Introduction to Abstract and Linear Algebra
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Lecture - 33
System of Linear Equations

So, we will be talking about System of Linear Equations. We have a equations.

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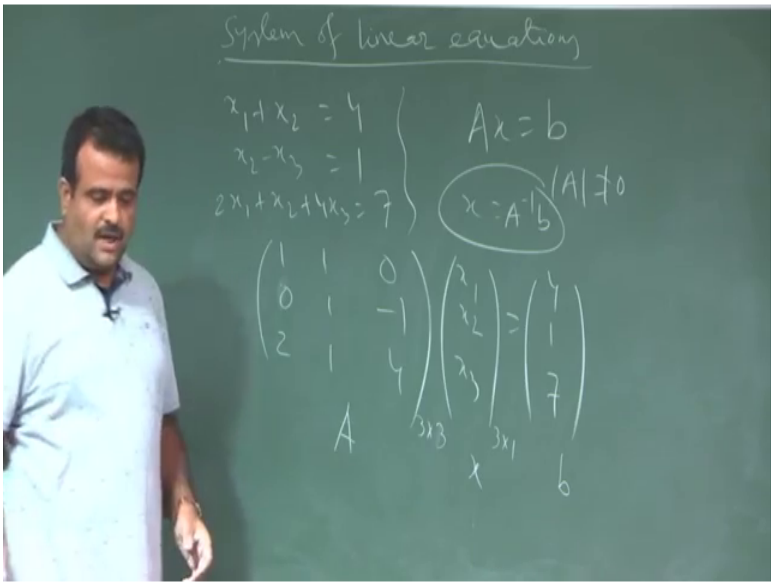

Now, the question is how to solve by equation. For example, suppose we have this equation with 3 variables. So, x 1 plus x 2 plus x 1, x 2, x 3; so, if this is 1, x 2 minus x 3 is equal to 1, 2 x 1 plus x 2 plus 4 x 3 is equal to 7. So, suppose you have this system of equation, now the question is how we can solve this.

So, you have to write this system in the matrix form, how to write in a matrix form? So, you can just write 1 1 0 1 sorry, 0 1 minus 1 and 2 1 4, x 1 x 2 x 3 this is equal to 4 1 7. We can write any system of equation is of the matrix form like this. So, because if you take this first row this is 3 by 1 matrix, this is sorry 3 by 3 matrix, this is 3 by 1 matrix. So, if you multiply these 2, it will give us 3 by 1 matrix that is same.

So, what is the first row of that? So, this into this. So, x 1 plus x 2 is equal to 4 that is the first equations. Then the second one; x 1 into 0 x 2 minus x 3 equal to 1, x 2 minus x 3 equal to 1 and the third one, 2 x 1 plus x 2 plus 4 x 3 equal to 7. So, this is any system of equation can be written as matrix form like A x equal to b. So, this is our A matrix, this is our x variable and this is b ok, b is the this is called coefficient matrix and this is the variable vector and this is the b.

So, now the question is how we can solve this, how we can get the solution of this system? Whether the solution exists or not? All this type of question we will discuss here. So, whether the system is if it is having no solution then it is called inconsistent system. If there is a solution then it is called consistent system.

So, if they if there is so, now, the question if there is a solution if it is infinitely many solution, finitely many solution, unique solution. Now, suppose we have a non singular matrix, if A is determinant of A is 0 a non-zero then we have a inverse of this. So, then you have a unique solution, then we can multiply A inverse of suppose A is a here is singular A, A is a square matrix. So, if A is a square matrix we can think of determinant of A.

So, if the determinant of A is not equal to 0 and if A is a square matrix then determinant question will come. Then it is a non singular matrix then the inverse exist then you multiply x is nothing but A inverse b. So, if this is a non singular square matrix. Otherwise, you have to go for different way because this may not be a always square matrix; it may be 4 by 3 or it may be any order.

So, for that we need to take what is called augmented matrix. So, what is the augmented matrix?

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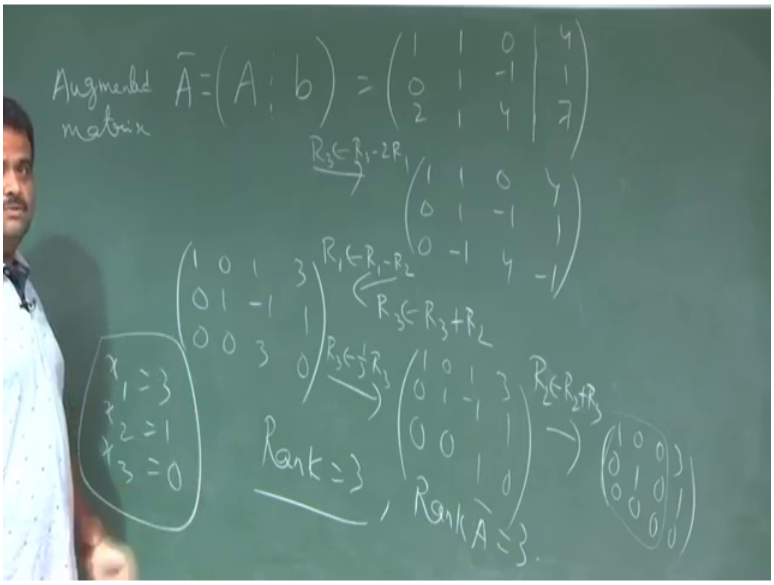

Augmented matrix is we take this matrix A along with this vector b; so, this is called augmented matrix ok, this is called augmented matrix. So, now, the question is how we can use this augmented matrix to. So, suppose we denote this matrix by A bar, now we will just apply the row operation on this augmented matrix and we will reduce to a some simpler forms. So, how we will do that we will see.

So, what is the augmented matrix corresponding to this equation? This is nothing but we have 1 0 2 1 1 1 0 minus 1 4 and we have a b component over here 4 1 7 ok. So, this we can have a space over here to indicate that that is coming from the b.

So, now we apply the row operation on this, we reduce this to the row equivalence echelon form. So, this is how we do that? So, we just replace this R 3; so, if we multiply so, we want to make it 0, so, we will do the R 3 is going to R 3 minus 2 R 1. If we do that, so, this first row will remain same, second row will remain same and the third row this will become 0 and this will become this is minus 1 and this will become 4 again and this is 2; so, 7 minus, so, this is minus 1 like this.

So, again if we do this another row operation, so now, we have to if we add this 2; so, if we just or if we subtract this. So, R 2, R 1 is going to R 1 minus R 2 and also if we do the R 3 is going to R 3 plus R 2. So, this will give us what? So, we are just R 1 is going to R 1 minus R 2. So, this 1 0, then we have a 1 3 R 2 we are not changing anything 0 1 minus 1 1 and R 3 we are changing, we are just doing the addition 0 0 then 3 0.

So, this is the one you are doing. So, now, this is the R 1 is going to this. Now, if we divide this by again if we divide again R 3 is going to say R 3 is going to half of R 3, then this will become how much? 1 0 1 3 0 1 minus 1 1 0 0 1 0, ok.

So, now, if we just add these 2 off then this will become 0; that means, R 2 is going to R 2 plus R 3; R 2 is going to R 2 plus R 3. So, then this became so, R 2 plus R 3 so, this will. So, first one is remain unchanged 1 0 3 then this will change, this will be of R 2 plus R 3 is not a good idea because, now not R 2 plus R 3 we want to this we want to make this to be 0; so, we can add this.

So, R 2 is going to R 2 plus R 3. So, now, this will be 0 1 0 1 0 0 0 0 ok. So, this is the coming from A part. So, rank of A is nothing but rank of A is 3 and what is the rank of the augmented matrix? Again rank of the augmented matrix is also 3 that is important rank of A bar is also 3 ok.

Now, if the rank is 3 then that means, it is a non singular matrix. So, that means, inverse will exist then we can directly go for the inverse way to get the solution. So, A inverse b, but before and we do not know the rank. So, we have to get the rank we have to do like this. Now, from here how we can get the solution? This so, this has the unique solution. So, from here we can just say x 1 equal to 3 x 2 is equal to 1 and x 3 is equal to 0 ok.

I think this will become 0, if please check the calculation and this will become 0 ok. So, now, this is the unique solution. Now, here there is a observation that rank of the matrix and rank of the augmented matrix should be same in order to have the solution, otherwise there will be no solutions. So, we will write that, if the rank of the matrix and rank of the augmented matrix are not same then there will be no solution because, we will end up with a if we have a augmented matrix A, so, why is so?

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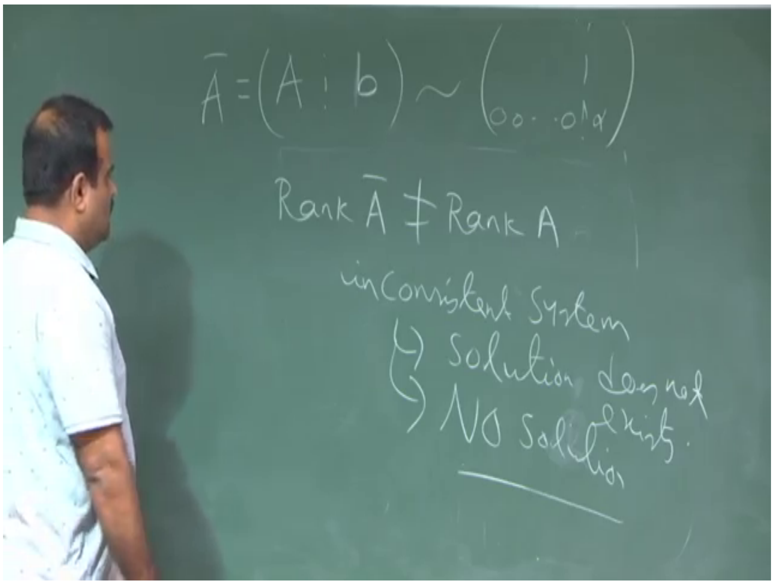

Because if we have a augmented matrix A and if we have a coefficient matrix A and if we have a b. So, this is our augmented matrix.

Now, if the rank of the augmented if the rank of A is less than the rank of the augmented matrix say then eventually what it will end up? Say suppose rank of for example, rank of rank of A is say 3 for some example and suppose rank of augmented matrix say some 4. So, that means, what? That means, if we apply a row reduced form of this. So, it will eventually give us like this. So, last row will be became 0, but we still have a coefficient over here. So, some alpha 1 over here and we have.

So that means, it cannot have a solution for that because, to have the solution there is no x is which will give you the alpha because all are coming 0. So, that is that means, the rank of so, rank of anyway there is a proof for this I am not discussing the proof now, we can we may be giving the proof in the note.

So, rank of A has to be same as rank of the augmented matrix. If it is not same then the there is no solution because if you take this augmented matrix if we keep on applying the row operations elementary row operation on this then if the rank is not same then eventually we will end up with a 0's in the last row with some non zero value in the b part so, like this.

So, this will be end up with some 0's over here if the rank is not same and here this is non zero some alpha. So, that is telling us that is not possible. So that means, there is no x i which will give us this because all are coming to be 0. So, that is the problem with this. So, this has to be there for having solutions and then it is called a consistent solution.

So, if the rank is so, just to recap this. So, the rank if the rank of A is not same as rank of the augmented matrix then we call this is a inconsistent system. Inconsistent system means no solution, solution does not exist or solution does not exist, this means no solution, inconsistent no solution.

So to have a solution we need to have this condition, the rank of a matrix should be same as rank of the augmented matrix then we have a solution. Now, the question is whether solution is unique or not those we will discuss. So, if the rank of this then we say the system is consistent.

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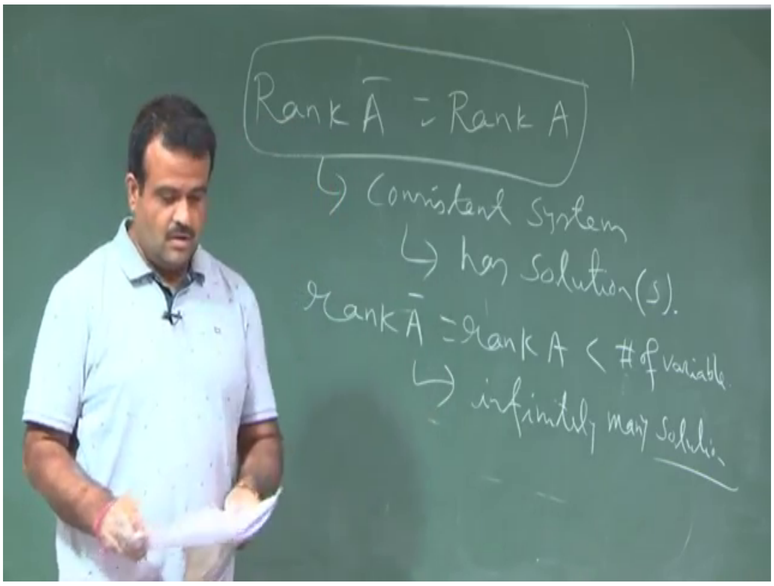

If the rank of A bar is same as rank of A then the system has a solution then it is called consistent system, consistent system of linear equation. This means it has a solution, it has a it may be infinite number of solutions so, but it has a solution has solutions or solution.

Ok so, now, the when it will have unique solution? If you have a system A x equal to b, when it has unique solution? If a is non singular; that means, then we have unique solution and that is nothing, but A inverse b ok. Now, if it is not then we have a if there is A then we have a infinitely many solutions. If there is if it is not unique if the rank of, so, that means, if the rank of if the rank of augmented matrix which is same as rank of A is less than the number of variables number of variables then we have a then we have a infinitely many solutions. It is consistent and we have a infinitely many solutions.

So these are the possibilities so, we will take some example to find the solutions of a system of equations ok. So, let us take an example.

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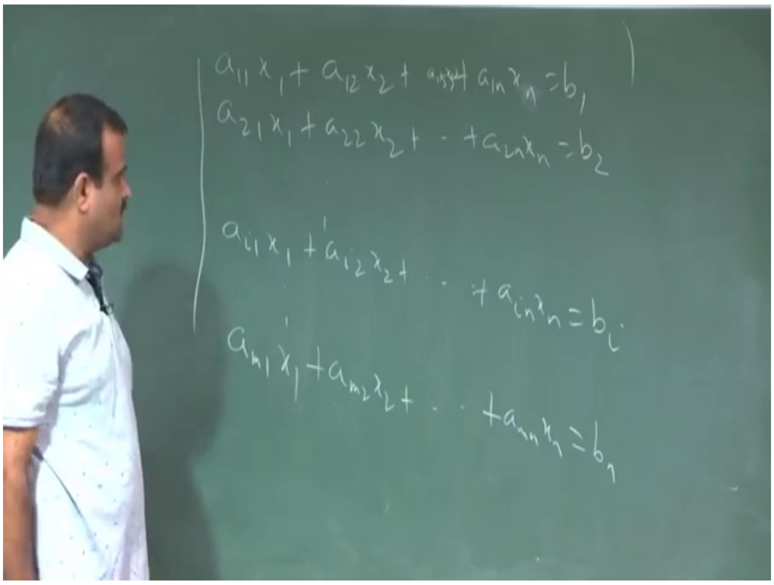

So, in general if we have to write in general. So, in general any system of equation is of this form a 1 2 x 2 a 1 n x n equal to b 1, a 2 1 x 1 plus a 2 2 x 2 plus a 2 n b x sorry x x n x n equal to b 2. So, dot dot dot a i 1 x 1 this is the i th row a i 2 x 2 we can have j th column a 1 j x j if you write anyway a i n x n equal to b i dot dot dot a m 1 x 1, suppose there are m equations in the system a m 2 x 2 a m n x n equal to b n.

So, there are this is called system of equations. So, this can be written as of the matrix form.

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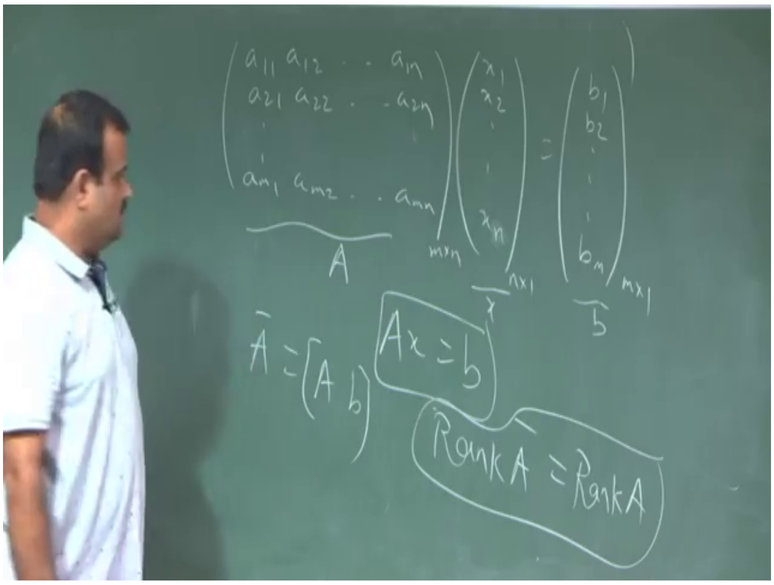

a 1 1 a 1 2 a 1 n a 2 1 a 2 2 a 2 n a m 1 a m 2 a m n x 1 x 2 x m x n actually n variables b 1 b 2. So, this is m cross n and this is n cross 1. So, this will be b m, this is m cross 1; this is a now this is our A matrix this is x this is b. So, A x equal to b of this form.

So, now the augmented matrix is A b, now we have we already talked about rank of this should be same for the having the solution inverse consistent system of equations. So, now, we will take some examples. This is a general form of any system of linear equations just we take example.

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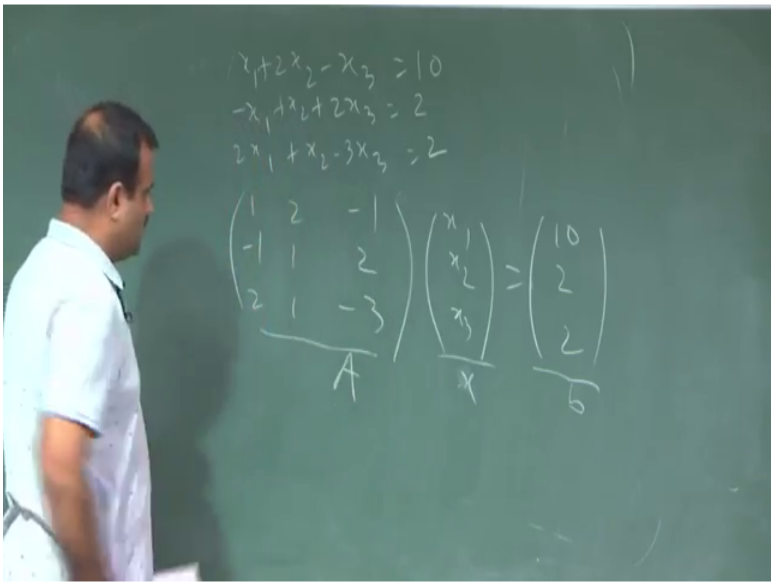

x 1 plus 2 x 2 minus x 3 is 10; minus x 1 plus x 2 plus 2 x 3 equal to 2; 2 x 1 plus x 2 minus 3 x 3 equal to 2.

So, this we can write in matrix form 1 2 minus 1 minus 1 1 2 2 1 minus 3 x 1 x 2 x 3 since b is 10 2 2 like this ok. So, now, this is our A matrix, this is our x variable vector and this is our b this is a coefficient matrix. Now, we have to find the rank of A and rank of augmented matrix. We have to first check whether this system is consistent or not for that rank of A and rank of augmented matrix should be same.

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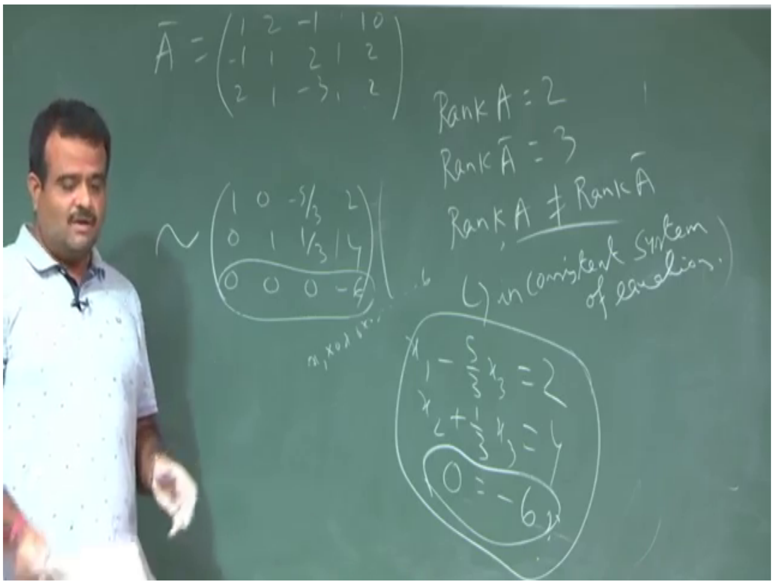

So, let us take the augmented matrix and we will do the row operation on this minus this is A 2 1 minus 3 and along with b vector; so, this is the augmented matrix. So, we will do the row operation. So, if you do the row operations, so, we first multiply this with 2 and subtract with this row and then we can add this and this. So, like this if it is the row operation eventually we will be getting this row reduced, row equivalence matrix of this 1 2 minus 1 10 0 1 1 by 3 4, you have to work out I am just skipping this part; 0 minus 3 1 minus 1 minus 18. So, by doing just the row operation on this matrix we will get this matrix ok.

So, now from here what we can say? What is the rank of A? Again we can do another row operations on this by adding these 2. So, eventually we will be getting this is the intermediate one; so, but again if we do the row operation eventually we will be getting this matrix 1 0 minus 5 by 3 2 we have to just subtract by 5 and then add with the another row 0 1 1 by 3 4 0 0 0 so, 0 0 0 and minus 6 anyway you can just work out with this.

Here we have a problem what is the problem? So, this is equivalence to this matrix under row operations. So, this is our; we have problem, now what is the rank of A? Because, these we get just by applying the row operation on this. Because, row operation we will not for row operation we will not mixing this part with this; if you have to do the column operation then we so.

So, rank of A is 2, but what is the rank of the augmented matrix? Rank of augmented matrix is 3; it is not 2 because if we take this one then the determinant is non 0 there is a problem so, this rank is not same. So, rank of A is not same as rank of A bar. So, the so, this is inconsistent why? Because, see for this if you consider this row there is no such x 1 which will give us 0 into x 1 plus 0 into x 2 plus 0 into x 3 equal to minus 6.

So, this is has no solution basically so, the we are reducing this to a another system of equations, well we are doing row operation; that means, we are say we have a system of equation x 1 plus 2 x 2 equal to 4 x 1 minus x 2 is equal to say 6 now row operation means we are just say it is a 6. So, we are just multiplying this with subtracting this with 2. So, this will be reduced to 3 x 1 plus x 2 is equal to 2, x 1 minus x 2. So, this is a row operation on the augmented matrix.

So, row operation while we are doing the row operation on the augmented matrix, we are achieving the equivalence system of equations which are same solutions. Because, we are just multiplying some we are exchanging some row, we are multiplying or we are just adding some row and subtracting with that. So, all the row operation we are doing on augmented matrix is giving us basically the same system of equivalence system of equations which are having suppose to have the same solutions.

So, now this system has equivalence with the original system, but this system has no solution because, these are all zeros coefficient matrix and here we have non zero. So, this system has no solutions. So, that is why this rank is not becoming same. So, this is another way to look at the why the rank of augmented matrix and rank of the matrix A should be same for the for the consistent system.

So, if the rank is not same; that means, this is the equivalence system will end up with like this, which has no solution because all are 0 there is will be no x is which will satisfy these will minus 6 ok. So, again I am telling you the while you doing the row operation on the augmented matrix we are we are just reaching to the equivalence system of equations which have same solution with the original system.

So, now this system has no solutions, that is why it is and again from this. So, that is why the rank of the augmented matrix would be rank of the matrix. So; that means, this system has is inconsistent. So, inconsistent system of equation because it has no; because rank of augmented matrix and the rank of this is because, the equivalence system is what? Equivalence system x 1 minus 5 by 3 x 2 x 3 is equal to 2 then x 2 plus 1 by 3 x 3 is equal to 4 and then 0 is equal to minus 6. So, this is the equivalence system corresponding to the original system because this system we got by applying row operation on the augmented matrix.

But, this system is inconsistent because this has no solutions, it is not possible. So; that means, and also it is reflecting from the rank so; that means, the for consistency we should have rank of a matrix is same as rank of the augmented matrix. So, we can I mean. So, we can work out on another example this I am giving you as a exercise.

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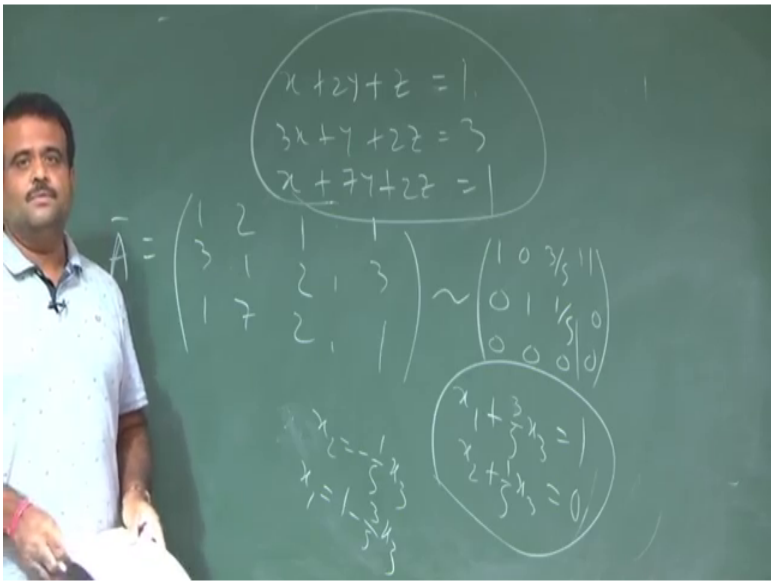

So, if you take this system x plus 2 y plus z is equal to 1 3 x plus y plus 2 z equal to 3 x plus 7 y plus 2 z equal to 1.

So, we can take the augmented matrix. So, this is the coefficient matrix 3 1 2 1 7 2 and if we take the b now, if we apply the row operation we will get the equivalence system which is having same rank. So, eventually we will end up with the row reduced echelon form 3 by 5 1 0 1 1 by 5 0 then 0 0 0 0. So; that means, rank of the matrix A and rank of the augmented matrix is same which is 2 which is less than the number of variables. So, it has infinitely many solutions.

So, we can just equivalence system is nothing, but x 1 plus 3 by 5 x 3 is equal to 1 and x 2 plus 1 by 5 x 3 is equal to 0 this is the equivalence system corresponding to this system. So, this has same solutions so, you can just take it is infinitely many solution x 2 is equal to minus 1 by 5 x 3 and x 1 equal to 1 minus 3 by 5 x 3 we can choose any x 3, any real number. So, this has infinitely many solution.

Thank you.