so row rank is same as rank so that means this is less than equal to minimum of rank of A and rank of B. This we can prove it, we will we will give the proof in the lecture note.

And then last theorem is rank of A plus B, this also we can prove using row rank or column rank is less than equal to rank of A plus rank of B. This we will prove using the row rank or column rank. So, rank of this is nothing but row rank of this. So, row rank of this is less than equal to row rank of A plus row rank of B. So, this is the, this proof will be given in the lecture note.

Thank you.