Introduction to Abstract and Linear Algebra
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Lecture – 29
More on Linear Mapping

So, we are talking about linear transformation. So, basically just to recap we have 2 vector space: U and V over the same field F.

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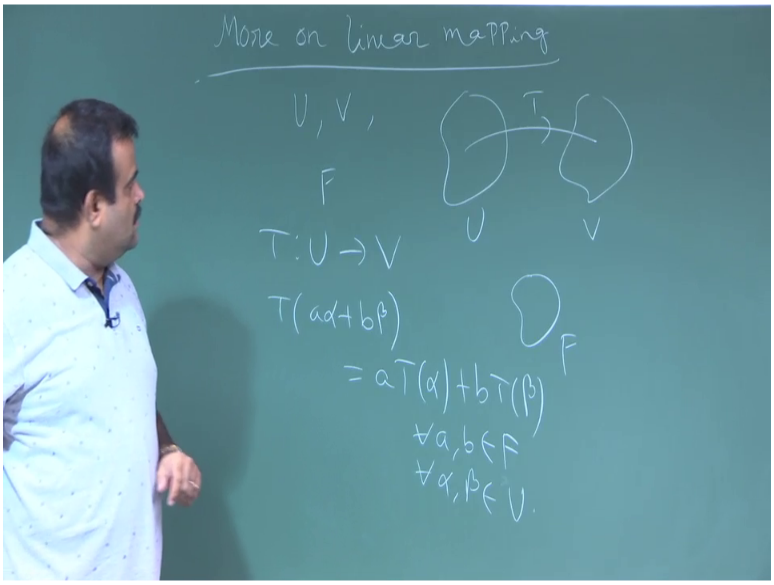

So, we have 2 vector spaces U and V over the same field F. So, a mapping from U to V is called linear transformation if the, if this property satisfy T of a alpha plus b beta can be written as a T alpha plus b T beta. And if this is true for all a b coming from the scalar and for all alpha beta or the vector form U. And this will be a vector in V and this will be a vector in V and that is why we need to have the same field, so we can operate this so this will be again a vector in V.

So, if this is true then we know this is the definition of the linear transformation. Now, we will talk about some operation on the linear composition up to a linear transformation ok. So, suppose we have 3 vector: space U V W over the same field, so you have a vector space U V and suppose you have a another vector space W and both are over the same field F.

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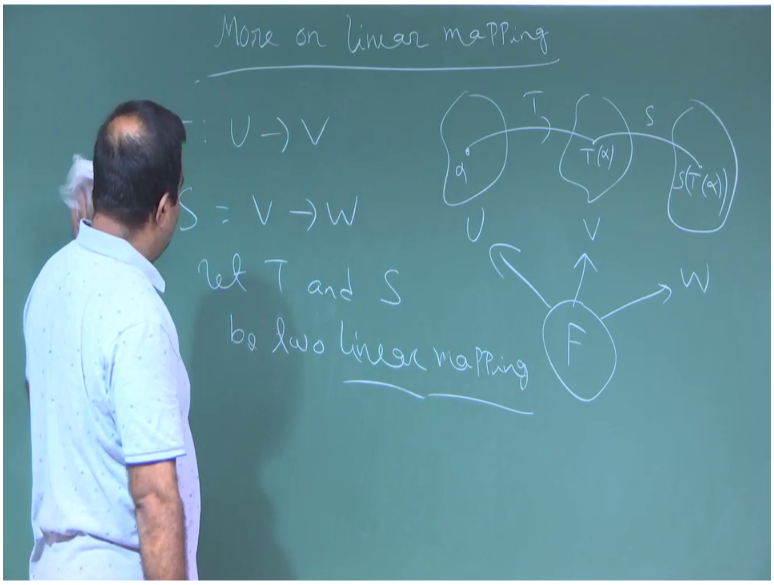

So, we have basically 3 vector space and we have a 2 linear transformation one is from say U to V and another one is say S from V to W. So, you have another mapping which is S from V to W. So, they let these to be let T and S be two linear mapping, linear transformation or mapping. So, now first of all this is a linear mapping, so it is a mapping T and U both are mapping.

So, T is a mapping from U to V and S is a mapping from V to W. Now, if you consider the composition of this is also a mapping. So, composition means we take a element alpha from here, so we apply T. So, this will give you give us T alpha and then on this is the element in V. So, if we apply S on this, so this is S of T alpha. So, now, if you consider this mapping say composition mapping which is denoted by S compost T, this is the mapping.

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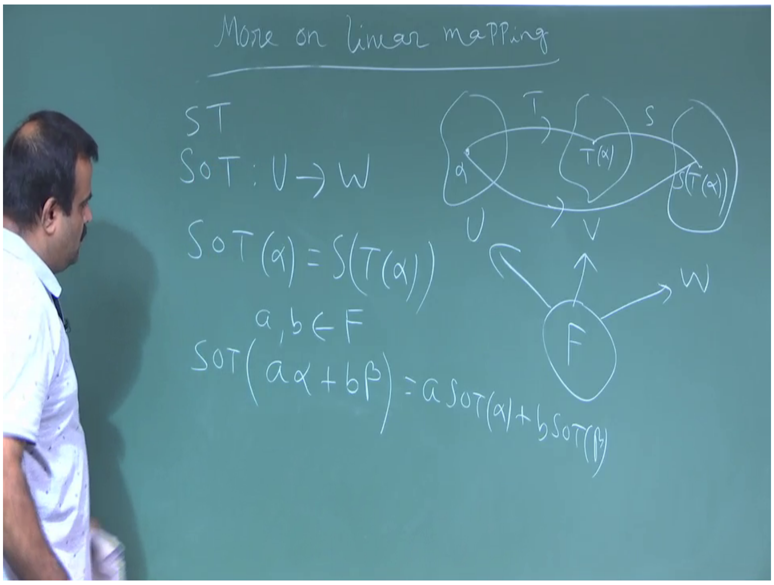

So, this mapping, so it is basically a mapping from U to W, so we take a alpha from U. So, S compost T alpha is basically we first applied T alpha then it will be an element in V, once it is a element in V we can apply the S on it. So, this is basically our composition, how we define the composition. So, this is the compos mapping, now we want to check whether this is also a linear mapping or not linear transformation, so that we have to check. So, for that what we need to check we need to check the, we need to take 2 ab from the scalar and we need to verify that.

So, this a alpha a alpha 1 plus b beta 1 or a alpha plus b beta a alpha plus b beta. If we can write this S compost T on this if we can be written as S compost T a, a S compost T alpha plus b S compost T beta. If we can show this then by definition this S compost T is a linear function, linear mapping sorry linear transformation yeah all are same basically. So, this in short we denote by S T basically, so this is basically S T in short we denote, so we want show this.

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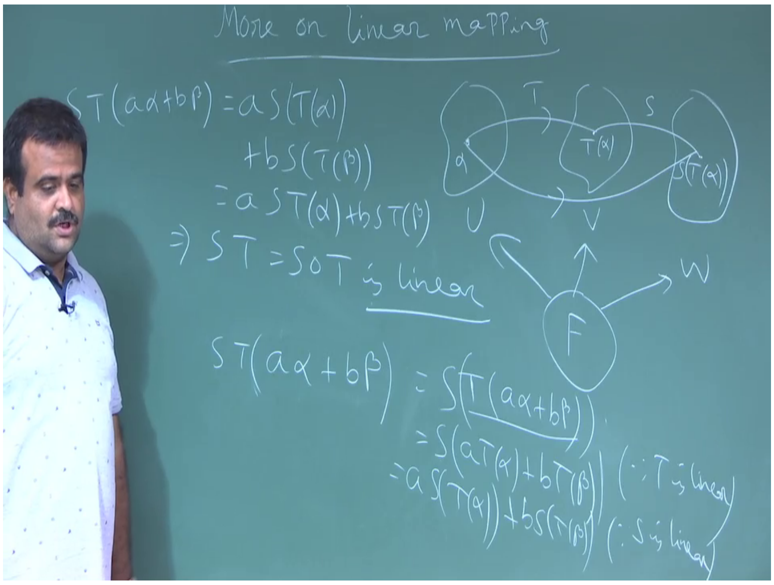

So, let us try that so S T of this is basically, so S T of this is basically by definition we first applied T on this a alpha plus b beta then we apply S on this. So, this is basically the composition of 2 mapping. So, now how to write this, so this can be written as so then T is a linear mapping. So, T can be written as S, so T can be written as a T alpha plus b T beta since T is linear, since T is linear transformation. So, by the property of linear transmission we can write that ok. So, basically we have this now S is also linear, so if S is linear we can write this as. So, this is some gamma 1 gamma 2, so basically a of S of T alpha plus b of S of T beta this is because S is linear, this is because S is linear.

So; that means, the S T of a alpha plus b beta is nothing but a S of a t a into T of alpha plus b S of T beta. So, this is nothing, but a S T alpha plus b S T beta and this is true for all alpha beta and all a b. So, these imply S T which is basically S compost T is a linear mapping is linear. So, this is the proof of linearity of this composition. So, the composition if S and T are both linear then the composition mapping is all composition function is also a linear function ok. Now, we will define the inverse of a transformation, so we just define the inverse linear transformation, inverse of a linear mapping ok.

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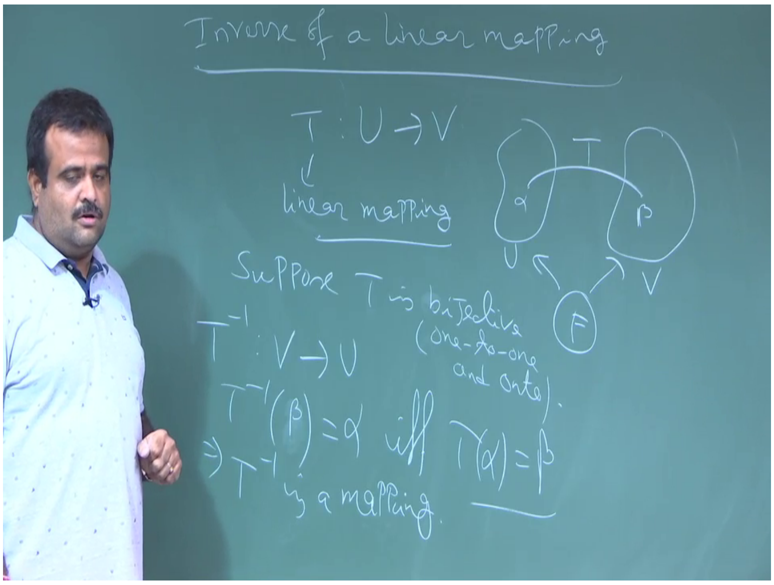

Now, we know the suppose you have a mapping from U to V, so suppose you have a to vector space U V over the same field F. So, this is over T now basic we suppose this is a this is a linear mapping suppose this is a linear transformation or linear mapping ok. Now this is a mapping now we know the mapping has a inverse if this is a this transformation is one to one and onto. So, suppose this T is bijective basically bijected mapping that means, it is one to one and onto if it is one two and onto then we can have a inverse.

So, how you define the inverse, so inverse is also a mapping. So, this T inverse which is a mapping from V to U, so how to define that so suppose this is beta so this is alpha so alpha is going to beta. So, T inverse of beta will be alpha if and only if T alpha equal to beta and since it is one to one T is one to one. So, there is only one alpha which is going to beta so the that means, this T inverse is well defined ok. So, that this means T inverse is a mapping T inverse is a mapping because this T is bijective T is one to one and as well as it is onto.

So, the range of T is covering the whole V so that means, if you take a any element all beta over here it has a pre image over here. So, this is a well defined mapping now we have to check whether this mapping is also a linear transformation or so or not yes this is a linear transfer, but; that means, is a proof. So, we have to check this inverse is also linear transformation is not or not. So, let us try that so for that.

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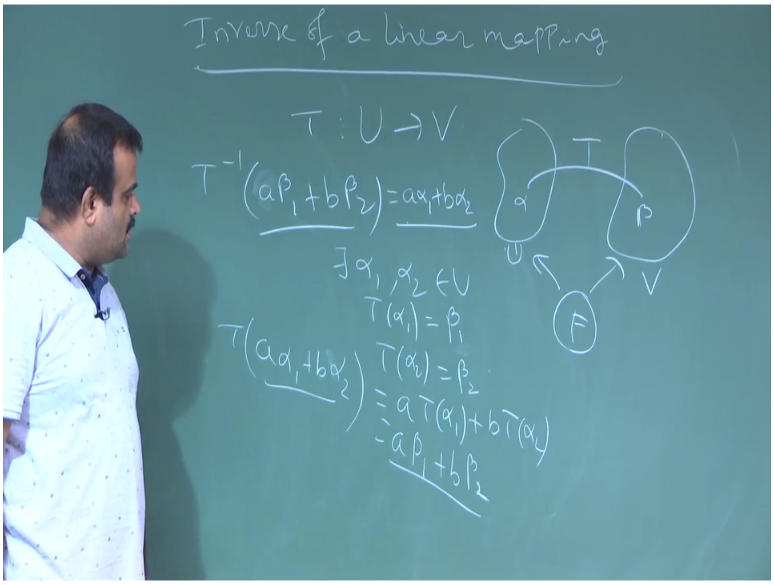

So, let us take T inverse of al a beta 1 plus b beta 2. So, we have say we take two point beta 1 and beta 2 in V and we want to write this as for linear transformation we need to write we need to. So, this is a T inverse of beta 1 plus b T inverse of beta 2 ok. So that means, this is basically how we can write that, so this beta 1 beta 2 belongs to V that means, there exists alpha 1 alpha 2 belongs to U such that T alpha 1 equal to beta 1 and T alpha 2 equal to beta 2 such that T alpha 1 equal to beta 1 T alpha 2 T is equal to beta 2.

So, now what we do we just take this now what is the a alpha 1 plus b alpha 2. So, if we apply T on me now T is a linear transformation. So, this will be written as a T alpha 1 plus b T alpha 2 now a T alpha 1 is basically beta 1 plus b beta 2. So, that that means, T of this is this so that means, T inverse of this is basically a alpha 1 plus b alpha 2 this is coming from this fact.

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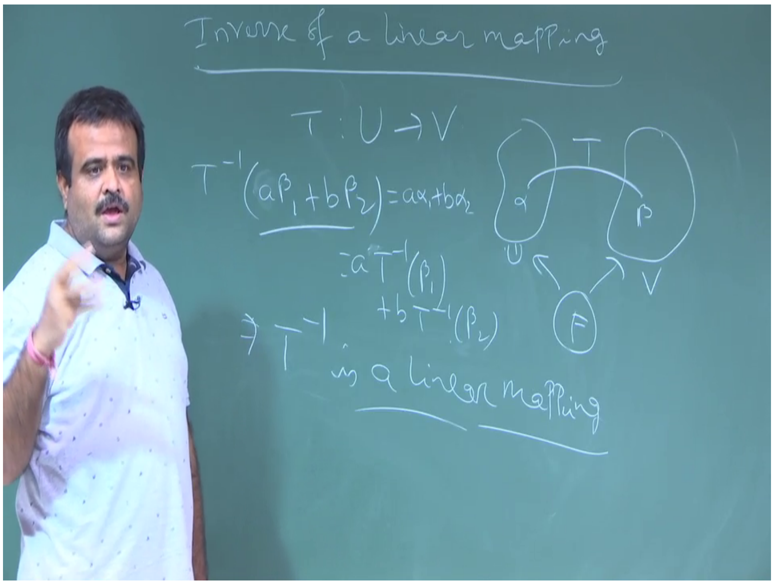

So, now this is nothing, but a T alpha 1 alpha 1 is equal to alpha 1 is equal to T inverse of beta 1. So, this we can write as a T inverse of beta 1 plus b T inverse of beta 2 because alpha 1 alpha T is coming alpha 1 alpha 2 is image is beta 1 and beta 2 that is it. So, this is the definition of this implies T inverse is a linear mapping say T is linear then T inverse is also linear transformation.

So, this is the inverse of a linear transformation is also linear, but for to exist the linear mapping we need to have this T should be a bijective mapping ok. So, now, we will define isomorphism or isomorphic between 2 vector space. So, first let us define the when you call a linear transformation is isomorphism.

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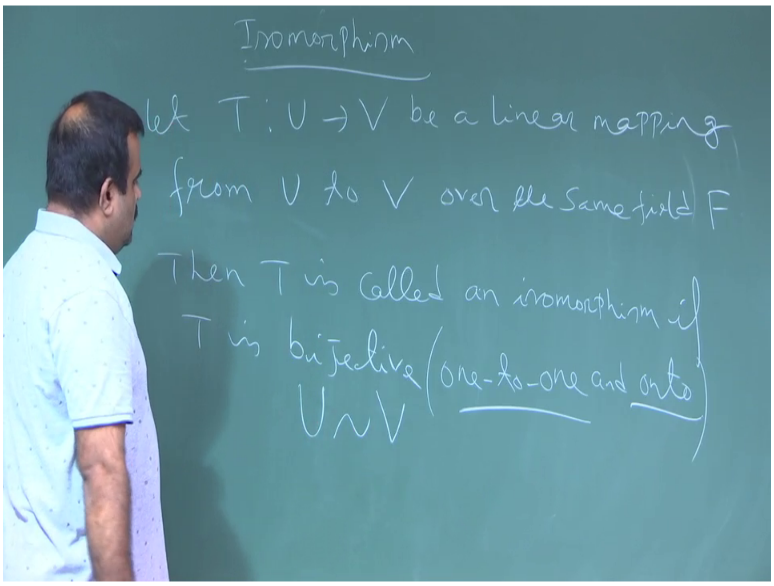

So, isomorphism, so it is basically a bijective mapping any bijective mapping is call isomorphism any. So, suppose you have a vector space U and V let this be a linear mapping, linear mapping between linear mapping from U to V from 2 vector space U to V. And these 2 vectors has to be over the same field over the that is most important same field F want the same scalar field F ok.

Now, this T is called then T is called a isomorphism and isomorphism presence if T is bijective if T is bijected. That means, it is one to one and onto one two one and onto mapping if both one two one and onto then we call a mapping is a bijective mapping. So, if T is bijective then we call T to be a isomorphism and then this vector 2 vector space are isomorphic if there exists a bijected linear transformation between U to V then we call a vector space to be isomorphic, and then it is denoted by U. So that means, if there is a linear transformation which is bijective then we call 2 vector spaces isomorphic.

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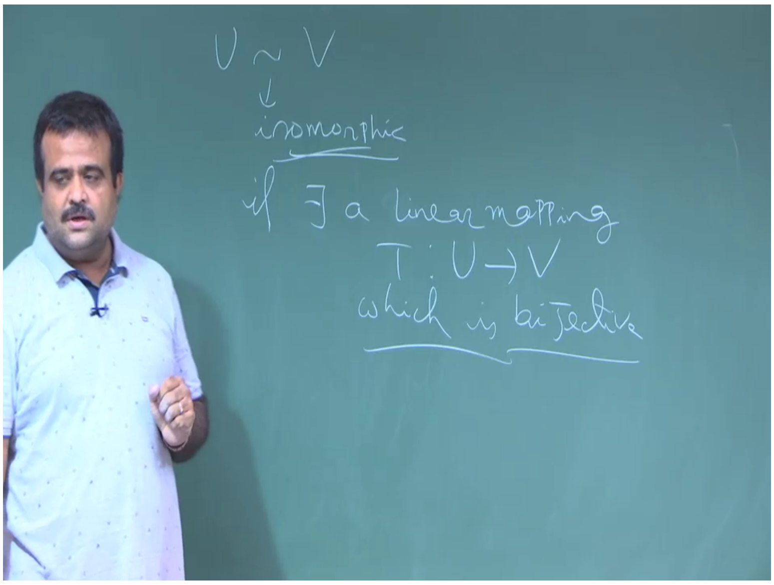

So, U and V this is isomorphic you can use this symbol isomorphic. So, even these isomorphic if there exists a linear transformation linear mapping T from U to V which is bijective so that means, if there is a isomorphism between U and V which is bijective then we call these to vector space are isomorphic U and V.

Now, we have some theorem, so if U V are isomorphic then they have a same dimensional vector space and conversely if they have a same dimensional vector space then it has to be isomorphic. So, we have we just write the theorem in the proper rate.

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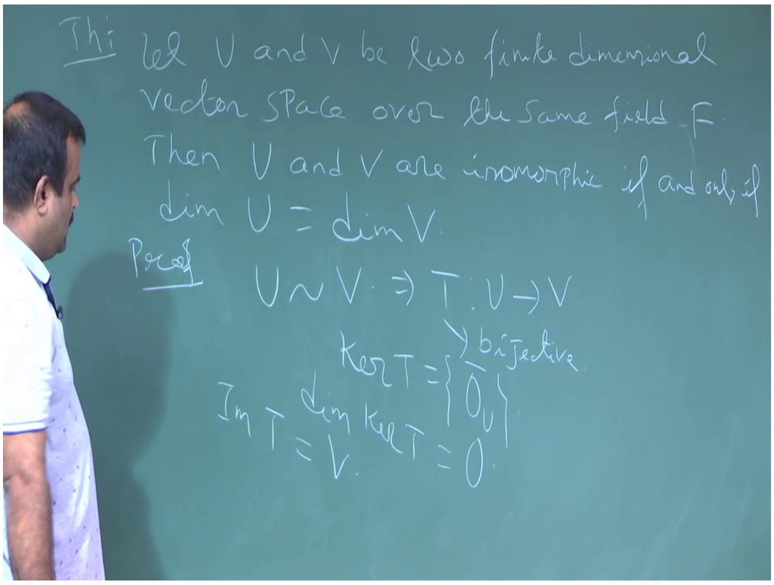

If this theorem is telling let U and V be to finite dimension vector space to finite dimensional vector space vector space, over the same field this is important otherwise we cannot define the mapping. Over the same field F then U fear isomorphic if the dimension of U is equal to dimension of V. Then U and V are isomorphic if and only if there dimension is same if and only if dimension of U is equal to dimension of V ok. So, this is the theorem, so you have to prove this theorem.

So, there are two parts one part is if there isomorphic then their dimension must be same first part and second part is if the dimension is same we have to prove they are isomorphic. So, let us try to prove the first part suppose they are isomorphic U and V. So that means, this imply there exist a bijective mapping U to V bijective linear transformation from U to V ok.

Since T is bijective then the kernel of T is basically only the 0 vector of U kernel of T means that with set of all vectors which are mapped to the 0 vector. Now, since T is bijective that means, T is 1 to 1 and we have seen in the previous class that for a 1 to 1 mapping the kernel is 0, so all these 0 is map to these ok. Now, kernel is 0 means dimension of the kernel is basically kernel is 0 vector dimension kernel is 0 and it is on two. So that means, it is T is on 2 also T is onto means the range of T is basically V because it is onto mapping ok.

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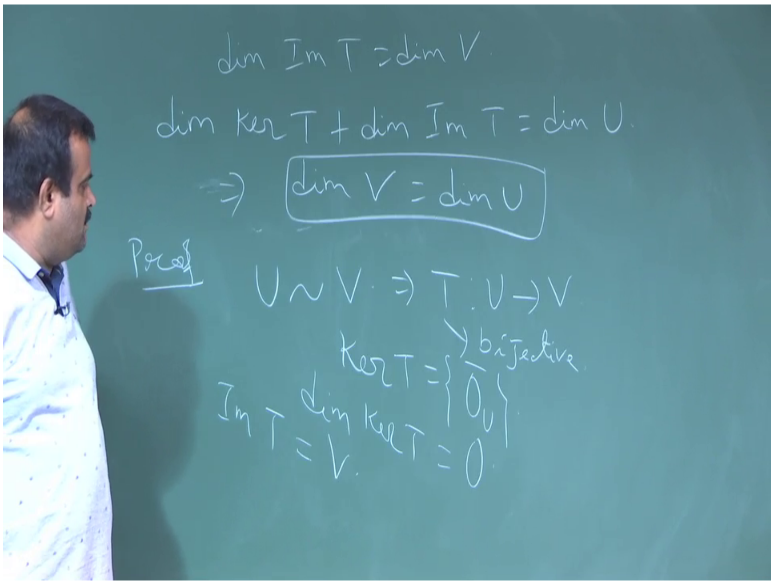

So, this means dimension of range of T is same as dimension of V. Now, we know in the last class that rank nullity theorem that is telling the dimension of the kernel of T large dimension of the range of T is equal to dimension of the domain dimension of V. So, now, this is 0 this implies now this is 0, so the dimension and this is the onto so dimension of V is equal to dimension of V.

So, this is the first part of the theorem, so if these two are if there is a bijective mapping then their dimension is same. And the second part of the theorem is if their dimension is same then we have to show that they are isomorphic that that means, we need to define a linear transformation which is bijective. So, let us try that this is the second part of the theorem.

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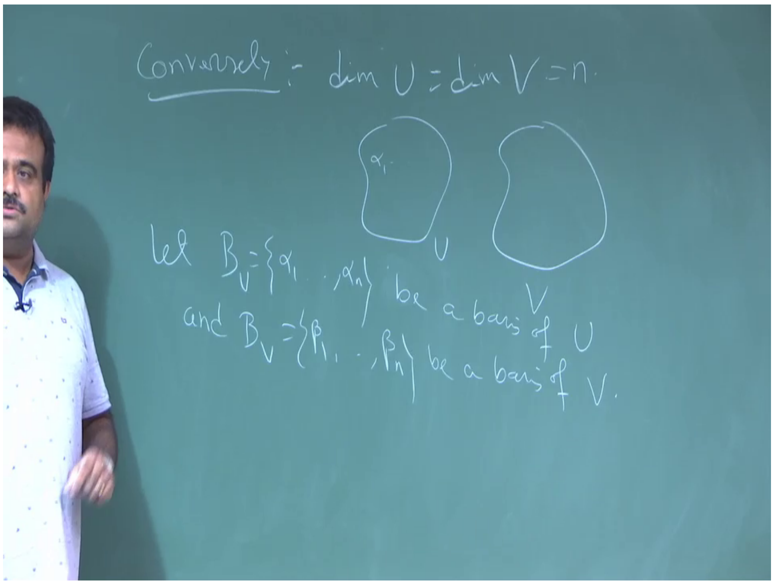

Conversely suppose dimension of U is equal to dimension of V, we have 2 vector space UV which are finite dimensional and their dimension is same. So, now let us take a basis on U, so say alpha 1 alpha 2 say their dimension is say n there are n number of vectors in the basis for both U and V.

So, say beta B U is equal to alpha 1 alpha 2 alpha n is the basis of U. So, this is U on that b U be basis of U and B V which a beta 1 beta 2 beta n be a basis of V their same dimensional. So, number of vectors in the basis will be same and that is a n, so this is a basis this is a basis. Now, we defined a linear transformation from U to V like this. So, T of alpha 1 is going to beta 1 like this.

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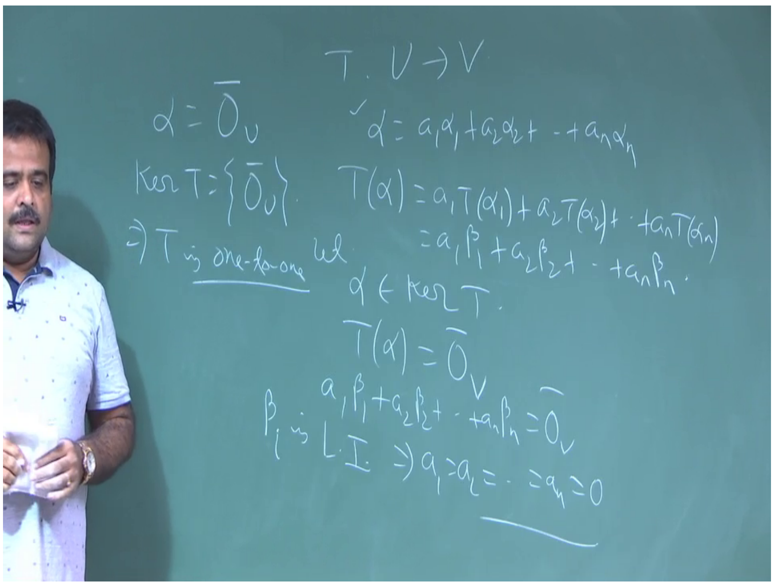

T of alpha 2 is going to be beta 2 and dot dot dot dot in general T of alpha is going to beta i. So, dot dot dot T of alpha n is going to beta n ok, so this is the way we define the linear transformation. Now, this is this is a so this is a linear transformation from U to V, now how we get this transformation suppose we take a alpha from here alpha is some a a 1. So, alpha is a member in U, so it can be written as the linear combination of the vector in the basis alpha n. So, how we define T alpha T alpha it is a linear transformation T alpha is defined as a to T alpha 2 a n T alpha n.

So, this is basically a 1 beta 1 plus a 2 beta 2 an beta n. So, this is the way we define the T from U to V we take any alpha from U and this is the way we define this ok. Now we will so this T is a bijective mapping, so for that first of all we need to show T to be a one to one. So, for one to one we need to show kernel of T is 0, 0 means the 0 vector of U so what is the. So, suppose alpha belongs to kernel of T let alpha belongs to kernel of T, so that means, T alpha is going to 0 vector of V now T alpha is nothing, but this. So, a 1 beta 1 plus so suppose yeah alpha is member of this suppose alpha is written as this.

So that means, this so that means, a a 1 beta 1 plus a 2 beta 2 dot dot dot an beta n this is 0 vector of V. So, now these are linearly independent beta as i are linearly independent LI, so this implies all the a i's are 0 ok, so all the a i's has 0. So, it seems all the a i's are 0 that means, alpha is basically summation of ai alpha i alpha is 0 vector of U. So, that that means, all these so the that means, kernel of T is on the consist of 0 vector of U ok, so that means, this T is this implies T is one to one. Now, we need to show that T is onto also for that we is you will use the rank plus nullity theorem.

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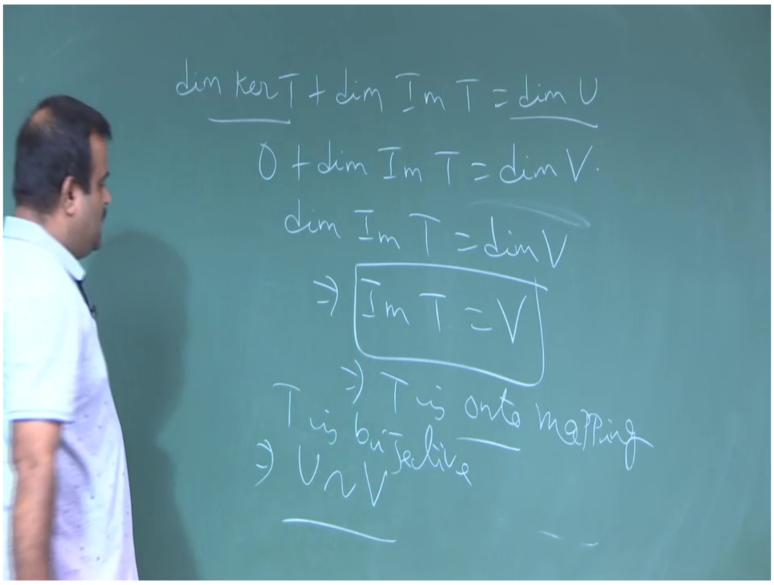

So, we know that kernel dimension of kernel of T plus dimension of range of T is equal to dimension of U this is the rank nullity theorem. Now this you have seen this is 0 because dimension is 0 because this T consists kernel of T consist all the 0 vector of U. So, 0 plus dimension of is equal to dimension of U which is same as dimension of V, so that is the assumption we made.

So, this makes dimension of range of T is basically dimension of V, so this implies range of T is basically the whole red V. So, this implies T is onto mapping, so this implies T is onto. So, you have seen T is one two and T is onto, so T is bijective and this implies that U and V are isomorphic if they have same dimension if they have same dimension they will also one. Now, I will quickly show another result on this isomorphism.

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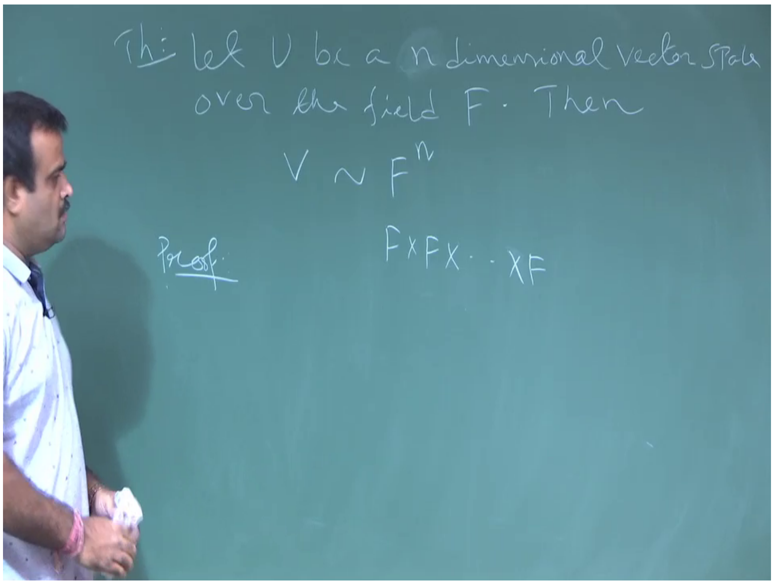

Suppose is other theorem let V be a finite dimensional be a n dimensional vector space mensional vector space over the field F, vector space over the field F. Then V is isomorphic with F n, F n is the basically Cartesian product F cross F cross is the Cartesian product of this. So, this is a this will, so this how to prove this ok? So, since time is over so we will prove it in the next class.

Thank you.