Introduction to Abstract and Linear Algebra
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Lecture – 31
Rank of a Matrix

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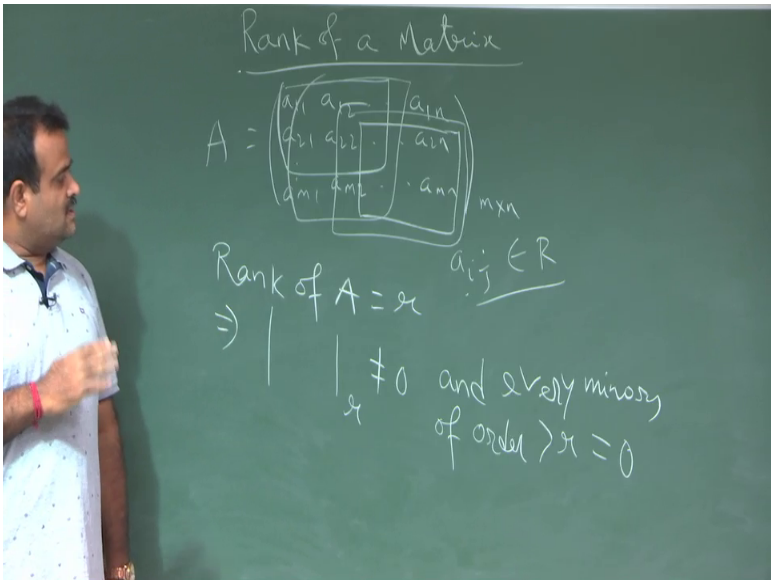

So we will talk about Rank of a Matrix. So, as you know matrix is a rectangular arrangement of the elements. So, suppose we consider a m by n matrix, a 2 1 if this is the second row, a 2 n, a m 1 this is the mth row a m n. So, this is an m by n matrix. So, where a i's are coming from a field, here we are taking a real number, but it could be a any field. So, now these elements are so this is a rectangular of size m into n. And the entities are the elements coming from a field. So, here we are taking an example of a real field, so this is a real matrix.

So, what is the rank of a matrix? So, rank of a matrix is so if you take a matrix, and we know the minor. So, it is the minor of say rank is rank of A is r means, implies that we have a minor of size r, which is not equal to 0 at least 1 minor. We need to get at least 1 minor of size r not necessarily principal minor any minor, so it could be here also. So, minor means we take a sub matrix of it, and take the determinant of that. And that value should be not equal to 0. So, it should have a minor at least one minor of order r, which is non-zero. And every minor minors of order greater than r is equal to 0. So, this is two conditions.

So, we take a we need to have a minor of order r at least 1 minor, which is non-zero. And all other minors of order more than r that means r plus 1 minor. If we take minor of r plus 1 that will be 0, r plus 2 minor will be 0, r plus 3 minor will be 0, so that is the definition of the rank. This is called determinant rank definition of determinant rank. We will talk about the row rank, column rank, but this is the rank of a matrix, this is the definition. So, what is the definition of a rank of a matrix?

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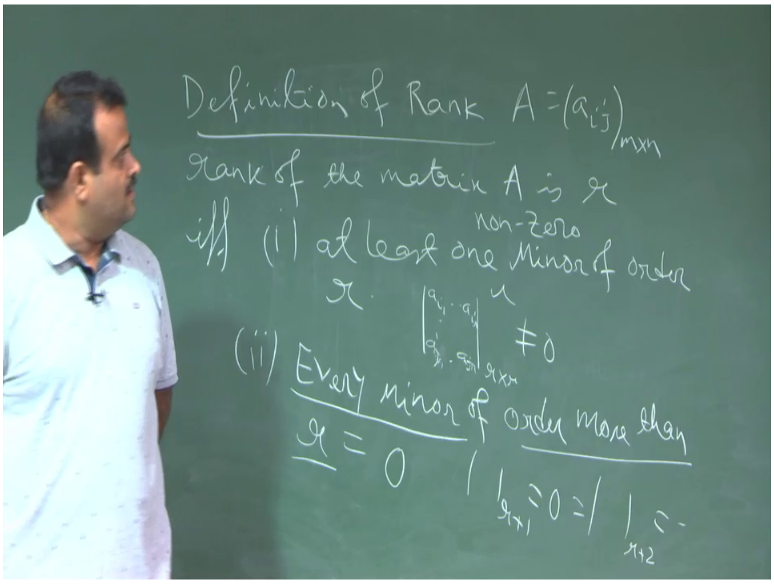

So, is a just definition of rank, definition of rank of a matrix. So, it is a r, so let r, so this is a rank of matrix A, which is of size say m cross n. So, the rank so we say the rank of the matrix A rank of the matrix A is r if and only if, there are two conditions. So, you should have at least 1 minor of order r, which is non-zero, at least 1 minor of order r at least 1 non-zero minor of order r, so that means, we should have a sub matrix of this matrix. I will take the determinant of that and size of that matrix is r cross r that one of such minor should exist. So, this is a sub matrix a i 1, a i r so, a so a j so i i so a i plus so it could be anything like j 1, j 2 like this a j r, but the size of this is r cross r.

And the second one is all the other minor of order more than r should be 0. Every minor of order more than r that means, r plus 1, r plus 2, r equal to 0. This is for every minor of order more than r, so that means, all the minor of order r plus 1 equal to 0. All the minor of order r plus 2 is equal to 0 like this, so on ok. So, this is the definition, this is called determinant definition of a rank.

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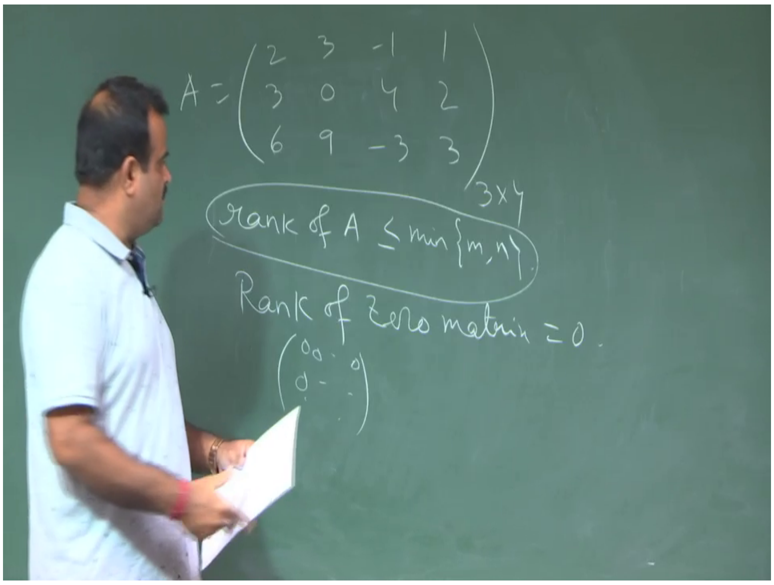

So, now we will take an example to see how to get the rank of a matrix. So, if you take a matrix like this, suppose 2 3 if these are real matrix that means, the elements are coming from the real field. But it could be general field also but, in that case we need to define the determinant about that field ok, but we know the determinant over the real field 3 0 4 2 6 9 minus 3 3.

So, there suppose we have given this matrix. What is the size of the this matrix, how many row 3 rows and 4 column. So, this is a 3 by 4 matrix. So, definitely rank will be so rank of a matrix rank of A is always less than or equal to minimum of m comma n. If it is a m by n matrix, because that is the we need to take a sub matrix whose minor is not equal to 0. So, we cannot have a sub matrix of size more than m or more n, because that has to be a square matrix. The sub matrix has to be square matrix.

So, if we have a m number of rows n number of columns, if m is less than n say. So, we cannot construct a some square matrix of more than m, so that is the that is the condition. So, the rank has to be less than 3, so rank is either 3 2 1 0, 0 that the rank of the zero matrix is so this is the convention rank of zero matrix is equal to 0. This is the convention that means, if you have all the 0's entries all are 0's, so rank of this is 0, because the there is no such minor exist, which can give us non-zero value, so that is the convention. We take the rank to be 0 fine.

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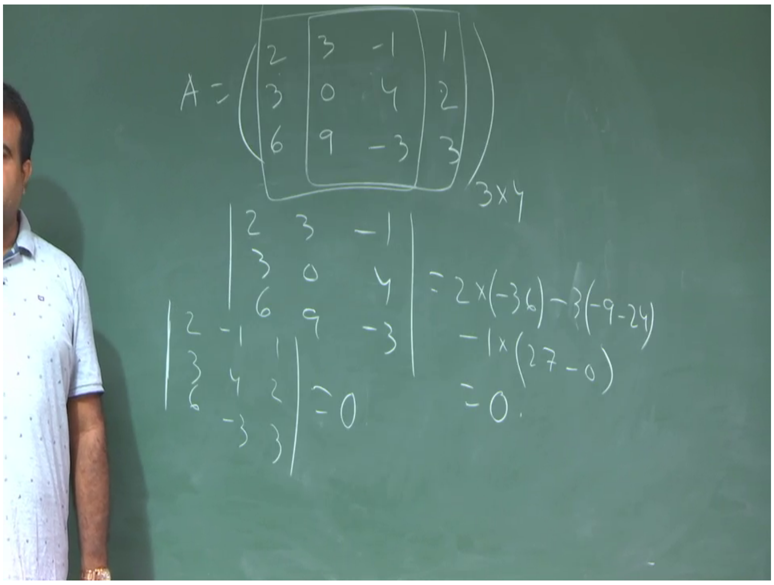

So, now we want to get the rank of this matrix, so how to get the rank of this matrix. So, we have to consider all the minors. So, what is the maximum minor we can have so 3 by 3 this so we can consider these minor. So, what is the value of this minor, 3 0 4 6 9 minus 3, so it is 2 into this 2 into this. So, it is minus 3 and then minus then minus of just calculate the determinant minus of this into these 2, so minus 9 minus 24. So, this is 3 into minus 9 minus 24, then minus 1 into so minus 1 into this into this, so 27 minus 0, so if we calculate this, we will get 0 over here.

So, not only this, if you take this minor, you can find this is also 0. If you take these, these, these minor like if you take 2 3 6 and then minus 1 4 minus 3 then 1 2 3, you will see this is 0. So, all the if you just find out, all the minor of order 3's are coming to be 0. So, rank cannot be 3.

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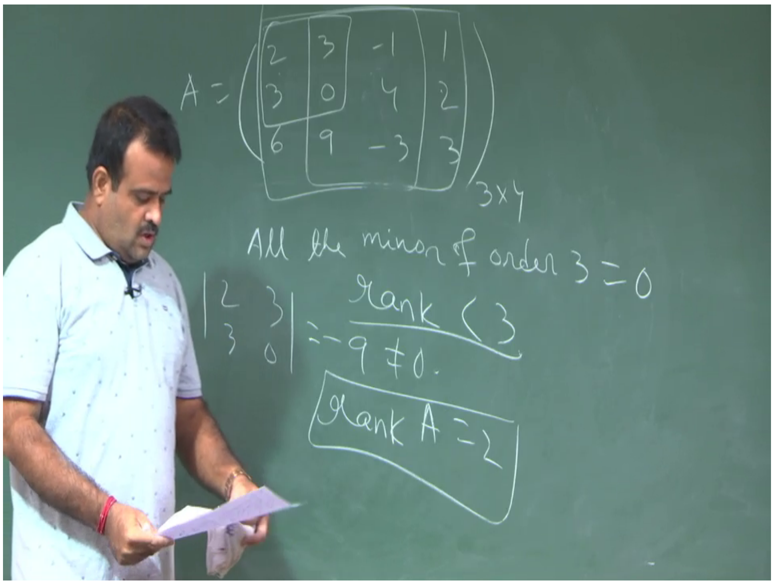

Now, you have to check whether rank could be 2 or not. So, for that we need to find at least 1 minor of order 2, which is non-zero. So, let us try to get that. So, the all the minor so all the minor of order 3 is 0. So, this we have checked, I mean we have check two, but you can work out for others minor of order 3 cross 3, so that means, rank must be less than 3. So, rank is either 2 or 1, so now we have to check whether the rank is 2 or not.

So, now we have to consider all the minors of order 2. So, let us check that whether we can have a minor of so we just take this minor of order 2. So, what is the value of this, so this is minus 9, so this is not equal to 0. So, we got a minor of order 2, which is non-zero so that means, from here we can say rank of A is 2 ok, so that is the definition of the rank. Rank of a matrix is just the index r such that we should have a minor at least 1 minor of order r, which is non-zero. And all the other minor, which is more than r that is r plus 1, r plus 2, those will be the 0 ok. So, now this is the definition of the rank.

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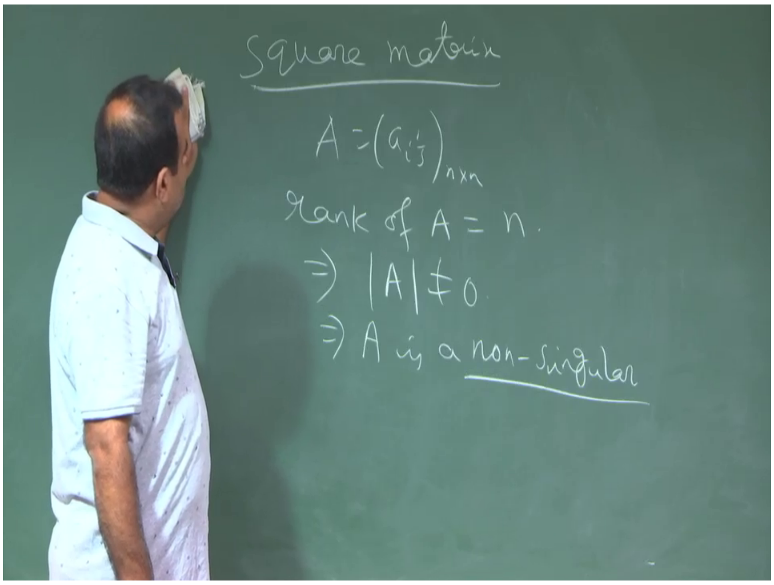

Now, we say if we have a square matrix, square matrix means the number of rows and number of columns are same square matrix. So, if you have a square matrix, say n cross n suppose number of rows number of columns are same. Now, the rank would be n here, because this is a square matrix.

So, if the rank of this is n, this means the determinant of A is non-zero. So, then this is called a non-singular matrix. Then this implies it is a definition A is a non-singular matrix, where determinant is non-singular. Otherwise, if rank is less than n, then you say this is a singular matrix. Now, if we have a rank less than if we have a singular and non-singular matrix, then we know that it has a inverse. So, what is the inverse of this matrix, so suppose A is a non-singular matrix.

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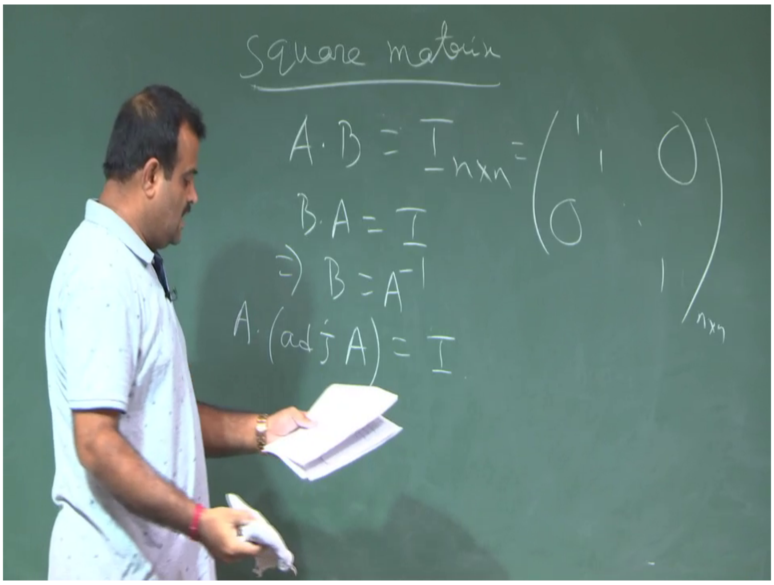

So, the definition of inverse is A into B should be the identity matrix. So, identity matrix means, all are 0 other than this diagonal element all are 0 element, this is a n cross n. So, you should have a we should have a this and the vice versa we should have we should multiply this in both side identity matrix. And then B is called inverse of A ok. Now, if A is a non-singular matrix, then inverse exist. And then we know the result A into adjoint of A is equal to identity A into adjoint of A is equal to identity matrix. So, this result we can follow, and we can get the that rank of A, we can get the that inverse of A matrix.

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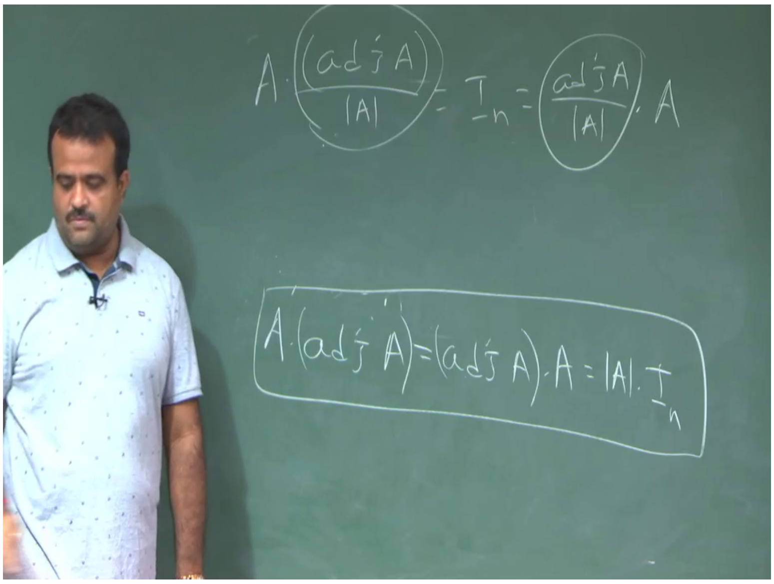

So, this is the result is telling no this is not the result is telling A into adjoint of A, which is other way also adjoint of A into A, this is telling us determinant of A into the this is determinant of A into identity matrix. This is the result we can prove. The adjoint of A means, we take a we take each element and we replace by its cofactor, so that is the we will come to an example that is the definition of adjoint.

So, now if A is non-singular, then the determinant is non-zero. Then from this is a result, this one can prove it. Now, if A is non-singular, then determinant is non-zero. Then we can do this thing we can divide this by determinant. So, A into adjoint of A by det of A is equal to identity, which is from both side, adjoint of A by det of A into A. So, this is our B, so this is the inverse of A from the definition.

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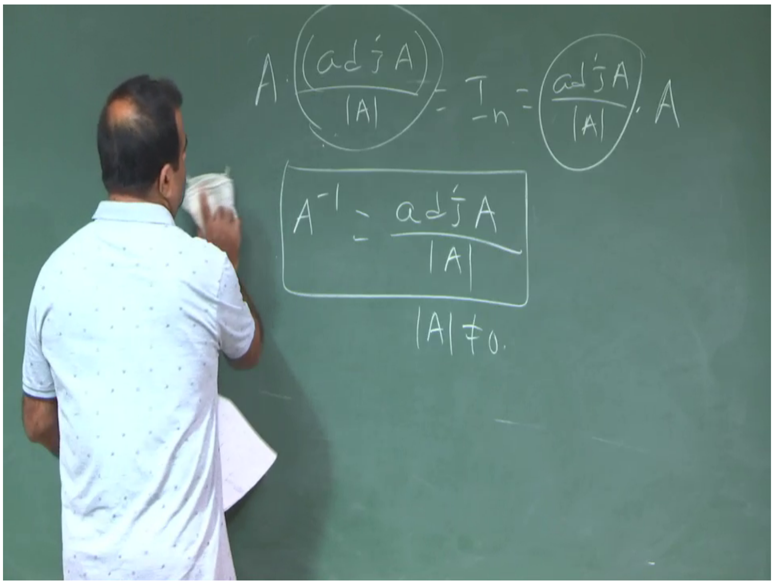

So, this is the A inverse. So, A inverse is nothing but adjoint of A. This is the matrix by determinant of A while A is a non-singular matrix while determinant of a is not equal to 0, so that means, if A is a non-singular matrix, then only inverse exist. And the inverse is of this form this ok.

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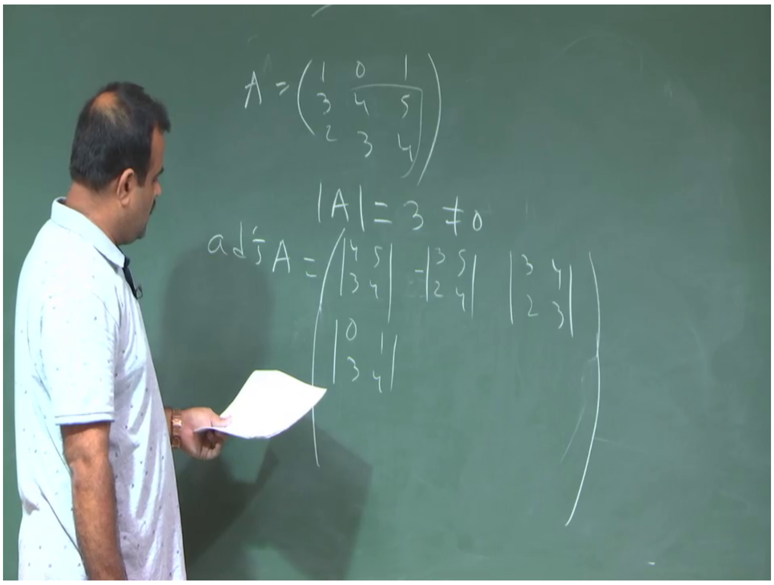

Now, how to calculate it? We can just have a quick example, so we can take a 3 by 3 matrix, so just a quick example. So, suppose we have we have to take a non-singular matrix. So, 1 0 3 4 5 2 3 4, now suppose this is our matrix. Now, we have to check what is the determinant value of this, so determinant of this is 3 (Refer Time: 16:23) you can easily workout on this.

Now, how to get the adjoint of A? Adjoint of A means, we just this one we are going to replace by the co factor this. So, this is basically nothing but determinant 4 5 3 4 this place. Then minus of this will be replaced by this minus means minus symbol, so this is 3 5 2 4 the determinant this is co factor. Then we have for this we have 3 4 2 3, and this is also a 3 by 3 matrix. And this one is this one is replaced by so we ignore this row and this column. So, this determinant 0 1 3 4 ok so, like this, so 0 1 3 4. So, this is the A or transpose of it.

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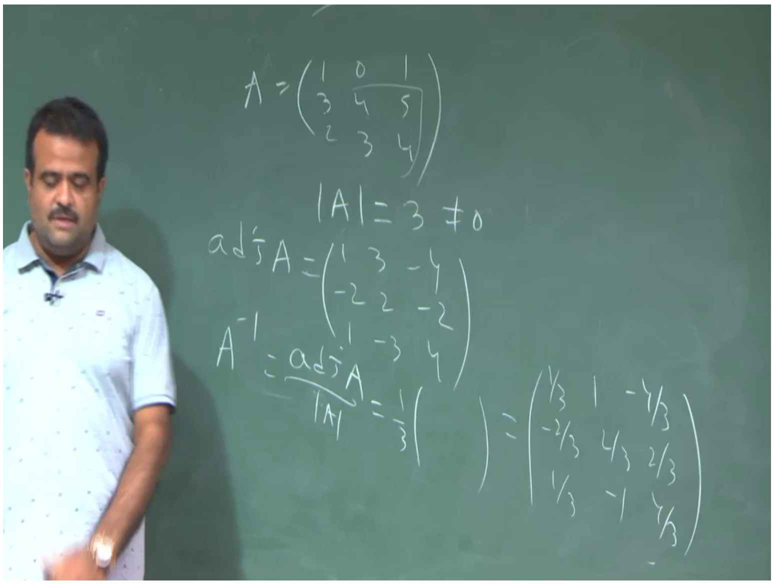

So, the anyway so this is if we calculate it, finally I am not going for details of this calculation, so finally we will get a matrix like this; 1 3 minus 4 minus 2 2 minus 2 1 minus 3 4 ok. Details calculation will be given in the note, but this is the calculation. So, and determinant of this is 3, so what is the inverse of this, inverse of this is adjoint A by det A. So, 1 by 3 into this matrix so, you can just take this 1 by 3 inside. So, it is 1 by 3 1 minus 4 by 3, then minus 2 by 3 2 by 3 2 by 3, then again 1 by 3 minus 1 and 4 by 3. So, this is the inverse of the corresponding matrix this. So, since the matrix is non-singular, then only inverse will exist ok. So, this is the inverse.

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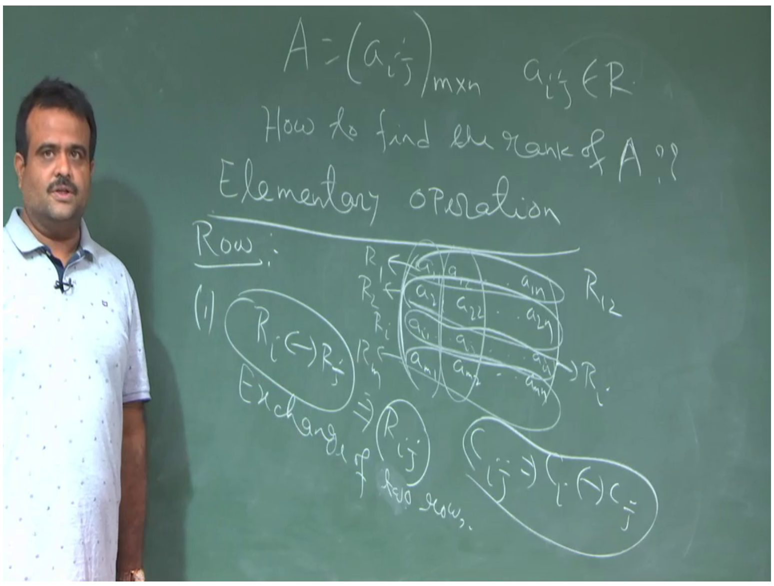

Now, we will see later on how you can get that because the finding adjoint is very difficult for the last matrix. So, how we can get the inverse for a large matrix? So, you have to do the some kind of elementary operation the row operation, column operation, so we will see that ok. So, let us just talk about how to find the elementary operations to find the rank of a matrix.

So, the question is suppose you are given a matrix n is order, it could be m by n. And the these elements are coming from the real field say. Again I am telling, it could be general field ok. So, now the question is how to find the rank of A? Now, one way to get the rank of A is to you have seen now you can check the all possible minors. And then we start with the minimum of this, then we start checking all possible minors. And once we get non-zero minors of some orders, then that will be the rank of the matrix, but that is difficult for a large matrix ok.

So, what is the solution? Solution is the elementary operation elementary operation, this is either row operation or column operation. So, there are three types of elementary operation. Suppose, we are talking about row operations on row, and same thing will be carried out for column also. So, row operation so we have a row like we have a matrix a 1 1 a 1 2 a 1 n a 2 1 a 2 2 a 2 n a m 1 a m 2 a m n. So, this is referred as row number 1 first row, this is referred as 2nd row like this, so this is referred as m nth row there are m row and n columns. So, this is the 1st column, 2nd column like this.

So, in general we have a jth row, ith row ith row is here a i 1 a i 2 dot dot dot a i n, so this is in general, this is the R j ith row. Now, row operation elementary operation means, we can do that three types of operation one is we can exchange two rows, see we can exchange 1st row with 2nd row or ith row with jth row, so that is denoted by ok. We can exchange the row, this is one operation. This is referred as this is referred as i R i j, we are exchanging the row.

If we exchange the 1st row, 2nd row, 1st row will come here, 2nd row will go there, so that is reference as R 1 2. So, we are exchanging the rows. 1st row is coming to the 2nd row, 2nd row is coming to the 1st row. So, in general this is R i j, so that means, R Rth row is going to the jth row, and jth row is going to the Rth row. So, this is one operation. So, this is exchange of rows exchange of two rows this is only on two rows. And similarly we can do for column also, exchange two column. So, we have C i j this means, C i and C j. So, jth column and ith column both are getting exchanged. So, this is the one column operation ok.

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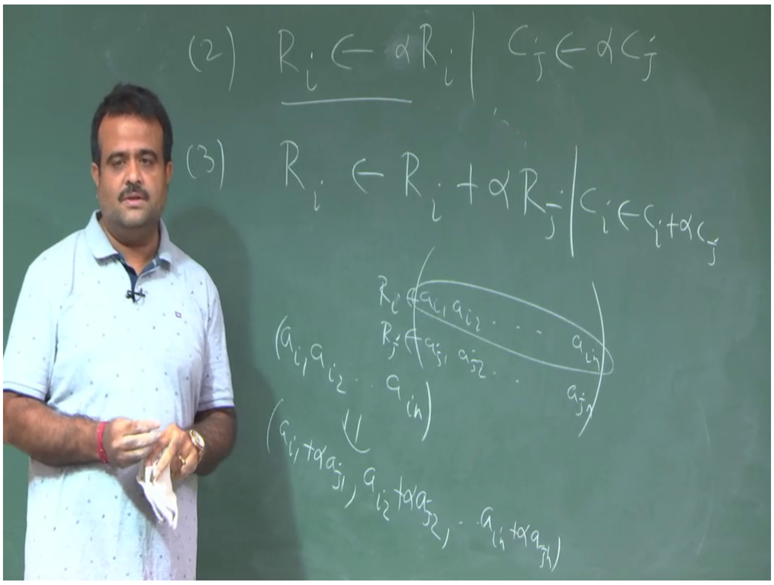

Another operation is we can add some constant on any of this row or for column any of the so this is the 2nd elementary operation is R i, we consider the ith row. R i you replace by some constant into R i say some constant say lambda, lambda or some alpha alpha into R i. So, this is referred as we multiply some constant with the ith row. So, if the ith row is if R i is this, R i is R a 1 i 2 i n. So, this will be replaced by we have to multiply a constant alpha a i 1, then alpha a i 2 alpha is another real number a i n.

So, this is referred as so these will be replaced by alpha times all these elements. So, this operation is referred as R i multiplying a scalar or the constant real number, here we are talking about real field. So, multiplying a scalar on a on the each element of the row that is referred as this ok. This is one operation. Similarly, for column we can have C j is going to some alpha times C j. We can take a particular column jth column, then we can apply constant on this, so that is the one column is going to replace by the constant time of that column. So, this is the another elementary operation.

And the last elementary operation is last this is last elementary operation is we can take a row, this is replaced by we can just do like this, so that means we can so we have a row say ith row and jth row. So, this is ith this is R i 1 R i 2 R i n sorry a i 1 a i 2 a i n a j 1 a j 2 a j n. Now, this is our ith row and this is our jth row. So, under this operation what we are doing, we are multiplying these by alpha and then this row we are replacing by this row we are replacing only R i. So, R i is replacing by R i 2 R i n. This is replacing by we are multiplying alpha with this, and then we are adding with this. So, r a i 1 plus alpha times a j 1 comma a i 2 plus alpha times a j 2 and so on, a i n plus alpha times a j n. So, this is the way we are continuing. So, this row is going to replace by, this row plus some scalar multiplication of the other row. So, we will do this type of so this is the another operation.

And similarly, we can do it for column also. So, the ith column can be replaced as C i plus alpha times C j, similarly we can do the same operation on the column. So, this will be elementary column operations ok. So, we will continue with this in the next model. So, next we will see the how this elementary operation can be, can help us to find the rank of a matrix ok. So, we will talk about in the next class.

Thank you.