Introduction to Abstract and Linear Algebra
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Lecture – 28
Linear Transformation (Contd.)

So we are talking about Linear Transformation between two vector space.

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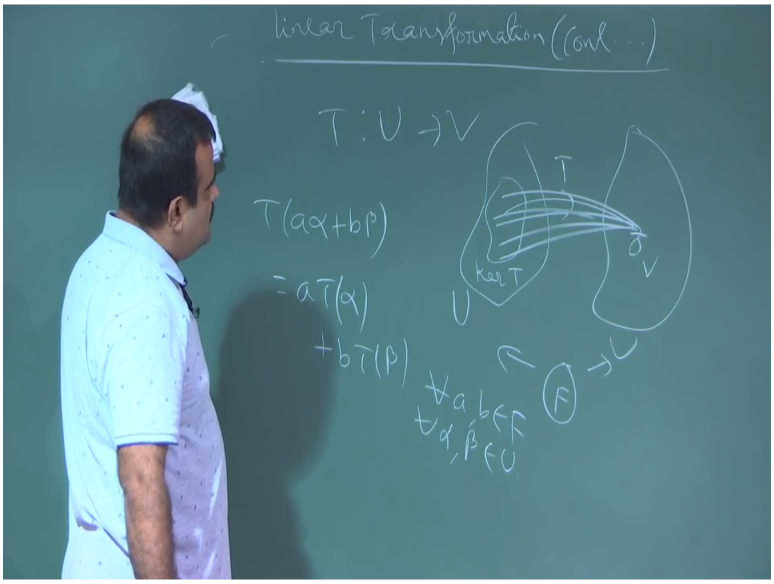

So, it is basically a mapping from the vector space U to V and these two vector space wherever the same field F. So, this is U this is V and it is basically a mapping from U to V which has some property like T of a alpha plus b beta must be equal to aT alpha plus bT beta and this is true for all a, b and for all alpha beta then you called this is a linear transformation.

And, then we defined the kernel of the linear transformation is basically set of all vector which are mapping to the 0 vector of V. So, that is the kernel of T.

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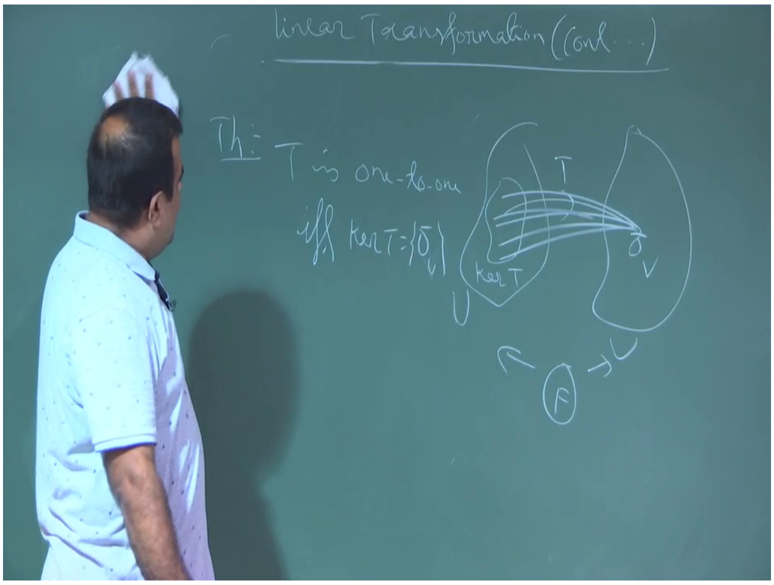

And, we have seen in the last class if T is one-to-one then the kernel of T is basically we have seen these theorems. So, T is one-to-one or injective if and only if both the way kernel of T is only the 0 vector of U.

So, this we have seen in the last class and also this is if, and only if the if the T is one-to-one then the kernel of T is 0 vector only and if the kernel of T is 0 vector then T has to be one-to-one this we proved in the last class.

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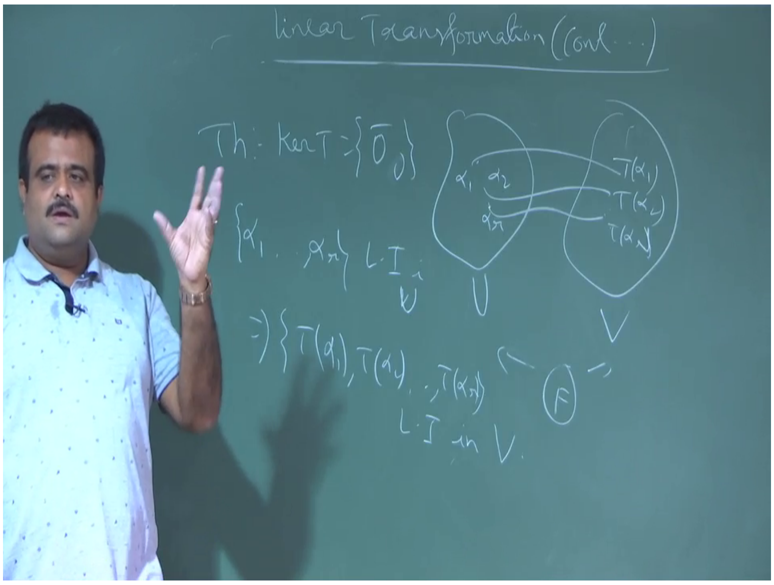

And, also in the last class we have seen another theorem like if. So, these two are the vector space over the same field. Now, suppose T is one-to-one; that means, suppose kernel of T is only the 0 vector of U; that means T is one-to-one. Then, if you take a linearly independent set of vector alpha 1 alpha 2 alpha 3.

So, these will map to the. So, T alpha 1, T alpha 2, dot dot dot dot T alpha r say. So, alpha 1, alpha 2, alpha r are L. I in U then this implies. So, T alpha 1 T alpha 2 alpha r are L. I in V.

So, it will it will just map to a linearly independent set of vector in V. So, if we take a linearly independent set of vector in U, provided this T is one-to-one; that means, provided kernel of T is 0. So, we will take some example of to get the kernel of vector space.

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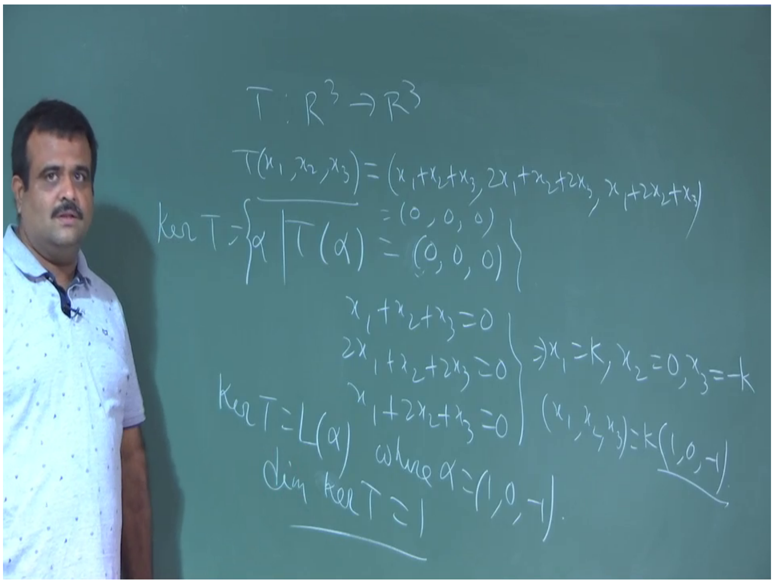

So, we take the example from the last class. So, we have the linear transformation from R 3 to R 3. So, this is basically T of x 1, x 2, x 3 this we discuss in the last class and we have you know this is a linear transformation x 2 plus 2 x 3 comma x 1 plus 2 x 2 plus x 3. So, we know this is a linear transformation. Now, we want to find the kernel of this.

So, kernel of this is basically the set of all alphas which are mapping to. So, kernel of T is basically set of all alpha which are mapping to 0 vector: so 0, 0, 0. So, that means, these are all mapping to these are all 0, 0, 0. So, from these we get three equations x 1 plus, x 2 plus, x 3 is 0 2x 1 plus x 2 plus 2x 3 is 0 and x 1 plus 2x 2 plus x 3 is 0 homogeneous equation and if you solve this we are getting x 1 equal to K some constant, x 2 is equal to 0, and x 3 is equal to minus k.

So that means, x 1, x 2, x 3 we can write as k of 1, 0, minus 1; so the that means, if you take this as a alpha so, it is a linear combination of I mean this kernel of T is basically linear span of alpha, where alpha is the vector 1, 0, minus 1. So, linear span of this with this is the linear span it is generate the kernel it is generate the subspace. And, we know kernel is a subspace ok.

So, this is one example of the kernel and the dimension of this kernel here is dimension of kernel of T is basically 1, because there is this is the only one vector which is generating this. So, it is dimension of this kernel subspace is 1.

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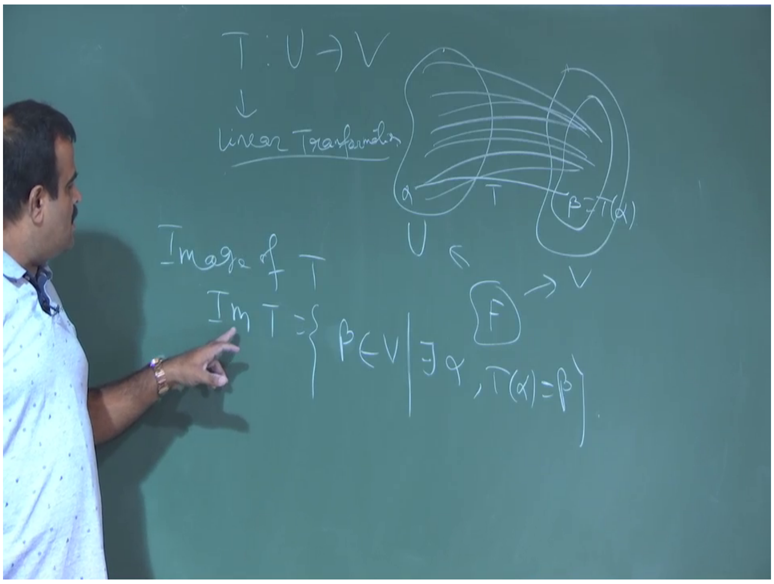

So, now we will define the image of a linear transformation, image or range also it is called. So, suppose you have a linear transformation from U to V. So, this is our vector space U, this is our another vector space V over the same field F and suppose, this is a linear transformation linear transformation so, we have those property of you know how to doing in the linear transformation.

Now we defined image of T image of T or this is also called range we know the range of the function, right. So, anyway we will write image of T. So, in short this is called ImT.

So, this is basically; we know this is a mapping from this set to this set. So, it may happen that from this set there are few vectors are covering not the all vectors, it is not need not be onto mapping. If it is onto mapping then all the element has a P image, but we are, but in general we just consider those elements from V so, those vector those beta from V such that there exists alpha for which T alpha is equal to beta.

So, we consider those element who has a T image over here. So, this is beta you have do alpha such that on the T this is basically T alpha. So, collection of those sets is called basically the range of the function or it is called also image of T. So, it is basically that collection of all the vectors of V which is having a P image; which is having a P image in use that means there exist a alpha such that it is that alpha is mapping to the vector beta. This collection is called range of T or image of T.

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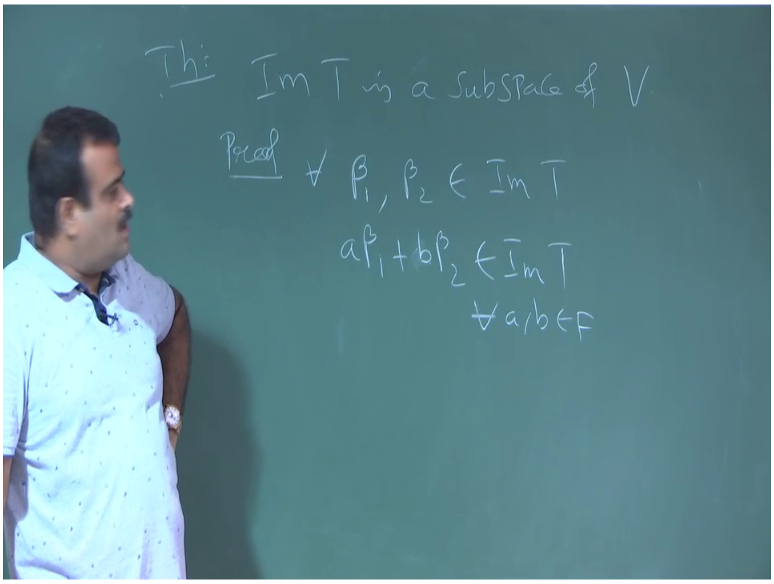

Now, also this is again in form a vector space subspace or under this V. So, how to show this? So, our theorem is. So, image of T is the subspace of V.

So, how to prove this? So, to prove this subspace we know we have to take beta 1 beta 2 from this set and we need to show that a beta 1 plus a beta 2 sorry b beta 2 is belongs to this and this is true for all these all a, b if you can show this then we can say the subset is a from a subspace of subspace. So, how to prove that? So, we take beta 1, beta 2 from the image of image set of or range set of T.

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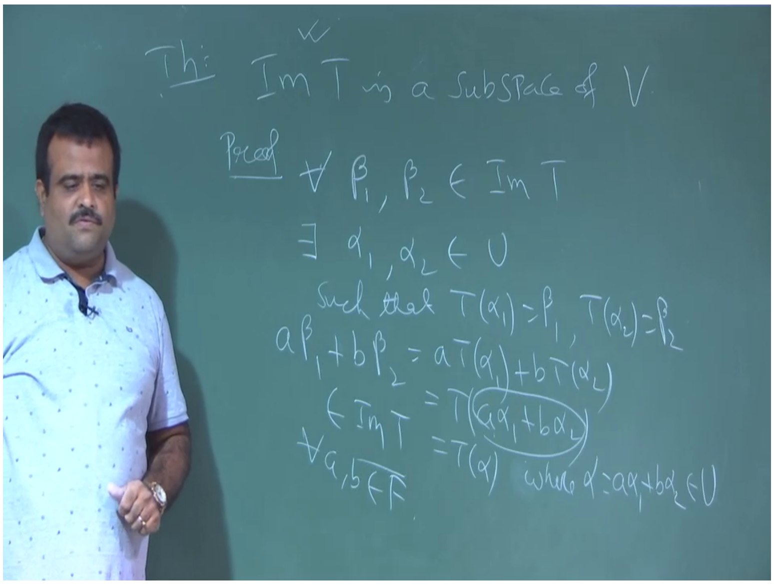

That means, there exist a since there in the image set there exists alpha 1 alpha 2 from U such that such that. So, there exists alpha 1 alpha 2 such that T alpha 1 equal to beta 1 and T alpha 2 is equal to beta 2 since there in the image of T, ok. So now, we consider a beta 1 plus b beta 2.

So, this is nothing, but what this is nothing, but a T alpha 1 plus b T alpha 2. Now, this is a T is a linear transformation. So, this is nothing, but a alpha 1 plus b alpha 2. So, now, the since so, this is a, this is some alpha this is T alpha. So, this, that means, where alpha is equal to a alpha 1 plus b alpha 2 and this is a member in this is a vector in U. So, it is mapping to some vector in V. So, that means, this belongs to image of image of T. So, this is true.

So, for any two alpha for any two beta 1 beta 2 we can say a alpha a beta 1 plus b beta 2 belongs to the same set for all a, b. So, this implies this is a vector space. This implies this is a, this form a subspace of V.

So, we will saw some properties on this subspace.

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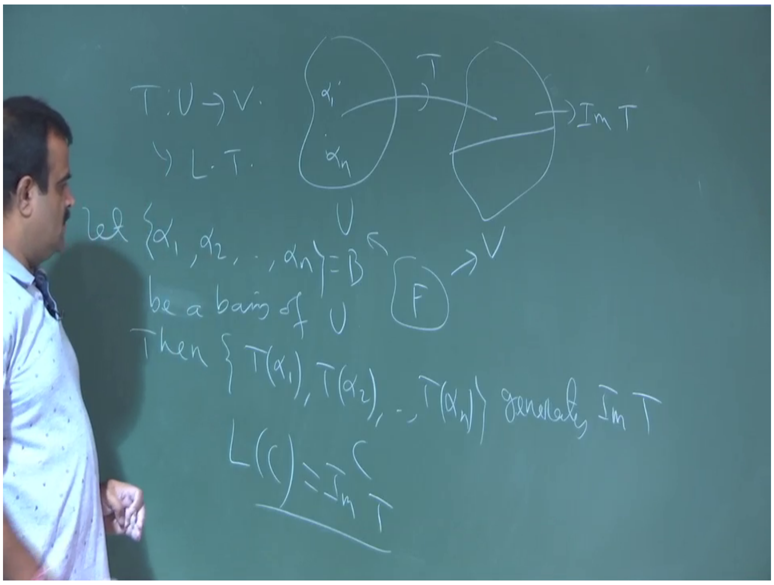

So, like first property is. Suppose, we have a vector space form U to V U to V over the same field. Now, we have at linear transformation say from U to V, we have two vector space U, V and we have a linear transformation of from into V.

Now, suppose this is we have a image of T. So, this part is basically there is no P image here. So, now say a this is a linear transformation. Now, if we get a basis if we have a basis in U so, let this theorem statement is that alpha 1 alpha 2 alpha n is a be a be a basis of U, then T alpha 1 T alpha 2 T alpha n generates that set, that image of T. Then this generates generate means generates means the linear span of this L of the set this is a this is a some set say this is b this is say some w or some C. So, L of C is basically image of this.

This need not be a linearly independent set, even they may not be unique, they may not be distinct also because, we have we are not saying that this kernel is it say one-to-one. If this T is one-to-one then we know that this will map to a linearly independent set of vector ok, but we are we have proving it we are not assuming this is one-to-one so far. We just take a basis of this for any T for any linear transformation we are claiming that this set will generate the image of this image, image of T.

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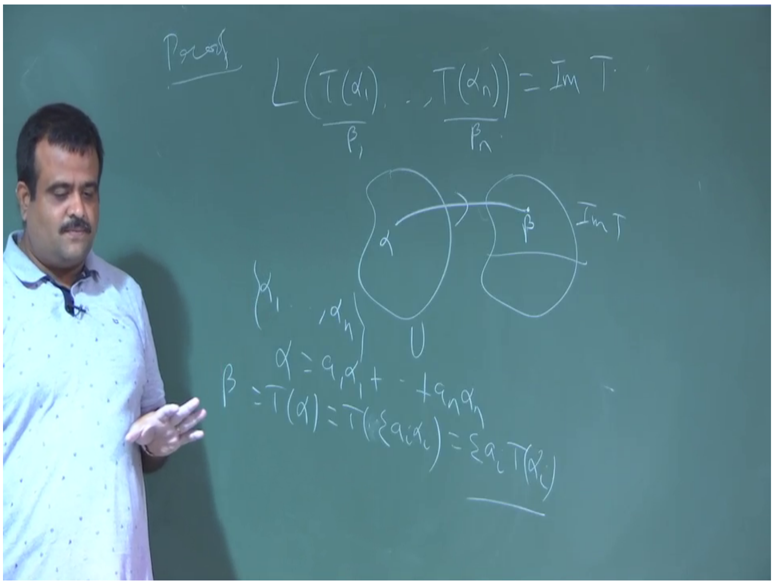

So, how to prove that? So, prove this trivial. So, basically we need to show linear a linear combination linear span of this vector is basically image of T.

So, now this is a subset of this because if we take this, this is a we take this is a beta 1 beta 2 beta n. Now, if we take any linear combination of this, this will be subset of this. Now, only thing we need to show that if we take a this is a image of T, if you take a beta here then we have to show that this should be written as beta should be written as some linear combination of this. So, that is important ok.

So, if you take a beta here we know there will be a alpha here which is mapping to beta because that is the way we defined the image of T; so for a given beta there is alpha. So, alpha is here, now we have seen that this is a basis of U this is the basis of U. So, alpha must generate by this. So, alpha must be written as some a 1 alpha 1 plus an alpha n.

So, now, we should write this as T of alpha must be written as T of this set summation of a i alpha i. Now, this is nothing, but beta and this is nothing, but summation of a i T of alpha i. So, this is the beta can be written as a linear combination of this. So, this is true. So, that means, if you take a basis from this then it will the image of this will also generate this range of T.

Now, these need not be a distinct set, it will be a distinct set and not only a distinct it will be a linearly independent set if this kernel of T is only 0 vector, if T is one-to-one then it will be a it will be it will form a basis of that image of range of T function in a range of T set.

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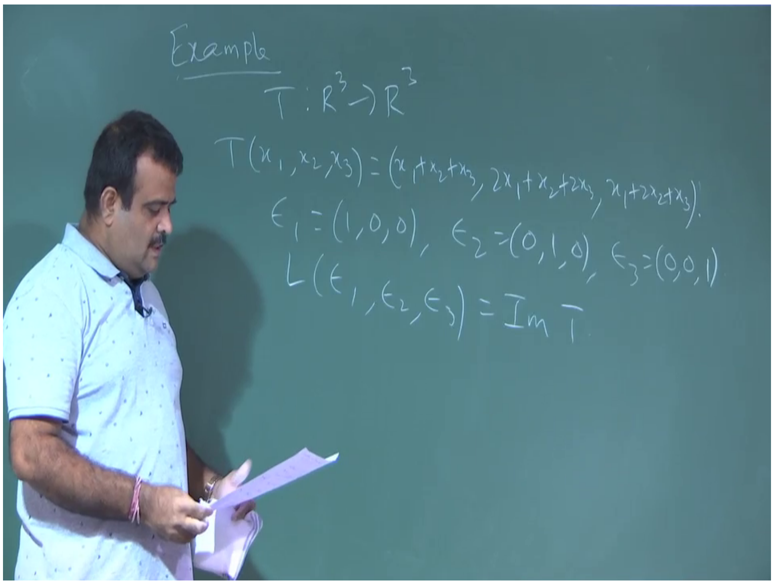

We will take an example. So, we will take an example. The same example we will be working on which we. So, we take a linear transformation from R 3 to R 3 and this was a linear transformation this is going to x 1 plus x 2 plus x 3 then 2x 1 plus x 2 plus 2x 3, then x 1 plus 2x 2 plus x 3 ok. So, this is we have seen this is a linear transformation. Now, we going to find the range of this.

So, further we take basis of this U this is our U. So, we take the standard basis and this is 0, 0, 1. Now, we know this linear span of this will form the range of T.

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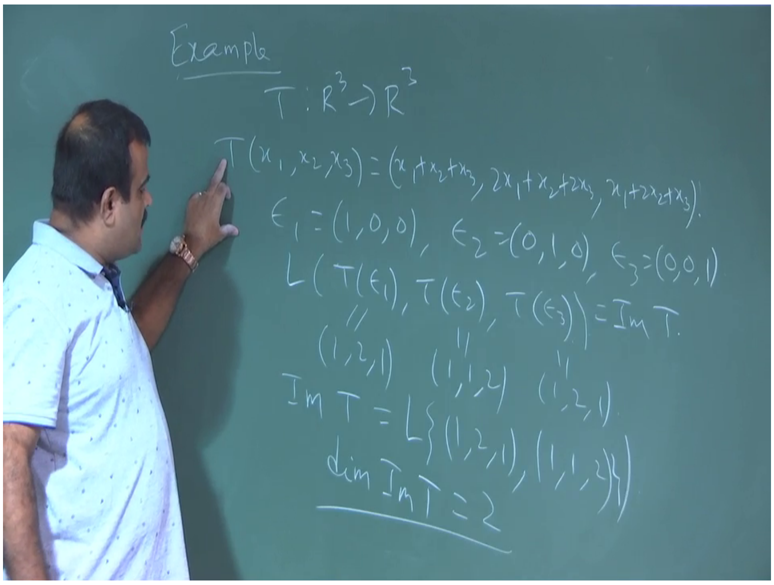

Now, so, linear span of T of that sorry T of T of epsilon 1 epsilon 3 this will form the range of T, just now we have seen in the last theorem ok. Now, we have to find this. So, the if we calculate this will be basically 1, 2, 1 and this will be basically 1, 1, 2 and this will be again 1, 2, 1. So that means, image of T or range of T is basically spanned by these two vector 1, 2, 1 and 1, 1, 2 and they are linearly they are distinct and also they are linearly independent vector setup vector.

And this is the dimension of this is basically this is a subspace dimension of this is 2. So, dimension of this is 2. Now, we have seen the dimension of the kernel is 1 and dimension of this range set is 2 this is also written this is also denoted by R T the range said of range of T R T and we know the null space of T, this is the R T a null space of this we have seen the dimension of the null space is 1; so 2 plus 1, 3 which is the dimension of the U.

Now, in general we want to prove that. In general whether this result is true or not so, we want to see that, so that we will see in a next result. So, that is called rank nullity theorem.

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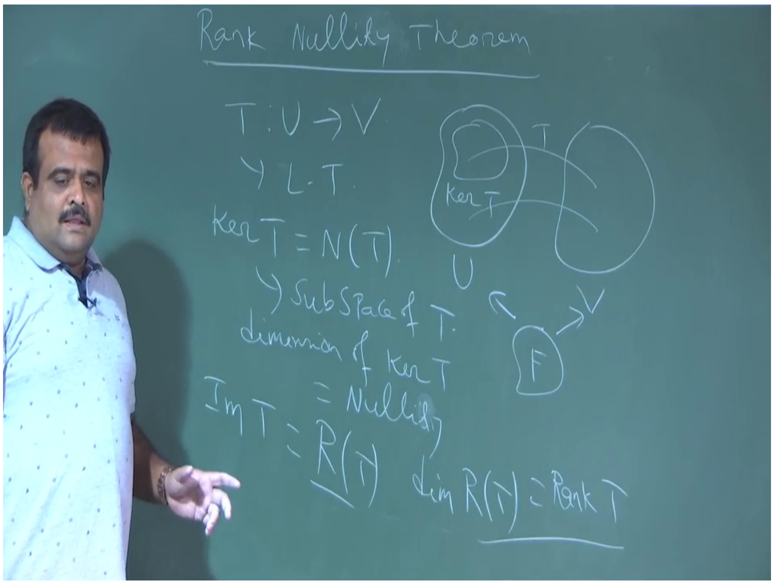

So, this is called rank nullity the rank nullity theorem. So, what this theorem is telling. So, suppose we have a linear transformation from U to V. So, you have two vector space U and V over the same field F and we know the kernel of T this is also so, we know this is a linear transformation this is a linear transformation. So, it is a mapping form here to here.

Now, we know the kernel of T this is also denoted by N T and this is a vector space we know this is a subspace and the dimension of this subspace is called nullity. So, the dimension of this kernel of T is basically called nullity.

And, we have seen that image of T this is basically also range of T, this is also a subspace and the dimension of this subspace is called rank of T dimension of this subspace is called rank of T.

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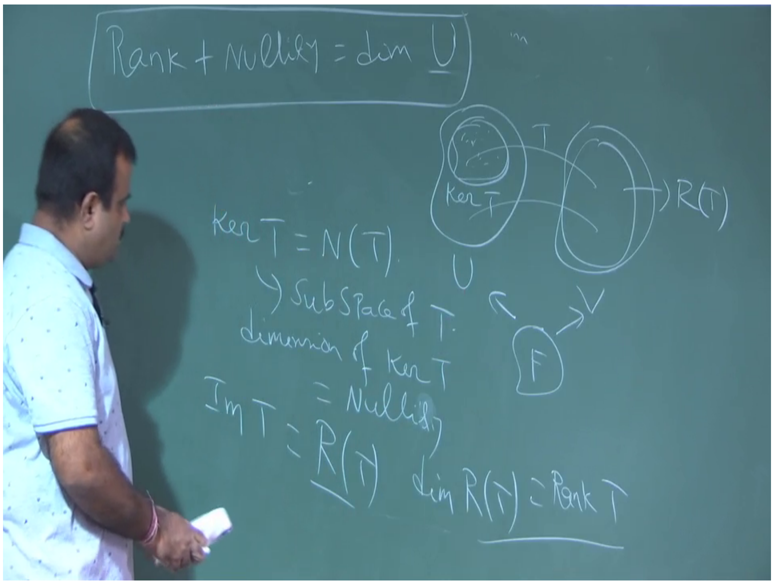

Now, this theorem is telling so, this theorem is telling a nullity rank plus nullity or nullity plus rank is equal to dimension of U this is called regularity theorem. So, rank plus nullity is equal to dimension of U we are going to prove this but we have to understand this.

So, this is a subspace we have a dimension for this or some basis in this there are some vector in the basis there are number of vector in the basis this is called dimension of that and we have here the range R T now this is also a subspace. So, this has a dimension and if we add these two dimension it will give as a dimension for the vector space V. So, how to prove that?

Let us quickly try to prove this. So, we will just give the rough idea of the proof.

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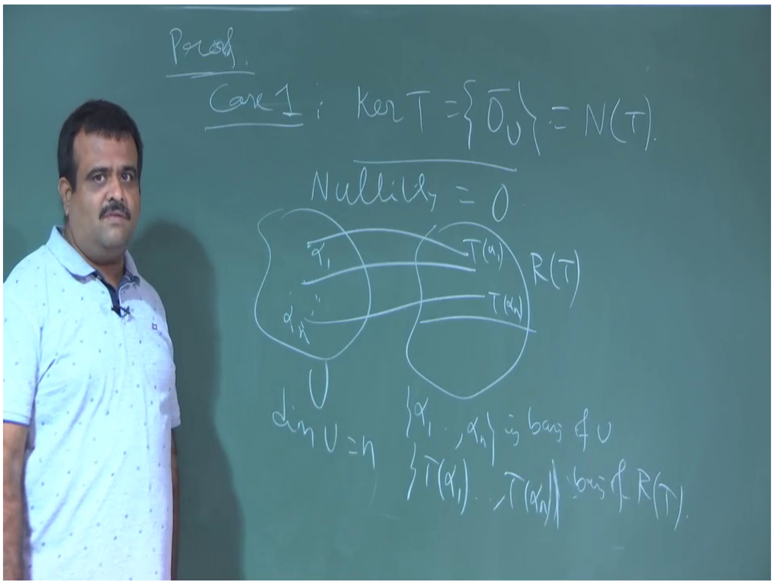

So, there are few cases like case 1: if T is one-to-one ok; that means, kernel of T is basically this one; that means T is one-to-one. So, if T is one-to-one then we have what we have? So, we have the dimension of. So, this is basically N T this is basically N T.

So, the dimension of so, nullity is here 0 nullity equal to 0 and then what is the rank? So, if it is one-to-one. So that means, any base for suppose you have a this is U. So, you have a basis for suppose dimension of U is say dimension of U is say in. So, that; that means, there are n vectors in the basis which are mapping to n victors over here.

So, T alpha was T alpha n and since they are forming basis. So, this will be also linearly independent set of vectors and they will generate the vector space R T. So, that means, this set if this is a basis alpha 1 alpha 2 alpha n is a basis of basis of U then this set will also be a basis of R T also be a basis of R T. So, the dimension of R T will be then n. So, rank is n and nullity is 0. So, rank plus nullity will be the dimension of V. So, this is case – 1.

Now, case – 2 is if the image is the whole V.

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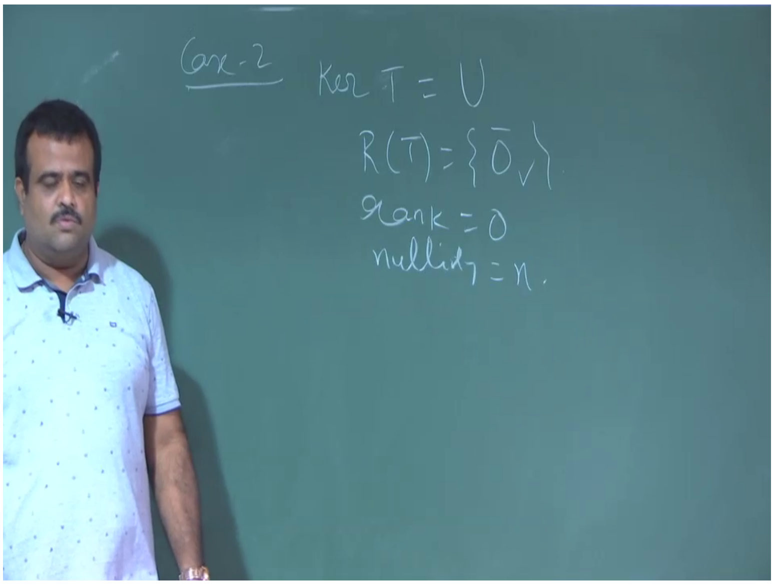

So, another case is case – 2 or this is the most general case like if this is a proper subset of this. So, so, this case – 3 is if this is the kernel of T is basically U; that means, everybody is mapping to 0 vector then this is also trivial. Then the rank is basically so, then R T is basically only 0 of V. So, the rank is 0 and nullity is nullity is basically dimension of U n. So, rank plus nullity is basically dimension of U.

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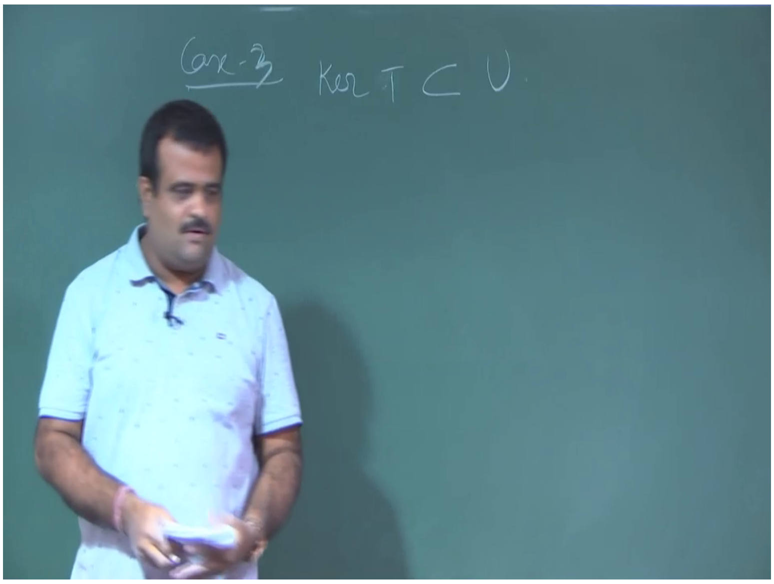

Now, case – 3 which is the proper subset of if the kernel is a proper subset of U. So, it is not I mean non trivial subset. So, this is a most general case to prove this.

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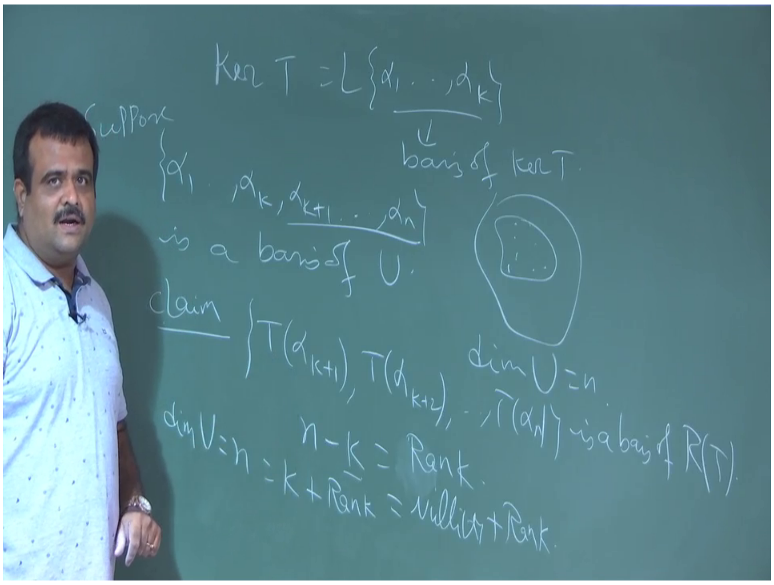

So, for that kernel of T we take some basis of this. So, we take some basis of this suppose alpha 1 alpha 2 alpha K. So, this is a basis of kernel of T. So, this is our U set and we have kernel of T and these are now this kernel of T is a subset of U. So, we can extend this to have a basis in U and U is a suppose dimension of U is a n. So, we can extend this by adding some vector to make a suppose this is a basis of U ok, this is the basis of U.

Now, we claim that this vector image of this vector will form a basis in the R T. So, we claim that this is our claim T of alpha K plus 1 T of alpha K plus 2 T of alpha n this will this is a basis of R T. So, if this if we can proves this if we can prove our claim if this is a basis of R T then what is the. So, how many elements are there here? There are n minus k element. So, that is the dimension of that is the dimension of R T which is basically rank.

And, what is the basis how many elements are there in the kernel. So, that is K. So, that is the K is the nullity so, that means, n is equal to K plus rank. So, K is the nullity. So, this is basically nullity plus rank. So, this is dimension of U.

So, dimension of U is basically nullity plus rank. Only thing, we need to prove our claim that this will form a basis of R T.

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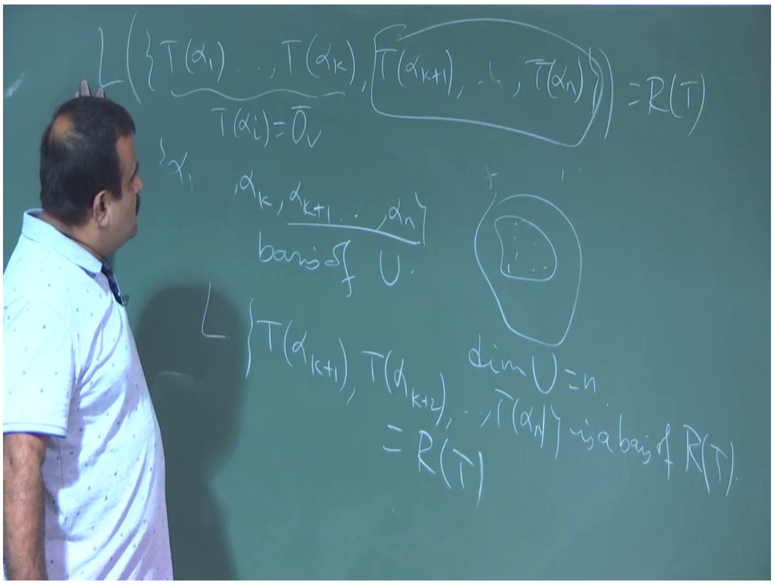

Now, we know how it will be form a basis of R T because we know this is the basis in U if this is a basis in U, then we know that since this is a basis we know T of alpha 1 T of alpha K and T of alpha K plus 1 to T of alpha n this will form a this will generate the R T. So, linear combination of linear span of this is basically R T.

Now, but all these up to here they are all belongs to kernel of T. So, these are all basically 0. So, they have no contribution in the linear span of this. So, basically this will contribute in the linear span of this. So, that means, this will be L of this is basically L of this is basically R T because these are all 0, 0 vector of V they are all going to 0 vector. So, they have no contribution in the linear span linear combination.

So, now only thing; so this is a linear so, this set generates the R T. Now, only thing we need to prove that this is a linearly independent set of vector. So, how to show this?

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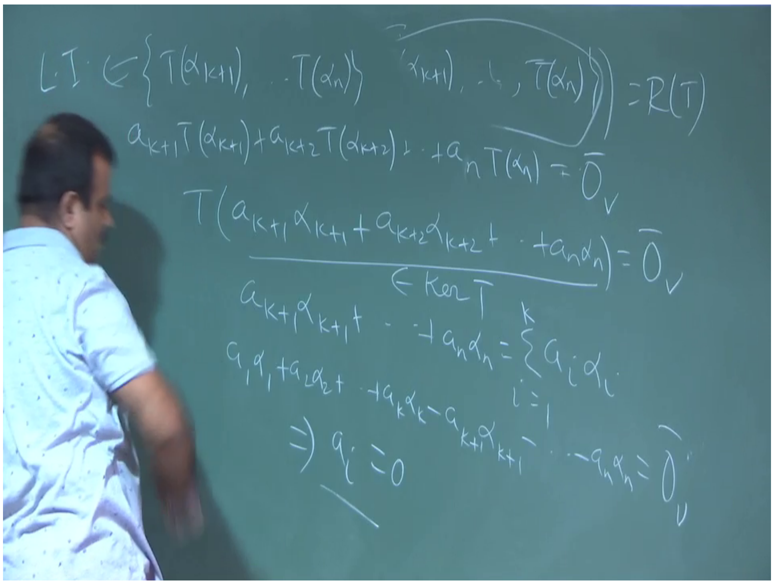

So, to show this, suppose we take some say a K plus 1 T of alpha K plus 1 plus a K plus 2 T of alpha K plus 2 dot dot dot a K a n T of alpha n. Suppose, this is a we take this to be 0, 0 vector in U.

So, now, this is basically if we this is the property of linear transformation an alpha n, this is basically 0. So, now this must belongs to kernel of T because kernel of T means all the vectors which are mapping to 0 vector. So, this must belongs to kernel of T now we know the basis of kernel of T this those are basically alpha 1 alpha 2 alpha r.

So, this must be written as the linear combination of linear combination of alpha i is 1 to K because alpha 1 alpha 2 alpha K up to alpha K forms the basis in the basis of the kernel of T. So, from here we can take this side or these that side. So, we can write that a 1 alpha 1 plus a 2 alpha 2 plus a K alpha K minus a K plus 1 alpha K plus 1 minus an alpha n is equal to 0 vector of U.

Now, this alpha 1, alpha 2 alpha these are basically the basis of U so, that means, they are linearly independent set. So, that means, if their linear combination is 0, then all the scalar has to be 0. So, this implies all the a is are 0. So, this implies this set is a this set alpha K plus 1 dot dot dot T alpha n this is a linearly independent set of vectors. So, that means, this will form a basis of R T the range of T. So, that will that is the proof our claim. So, this is the proof of the rank plus nullity theorem.

Thank you.