Introduction to Abstract and Linear Algebra
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Lecture - 30
Linear Space

So, in the last class, we have defined the isomorphism between two vector space and we have seen that we the definition is if two vectors finite dimensional vector space are isomorphic, if there is a Bijective mapping, bijective linear transformation from one vector space to another vector space.

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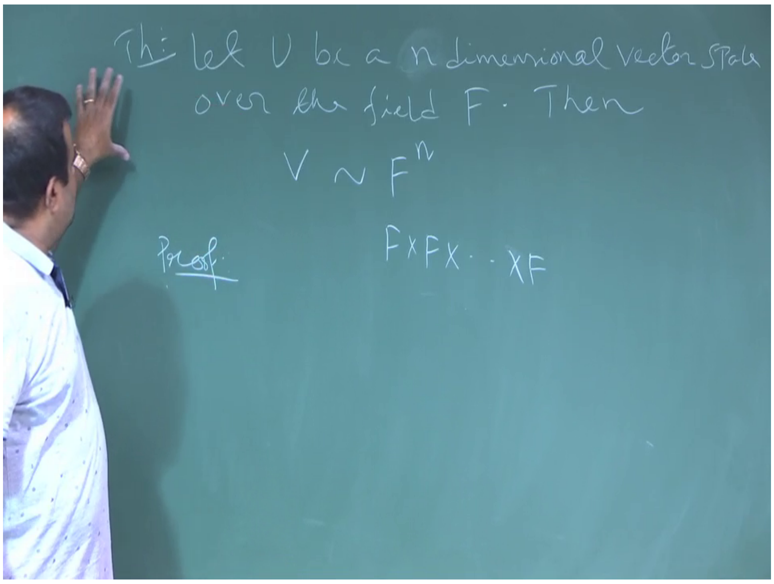

So, and we have stated this theorem in the last class that if V is a finite dimensional vector space over the field F, it could be real field also. F is any general field; it could be over r also. Then we can show that V is isomorphic with F to the power n. So, this we are going to prove now. So, how to prove this? So, suppose so, V is dimensional n that so, yeah so.

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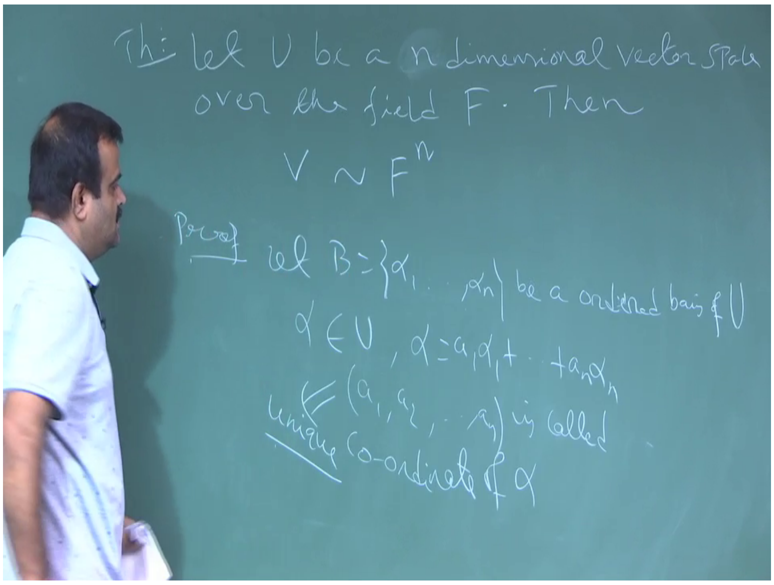

So, we consider a ordered basis of V be a ordered basis of V. So, ordered basis means we are not changing the position of this. So, alpha coming before alpha, alpha 1 coming before alpha 2 like this.

So, there is a while ordering like this. So, why we take this because we know the coordinate right coordinator of a vector. Now, if we take a vector a sorry alpha belongs to V, now alpha will be written as linear combination of this alpha n and we know this coordinate, this alpha this a 1, a 2, a n these are called coordinate of this alpha with respect to these ordered basis.

Now, this is unique. This is called, we discuss this one in one of the class, this is called coordinate of alpha with respect to this ordered basis. So, we take a ordered basis, then we take the coordinate and the since this is a ordered basis so, it will not change the position so; that means, coordinate will be the unique. We know this is unique, that is why you need to take the ordered basis. So, now we define a mapping and this is coming from basically, F to the power n. So, that is our mapping because this is a.

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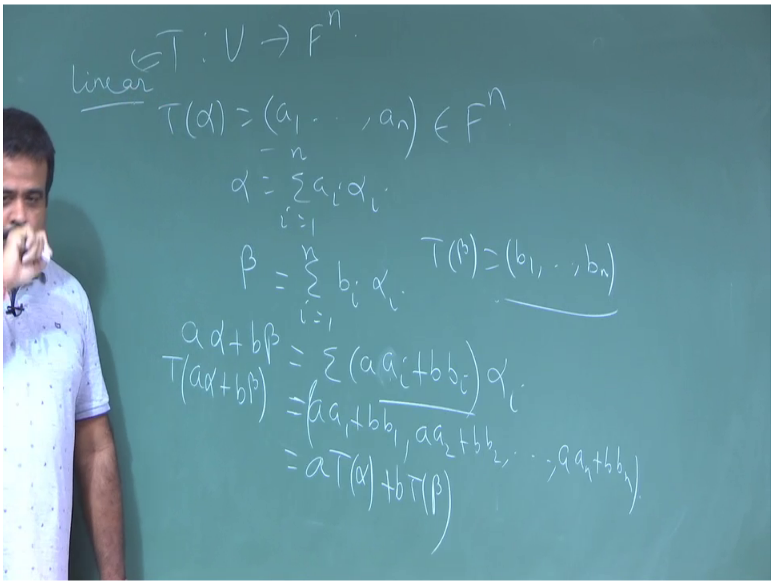

So, now we define a mapping from U to V, U to F to the power n as. So, T alpha is going to basically this coordinate of this. So, we take a alpha, we write this alpha with respect to the linear combination of this ordered basis and that coefficient will give us the coordinate and we take that transformation from alpha to this field. So, this belongs to F to the power n so, this is the.

Now, we need to show that this is a, this is the mapping because this is since this is the unique. So, this is a mapping. So, this is only find mapping. So, we need to just show that this is a linear transformation and also it is a bijective mapping. How to show this is a linear transformation? Suppose, we take a another beta from here. So, our ordered basis is alpha 1, alpha 2. So, so alpha is basically summation of a i alpha i; i is equal to 1 to n. We take say another beta, this is summation of say b i alpha i, i is equal to r to n.

So that means, now if you take a alpha plus b beta, so this is nothing but we have to find the coordinate of this, then only we can get T of a alpha plus b beta where it is going. So, a alpha plus b beta we can just write as summation of a a a i plus b b i alpha i. So that means, this is the core so that means, this is nothing, but a a 1 plus b b 1 first coordinate then a a a 2 plus b b 2 like this. So, a a n plus b b n like this. So, this is the coordinate of this. So, that that means, T of this is going to this.

Now, this is basically written as a of T alpha plus b of T beta because this T alpha is nothing but this and T beta is basically b 1, b 2, b n. So, this is the linear so, T is linear.

So that means, T is a linear mapping. Now, we need to show that this is a bijective mapping then only we can say this two are isomorphic. So, how to show this is bijective? So, for bijective, first you have to show the one to one.

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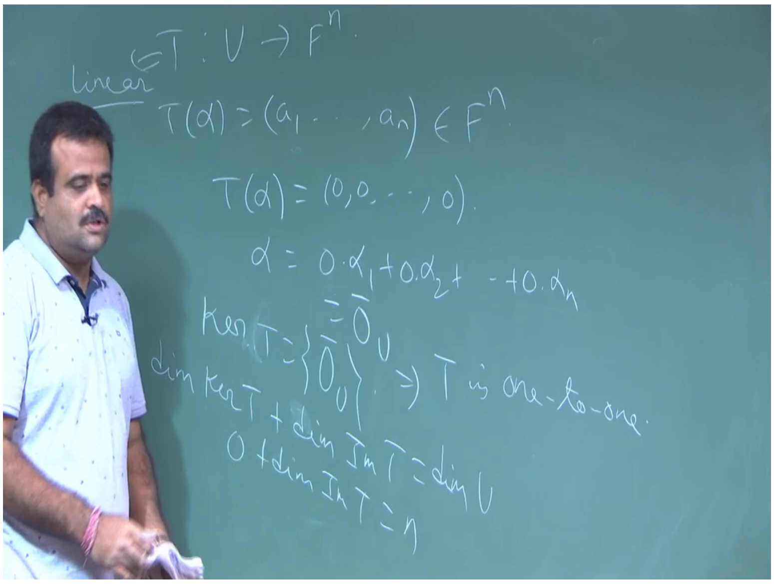

For one to one, just we can just have kernel of this. Suppose, we take a T alpha is equal to 0, 0 vector. So, this is 0, 0, 0.

So, now alpha will be written as what? Alpha will be written as 0 into alpha 1 plus 0 into alpha 2 like this, 0 into alpha n. So, this is basically 0 of U. So, the kernel of T is nothing but only 0 of U. So, this implies T is 1 to 1 and also similarly which one show T is on to by that angularity theorem because this, this is rank nullity theorem is telling us what? Rank nullity theorem is telling us dimension of kernel of T plus dimension of range of T is basically dimension of U. Now, dimension of U is, so this is 0. So, 0 plus dimension of range of T is basically this is n. So, dimension of range of T is n. So, that is basically F of n powering. So, this is basically one to also. So, this is a one to one mapping. So that means, this F a U and F to the power n is isomorphic.

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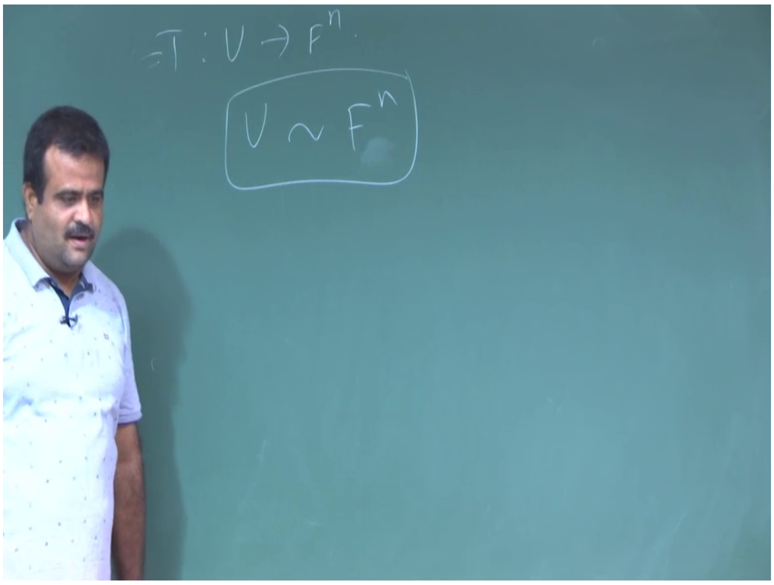

So, U is isomorphic with F to the power n, sorry F to the power n if U has a dimension n and F is the underlined field. So, this is the isomorphism. Now, we define the linear set up, linear space basically.

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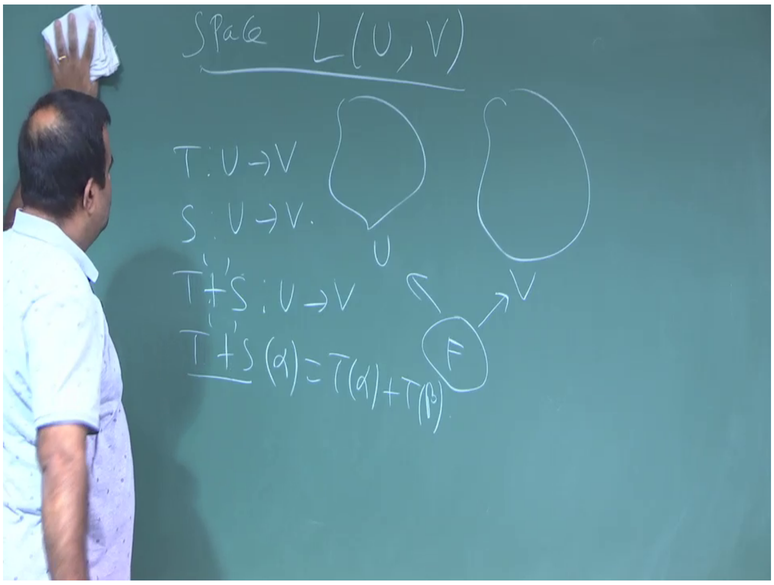

The space this is phase U comma V, we consider set of all linear transformation from two vector space U, V, so over the same field.

Now, let, so let U V is the U V are two vector space U and V, the two vector space over the same field and if we consider two linear transformation T of T and S say. These are two linear mapping say. Now, we define a plus mapping how to define S T plus S, T plus S is also a mapping from U to V. The definition of T plus S is basically we take alpha on this. So, this is basically T alpha plus T beta. This is the definition this is how we define plus operation, this is the plus on the this is one operation on the linear transformation.

And we will see this will form a vector space basically over the same field F. So, that will see slowly. So, now, first of all you need to show this is a linear transformation, this T plus S. So, how to show that T n is both a linear transmission from U to V. So, how to show T plus S is also linear transformation?

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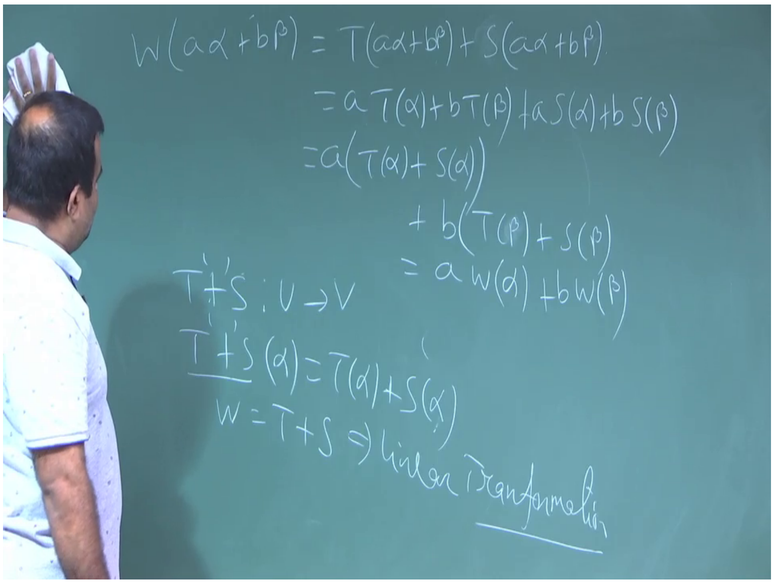

So, for that we need to show so, suppose T plus S is W. So, we need to show the W of a alpha plus b beta is basically a W alpha plus b W beta.

So, this is nothing but what? This is nothing but T plus S. So, T of a alpha b beta plus sorry, this is no, this is basically W T plus S. So, this is S sorry this is wrong this is S of alpha yeah we are having alpha. So, we are first applying T alpha plus S alpha. So, this will be first we are applying T alpha, T of that plus S of that.

So, now we know T is a linear transformation. So, the this will be written as a T alpha plus b t beta and we know S is also a linear transformation. So, this will be similarly this will be written as a S alpha plus b S beta. Now, if we simplify this and these are all operation over the over the vector space V and this is commutative and associative. So, we can just write this as a of T alpha plus S alpha plus b of T alpha sorry T beta plus S beta.

So, this is nothing but a of W alpha plus b of W beta, but W is this. So, this implies this is a linear transformation, linear mapping. So, this plus we define on the two transformation. So, that plus is also a linear term. Now, we define the scalar multiplication of this linear transformation and we will see that is also a linear transformation.

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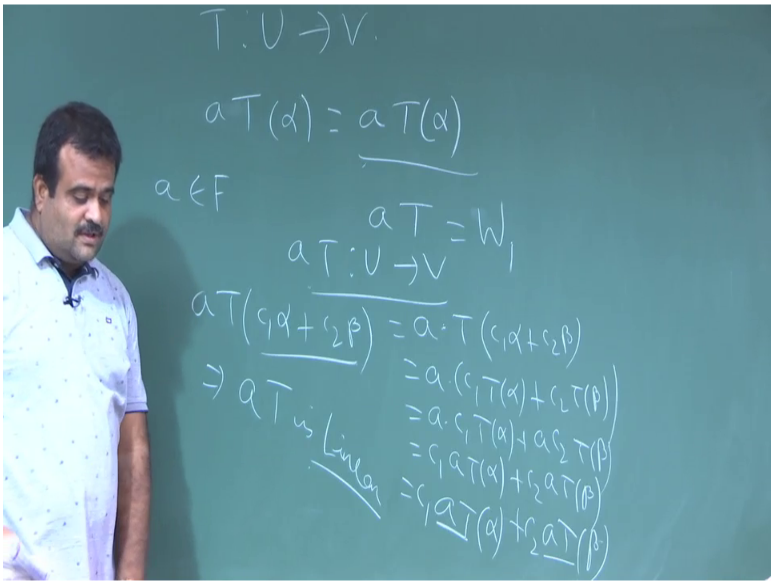

So, we take a linear transformation from U to V. Now, we defined alpha T, sorry a T scalar, a T where a is coming from the scalar. So, how to define this? This is some sort of a scalar multiplication, but not we are defining this scalar multiplication, but for simplicity we just writing at. So, a T of alpha is basically a of T alpha, this is the definition this is how we define a T.

So that means, a T we have to show this a T is also this is basically the referred as a T which is basically say some W we use some W 1 or something anyway this is a this is a mapping this is a mapping form U to V because this is a V vector and this is a scalar. Now, this scalar multiplication on the vector V. So, this will give you a vector in V.

So, this is a mapping. So, a T is a mapping from U to V. Now, only thing we need to show that this is a linear mapping. So, for that what we need to take, again we need to take some at of some c 1 alpha plus c 2 beta and this we have to write c 1 at alpha plus c 2 at beta. So, how to write this? So, this is nothing but by definition this is a element in U. So, this is basically a of T of this, c 1 alpha plus c 2 beta. This is the multiplication in the scalar multiplication in V. So, now, T is linear. So, this is basically c 1 T alpha plus c 2 T beta.

Now, this we can write here a c 1 T alpha plus a c 2 T beta. Now, this is this is from the associative property of the, this is the scalar multiplication property of the vector space, we know the it distribute. So, now this is again we can write because this is a coming from F this is this is also field. So, this is commutative under that field multiplication. So, this will be written as c 1 at alpha plus c 2 a T beta. So, this is nothing but c 1 a T alpha, a T is the operation, a T is the transformation plus c 2 a T beta. So, this implies a T is a linear transformation, a T is a linear mapping is linear. So, if T is linear then the way we defined a T as this, this is also linear.

So now, it is time to see whether this set of all linear transformation will form a vector space over the field same field F under these two operation like that plus operation, plus of two linear transformation and the scalar multiplication. So, that is the space called L U V.

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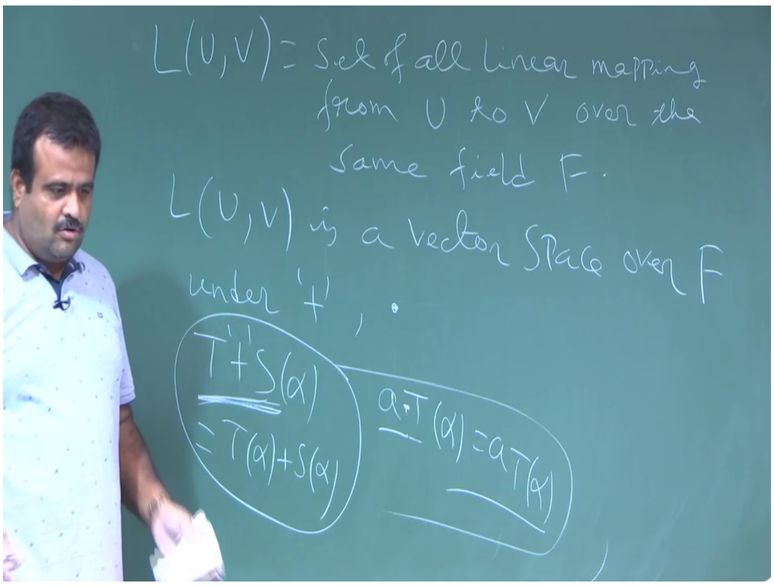

So, L U V is the set of all linear mapping from U to V from two vector space U to V over the field same filed F.

And then, this U V is also is a vector space over the same field F under the two operation plus; plus means that T plus S. This if we take two linear transformation, just now we have seen how a different plus. So, plus of alpha is basically this is itself is a linear transformation we have seen, T alpha plus S alpha. This is the way how we define the plus.

Under plus and that dot; dot means we have seen that a T this a T, a T is a linear transformation again just now you have seen. So, at of alpha is basically a of T alpha. So, this is the our definition of these two operation. This is a vector space under plus and dot. So, for that we need to show certain properties of the vector space. So, for vector space we have to show this is this will form a Abelian group with this and also this scalar will be distributed over these are the property. So, we can easily one can easily verify that.

So, this space is called a space L V. So, later on we will see the dimension of this space when you bring the matrix into the linear transformation; how we can visualize a linear transformation as a matrix. So, just we will introduce that now. So, now, it is time to see the relationship between the linear transformation and a matrix.

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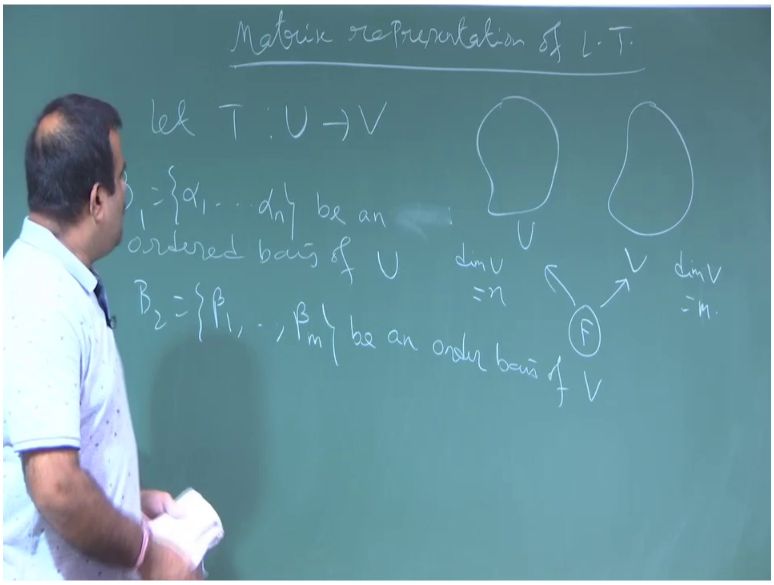

So, this is called Matrix representation of a linear transformation, of a linear mapping.

Now, suppose we have a linear transformation; let T be a linear transformation between two finite dimensional vector space U and V over the same field F. So, we have a two vector space U and V over the same field F and they are finite dimensional and suppose dimension of U is say n. So, dimension of dimension of U is n and dimension of V is say m.

So, now if they are m n so, there then there has to be a basis. So, suppose b 1, b 1 is the is the ordered basis of U. So, it is n dimensional, let B 1 be a be an ordered basis be an ordered basis of U and we take b 2 which is basically beta 1, beta 2, beta m be an ordered basis of basis of V.

So, we take two basis; ordered basis means we want to have the coordinate, to have the coordinate which is unique we need to have the ordered basis otherwise the coordinate position will change if these are not in order. So, we take two basis; one is from U another, one is from v. So, alpha 1 alphas are from U and betas are from V. So, now we have a linear transformation from T to U to V. So, this will map these alphas to betas. So, let us write that.

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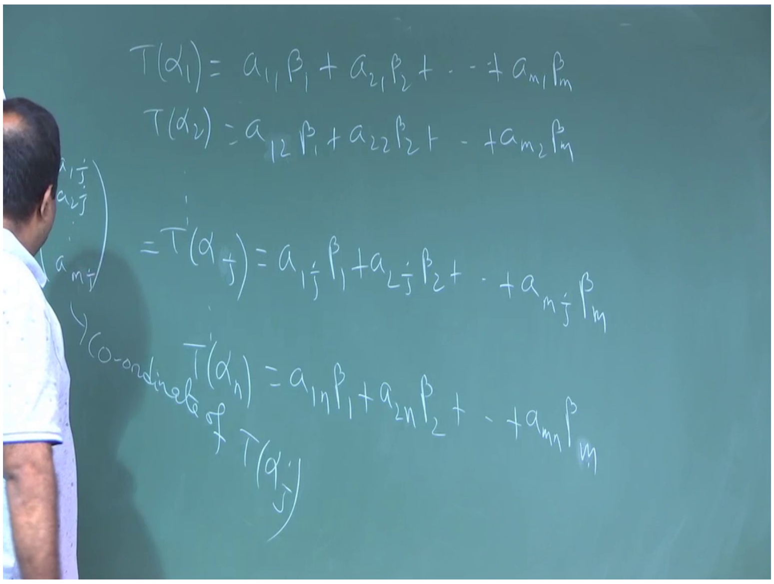

So, T of alpha 1 will map to will be a vector in v. So, vector in V will be written as some linear combination of the basis ordered basis. So, suppose that is basically a 1 1, beta 1 plus a 2 1 beta 2 plus how many elements, there are m elements. So, a m 1 beta m. It has to be written as a linear combination of the basis of this because this T of alpha 1 is a element in V.

Similarly, T of alpha 2 we can write as a 2 1 beta 1 sorry a 1 2 and plus a 2 2 beta 2 plus dot dot dot a m to b m this way. So, this way you continue T of alpha i or alpha j will be written as a 1 j beta 1 plus a 2 j beta 2 plus a m j beta m dot dot dot we have alpha 1, alpha 2, alpha n. Alpha n is equal to a 1 n, beta 1 plus a 2 n beta 2 plus a m n beta n.

So, these are the basically the coordinate of this is a 1 1, a 2 1 this is the coordinate of alpha 1. So, what is the coordinate of this coordinate of this alpha j; coordinate of alpha j is basically a one j a 2 j dot dot dot a m j. So, this is basically coordinate of coordinate of T of alpha j.

Now, we write this in a matrix form. The first column of that matrix is the coordinate of this, coordinate of this like this. So, this we have to remember. So, this is the way we are writing. So, now, we will write this as a matrix form. So, this is the coordinate, now we take any element alpha from here, so we will just yeah we can remember this. So, now, we take an arbitrary element from U.

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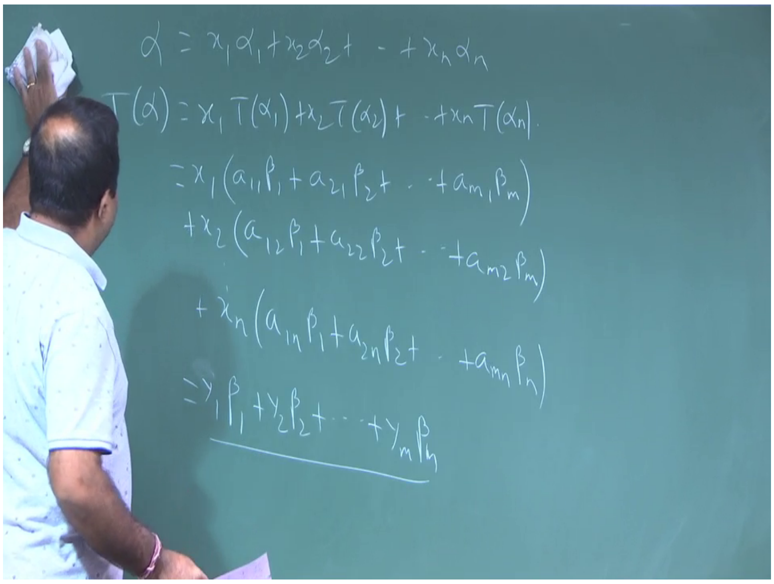

And this will be written as x 1 alpha 1 plus x 2 alpha 2 plus x n alpha n.

So, now we want to apply T of this alpha, T of this alpha is basically T of this. Now, T is a linear mapping so, x 1 T of alpha 1 plus x 2 T of alpha 2 plus x n T of alpha n. Now, alpha 1 we know alpha 1 is basically if we write that earlier form, it is basically a 1 1 beta 1 plus we want to write in terms of beta sighs because this has to be written is the linear combination of betas because betas are the basis in V.

So, a 2 1 beta 2 dot dot dot a m 1 beta m plus x 2 of like this a 1 2 beta 1 plus a 2 2 beta 2 plus a m 2 beta m. So, like this we continue plus x n of a 1 n beta 1 plus a 2 n beta 2 dot dot dot a m n beta n. So, this if we simplify this, this will be written as some form of y 1 beta 1 plus y 2 beta 2 plus dot dot dot y m beta m. So, this is the way we write this. So, now this y i are nothing but coming from the multiplying the matrix. So, this is basically y i are coming from if we put the matrix over here.

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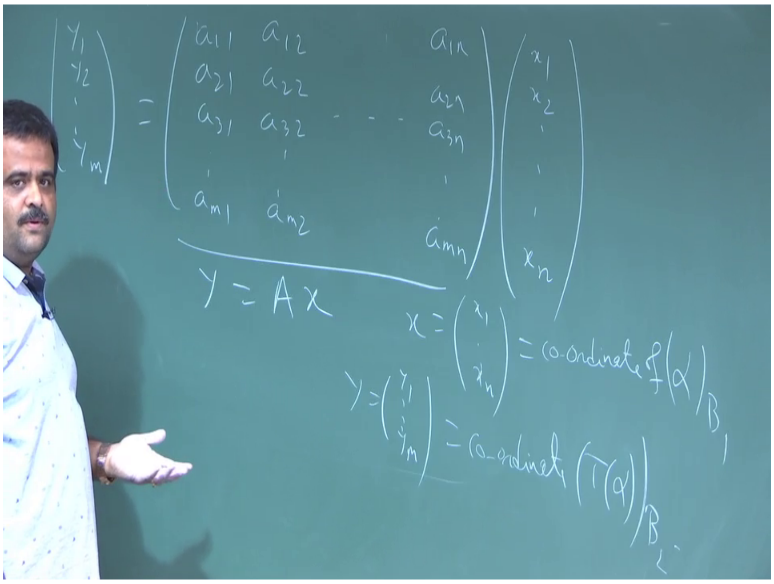

This a matrix by the coordinate of alpha 1, coordinate of alpha 1 is basically 2 1 a 3 1 dot dot dot a m 1. So, a 2 2 a 3 2 a m 2. So, last is alpha n. So, a 1 n a 2 n a 3 n dot dot dot a m n. So, this is a matrix a matrix. So, this is nothing but we take the first column as a coordinate of T alpha 1, T alpha 2, T alpha n.

Now, if we multiply this with that x 1, x 2, x n which is basically the coordinate of that alpha and that will give us this beta, this y. So, this is basically y 1, y 2, y m. So, this is the matrix a matrix. So, y is nothing but a x where x is the coordinate x 1, x 2, x n, it is the coordinate of alpha coordinate of alpha with respect to that basis beta 1, then the y is the y is basically y 1, y 2, y m which is basically the coordinator of coordinate of T alpha with respect to beta 2.

So that means, if we know this matrix, we can get the transformation. If we know this matrix, then any vector can be any vector can give us this coordinate. So, if you multiply this, it will give us the coordinator of the vector in V. So, that will take an example, the quick example and this matrix is a matrix, we will take an quick example on this.

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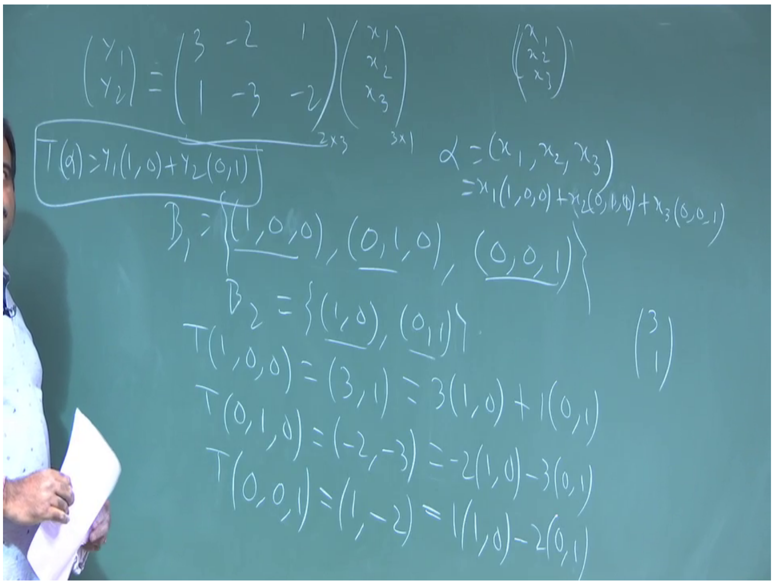

Suppose, we have a linear transformation from R 3 to R 2 we define T of x 1, x 2 x 3 as 3 x 1 minus 2 x 2 plus x 3, then x 1 minus 3 x 2 minus 2 x 3. Now, we want to get a matrix on this, we want to get a corresponding matrix for this linear transformation. Then if we know the matrix, then the transformation is given to us.

So, for that we take a basis for R 3. So, we take the standard basis 1, 0, 0; 0, 1, 0; 0, 0, 1 this is our B 1. We take the standard basis of R 3 and then we take the standard basis of R 2 also 1, 0 0, 1. Now, to form that matrices are alpha 1, alpha 2, alpha 3 and this is our beta 1, beta 2. Now, to get that where alpha 1 is going, so you have to take alpha T of 1, 0, 0 alpha 1 T of alpha 1 is basically if you calculate, it is basically 3 comma 1 just if you put the value over here and these we had into write as beta 1, beta 2. So, this is basically 3 of 1, 0 plus 1 of 0, 1.

So, basically 3, 1 is the coordinate for this T of this in terms of this basis. So, this 3, 1 will be the first column of the matrix. Now, similarly if we calculate this is basically minus 2 comma 3. So, this is basically minus 2, 1, 0, minus 3, 0, 1.

So, the matrix is first column is 3, 1. So, the second column will be minus 2, minus 3 and the third column will coming from T of alpha 3. So, this is basically if we do the calculation, 1 comma minus 2, 1 comma 0 minus of. So, that that means, 1 comma minus 2 is the coordinate for this. So, this is basically, so this is the matrix corresponding to that linear transformation.

So, if you know the matrix, then how will get the transformation; if you take any vector, if you take any vector alpha which is basically x 1 x 2 x 3, so, which is written as x 1 in to this under these basis, so x 2 into 0, 0, 0 plus x 3 into 0, 0 sorry 0, 1, 0, this is 0, 0, 1.

So, that means, this x 1, x 2, x 3 is the coordinate of this alpha with respect to this basis. Now, if you multiply this with the matrix, x 1, x 2, x 3. So, this will give us. So, this is 2 by 3 this is 3 by 1. So, this will give us 2 by 1. So, this is y 1, y 2. So, basically T of alpha is basically y 1 plus y 2 of 0, 1 because this we get the coordinate by multiplying this with this and this is our T of alpha. Any how you will talk about more details on this.

Thank you.