Introduction to Abstract and Linear Algebra
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Lecture - 26
Complement of Subspace

So we talked about basically; what do you mean by complement of a subspace.

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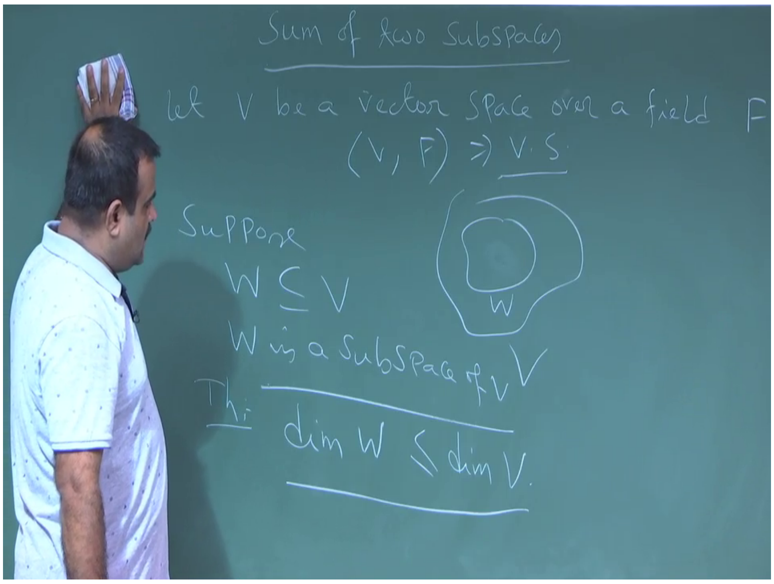

So, for that let us talk about the sum of 2 subspaces. So, before that let V be a vector space, vector space over a field F. So, this is a vector space over a field F, F is a field and V is the vector, set of all vectors. And now we have a say we consider a subset of V say U or say W.

Let W be a suppose W a; W is also form a vector space over the same operator and over the field F, then W is a suppose W is a subspace of W is a subspace of V. Then this theorem is telling the dimension of W must be less than equal to dimension of V.

So, dimension of any subspace must be less than equal to dimension of the less than equal to dimension of the vector space. So, how do justify that this is quite trivial to visualize. Now since W is a subspace. So, there are 2 cases, W may be a null space. So, null space means W may consider only the 0 vector.

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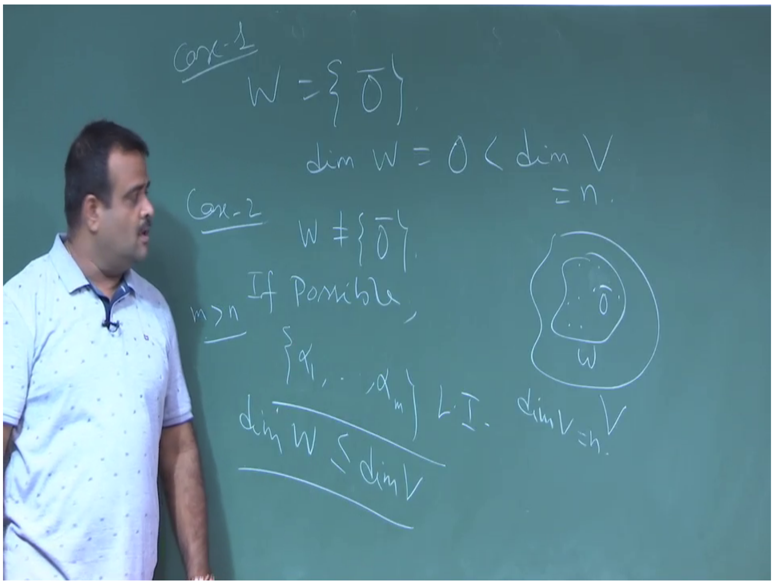

So, this is your case 1 so, W maybe the null space, W is consider only 0 vector.

So, then we know the dimension of this null space is 0 and which is less than dimension of the vector space. And suppose this dimension is say n suppose P is a n dimensional vector space.

Now, the case 2 when this is not a null space it is, it has other member other than the null space. So, this is our V and this is our W. So, other than this 0 vector we have other element also there in W and then we want to show that W is a dimension of W is a; this is case 2.

Case 2 is when W is not a null space basically; so W contained nonzero element; nonzero vector also. So, in that case how to convince, how to show that dimension of W is less than dimension of V; if it is not, if dimension of W is more than n. So, then if possible if possible dimension of W is more than n, n is the dimension of V. Then; that means, there is a basis of size say m which is m is greater than n.

Now, this is a basis means, this is linearly independent set, now this is also element, this is also a vector from W a vector from V which is not possible because dimension of V is n. So, the maximum number of linear independent set of V must be less than equal to n.

So this is not possible; so that means, dimension of W has to be less than equal to dimension of V. So, dimension of any subspace must be less than equal to dimension of the vector space.

Now, to we talk about sum of 2 vector spaces and we talk about their dimensions.

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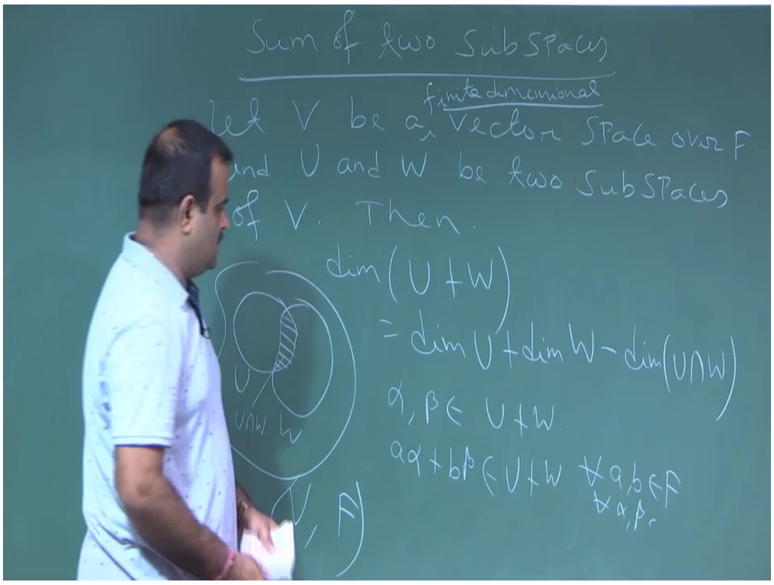

So, this is basically some of 2 subspaces, again the same story. Let V be a vector space over the field F and U and W be 2 subspaces of V over the same field, subspace has to be for over the same field. So, V and W are 2 subspaces of sorry U and W will be is 2 subspaces of V. Then the theorem is telling then the dimension of U plus W is basically equal to dimension of U plus dimension of W minus dimension of U intersection W. So, this is the theorem ok. So, before proving that theorem we need to understand this. So, let us draw the picture. So, we have a vector space V over a field F and we have 2 subspace U and W.

Now, we have seen the sum is also a also a vector space, it is also a subspace and the intersection this is the intersection is also a. If U is a subspace W is a subspace then the intersection is also a subspace this also we know this result or even we can prove that by the necessary and subsequent condition and U plus W is a subspace ok.

So, once this is a subspace and this V is a finite dimensional vector space V is a finite dimensional this is required V is a finite dimensional vector space ok. Then this theorem is telling if we take this subspace U plus W, why U plus W is a subspace?

We can show that we take 2 element from here alpha beta U plus W. Then we have to show that a alpha plus b beta is also belongs to V plus W, and this must be true for all ab from F and for all alpha beta all alpha beta coming from this U plus W this is for all alpha beta.

If we can show this then we are done then this is a subspace this is easy to show because we can take U is alpha is from U plus V. So, alpha can be written as sum x 1 plus x 2, where x 1 is coming from here x 2 is coming from here and beta is also can be. So, this way we can take the sum of this. So, this is also can break it into this part and we can prove that like this.

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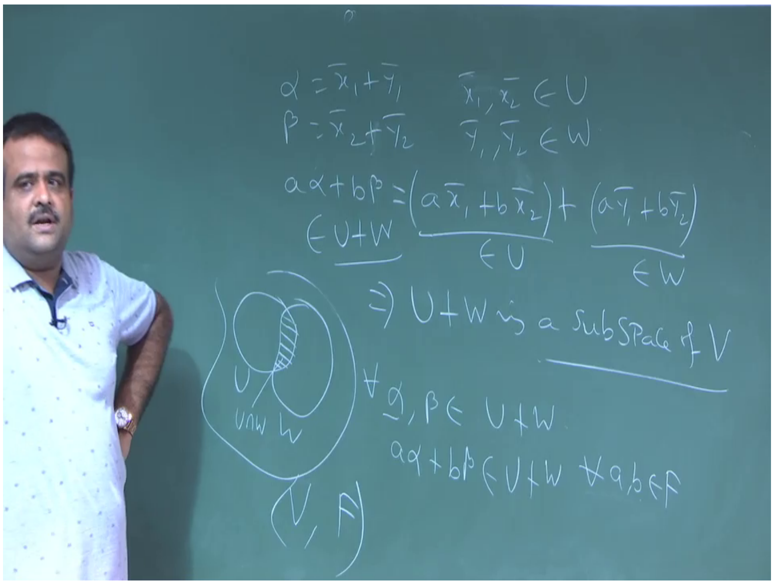

So, alpha beta both belongs to this alpha is equal to sum x 1 plus y 1, these are vector where x1 and beta is basically x 2 y 2 and x 1 x 2 coming from U and y 1, y 2 coming from W. So, then you have to show this because there now we take a alpha plus b beta. This is nothing, but we can write as a of x 1 plus b of x 2. We apply the associate property of this and then the commutative property of this because this is a vector space. So, the plus this vector plus is the associate and the Abilene also.

So, this plus a y 1 plus b y 2, now this is again a member in V U because this is U is a subspace so, closer property. So, this is a member of U and this is also a member of W. So, hence this is a member of U plus W and this is true for all ab. So, this is this implies U plus W is a subspace of V.

Since it is a subspace then and V is a finite dimensional vector space. So, we can talk about the dimension of the subspace. And the dimension of this theorem is telling dimension of the subspace is nothing but dimension of U plus dimension of W minus dimension of this intersection space, this is also subspace. Because, if U is a subspace W is a subspace we know the intersection is the subspace. So, this we have to proof now.

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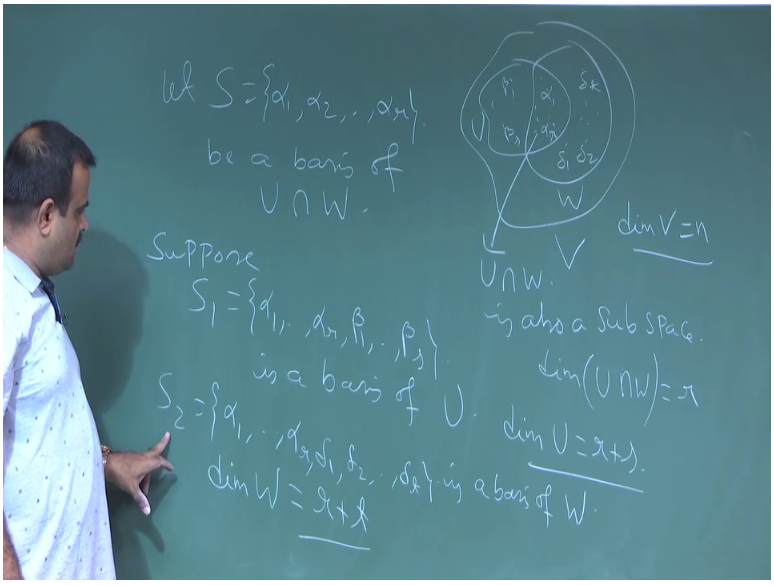

So, how to prove that? Ok, so let us draw this picture. So, this is V, this is U this is W and this is basically U intersection W, this is also a subspace, subspace ok.

Now, since this is a subspace and V is a finite dimensional vector space. So, read dimension is also finite less than n. So, suppose the dimension of this is r and let S be the basis of this. So, we take some vector r vector suppose dimension of this is r dimension of U intersection W is say r. So that means, there are r element in the basis and suppose this form a let this be a basis of U intersection W.

So, since this is a basis so, this is this elements are here this alphas now this alphas are here. So, we take we can so, this set is linearly independent set. Now to get a basis of U we have to add some more element from here. We can use the extension theorem algorithm to get that basis of this. So, suppose we add few more element over here and we get a basis of U suppose S1 we denote. So, alpha r will be there alpha 1, alpha 2 alpha r and we add say beta 1, beta 2 beta S.

Suppose this is a basis is of U. So, we have some alphas over here alpha r and we take some betas from here, beta S. So that means, dimension of U is we are taking as to be r plus S which is also less than n because n is the dimension of V, we can take dimension of V is say n. So, the dimension of dimension of U is basically r plus S. So now, a this is suppose this is a this is the basis of U ok.

Now, similar way we can take some element from here from delta S, delta1, delta 2 up to say delta t and you add this with this to get a basis of W alright and again say that said we denote by S 2 alpha. Because, alphas are common in both U and W and this is the linearly independent set. So, we have to add some more vector to have a basis of that set. So, this is basically delta1, delta 2, delta t this is a suppose this is a basis of W ok.

So, the dimension of W is we are assuming to be r plus t dimension of W you are assuming to be r plus t. So, this is our these are 2 basis of one we are taking one basis of U another basis of V W. Now, since it has already these alphas are linearly independent set we can add some more linearly independent vector to have a basis of U similar way for W. So, these are the addition vector for U and W respectively ok.

Now, we want to now show that if you take the vector like this setup vector we take the alphas.

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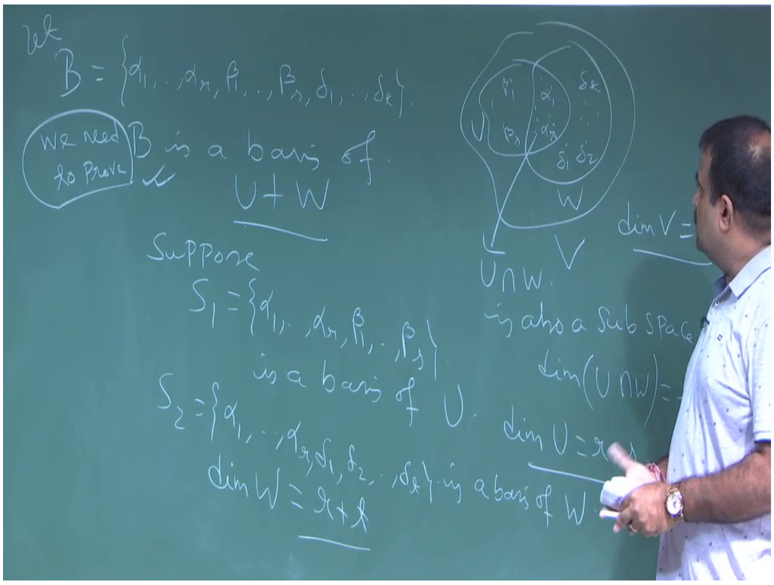

Then we take the betas, then we take the deltas and we claim that this is a basis of U plus W. So, this vector we denote by B say ok. Let B be the construct this B from this.

Now, we want to prove that B is a basis of U plus W that is it, if we can show that then dimension also we can get from here. So, dimension of U plus W will be r plus S plus t which is nothing but dimension of U plus dimension of V minus dimension of this because, 2 r is coming together. So, we have to subtract one r. So, now this we have to proof, we need to prove that we need to show that we need to prove this. If we can prove this then we are true, if you can prove that this is a B is a basis of this U plus W then we have true so, let us try that.

So, how to show B is a basis of U plus W? Now first of all; so this you have to, so let me write it again this B ok.

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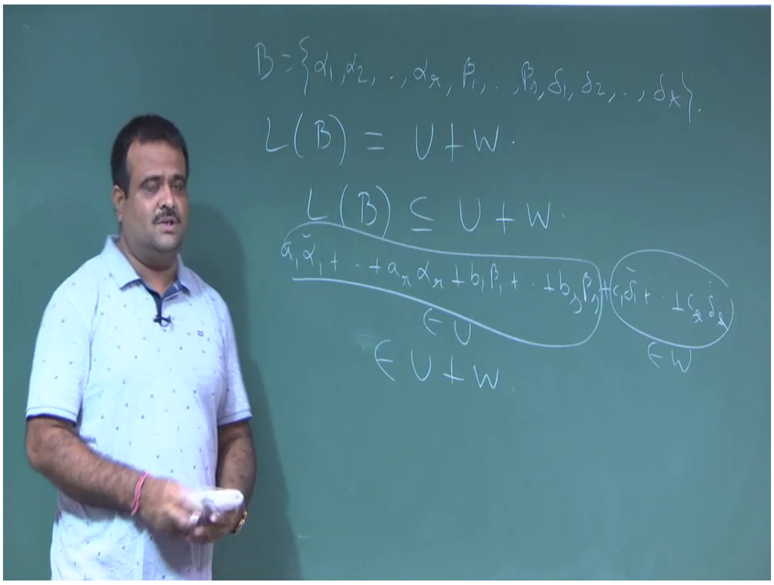

So, B is basically alpha 1, alpha 2, alpha r determine beta 2 beta S delta 1, delta 2, delta t.

Now, we want to show that b is a basis of this of this U plus W. So, for that you have 2 step first step is linear span of B has to be shown to be U plus B U plus W. And then you have to show B is a li, linearly independent set of vector. If these 2 can be solved then we can claim that V is a basis of U plus W. Let us first try to show the first one.

Now, first one part is trivial now. So, L of B is a subspace of U plus W. This is trivial because if you take any linear combination of this. This is a linear span of the vector this. So, if you take any linear combination of this like say a1 alpha 1 dot dot dot r alpha r plus b 1 beta 1 dot dot dot b s beta s plus c 1 delta 1 ct delta t if you take any linear combination. So, this can be written as so, we can take a.

So, these are the element from U. So, we can take this up to this part is coming from U and we again deltas are element in W, this part is coming from W. So, this is belong to U plus W so, quite trivial.

So, any element from the linear span of this is a element of this now the reverse is we have to check. That means, if you take an element from this whether that can be written as a linear combination of this so, that you have to check. Then we can say this is linear span of this is this.

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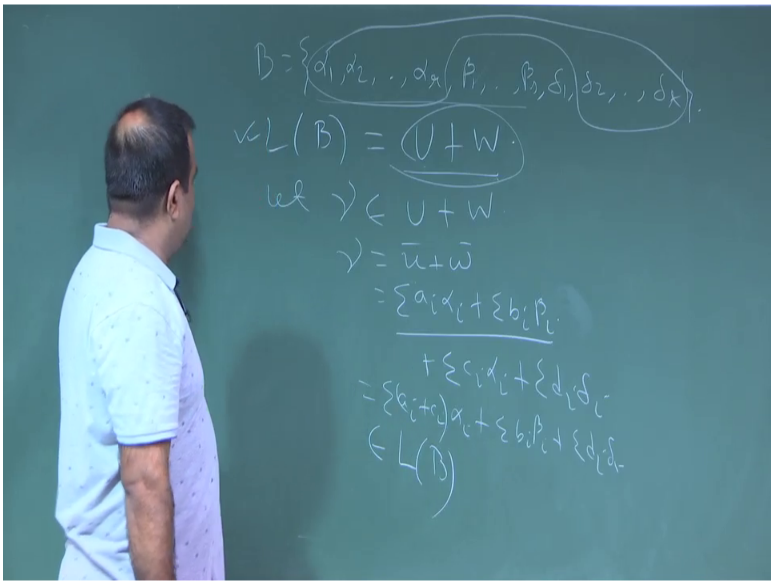

So, for that let you take a element from this. So, let us take a say an element from U plus W. So now, since this is an element from U plus W it has 2 part U plus W has a2 vector U vector W vector.

Now again since U is a vector so, it is coming from U. So, it must be written as a linear combination of the basis of this. So, basis of U is basically up to this. So, it is basically summation of a i alpha I plus summation of bi beta I, this is for U. And then W again will be basis of W is this along with this is a basis of W, so that means, this should be also written as summation of some c i alpha I plus summation of d i delta I.

So, this combined we can have summation of summation of a i c i alpha I plus summation of b i beta i plus summation of bi delta I. So, this is basically a linear span of B. So, any vector from here so, this is a subset of this and earlier you have seen this is subset of this. So, this is done. So, B generates this so, one part is done, now we have to show whether this set is linearly independent or not. So, let us try that. So, for that this is our next goal, we want to show that this is a set of linearly independent vector. So, for that let us take a linear combination to be 0 vector so.

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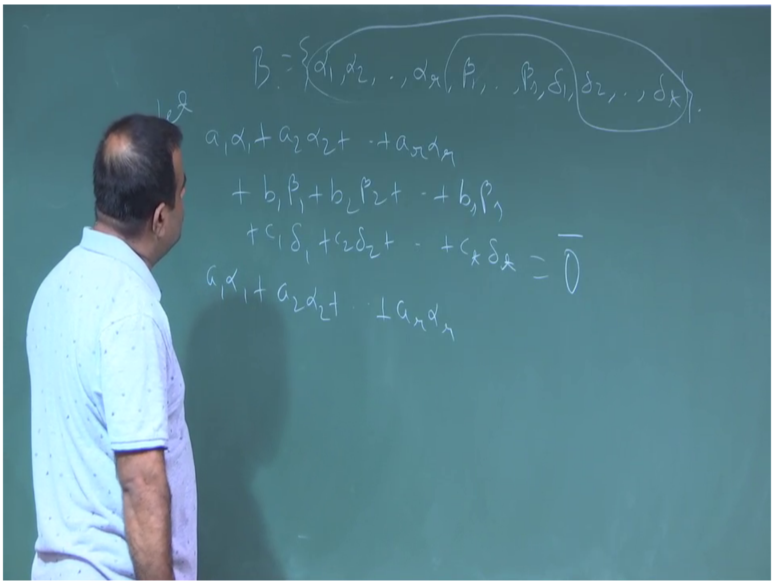

We take a1 alpha 1 let plus a 2 alpha 2 plus a r alpha plus I am writing here b 1, beta1 plus b 2 beta 2 plus up to b s beta s ok. So, this is b b s beta s, then we take c with c c 1 delta 1 plus c 2 delta 2 plus ct delta t. This we are taking to be 0 vector and then eventually you need to show these all the coefficients are 0.

So, how to show this? Now, this is basically we can take this from this side a 2 alpha 2. So, we can just r alpha r this is basically we can take this or we can just take this one that side.

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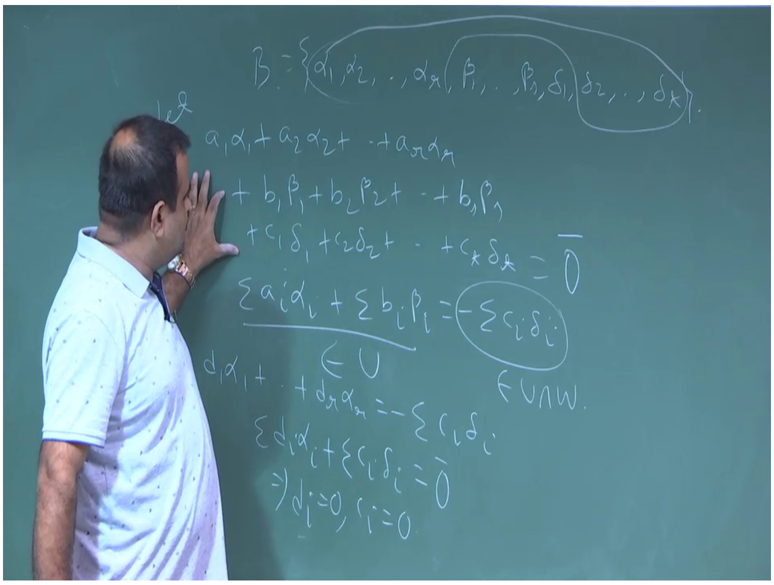

So, b 1 beta 1 plus b 2 beta 2 just a minute; so summation we use the sum summation of a i alpha i plus summation of bi beta i is equal to minus of summation of c i delta i and i is running here from 1 to r, i is running here from 1 to s, i is running here from 1 to this ok. So, this is a basically this is a member of this is belongs to W. So, this belongs to W. Now since, deltas are like this, now one minute. So, this is basically so, I want to take this part here sorry.

So, I want to take summation of summation of d i beta i alpha j so b i, b i beta i. So, sorry same thing; so, this part summation of a i alpha i plus summation of bi beta i is equal to same things c i delta i.

Now, this is belongs to U and this is belongs to W now. So, this is belongs to W because this is coming from W. So, this part must belongs to U intersection W. So that means, this must be written as some linear combination of alphas. So that means, d1 alpha 1 plus d r alpha r should be written as minus summation of c i delta i's.

So, from there we can take summation of d i alpha i plus summation of c i delta i is equal to 0 vector. So, from here we can say d i is equal to 0 for all i and ci is equal to 0 for all i.

So, one so c i we are getting to be 0 for all i. Now one c i is 0; that means, this is a this is also 0 because this is a this is a basis from U. So that means, a i is also 0 bi is also 0. So, this is the proof. So that means, this is a basis of U plus B. So, they are linearly independent and they are spanning the set.

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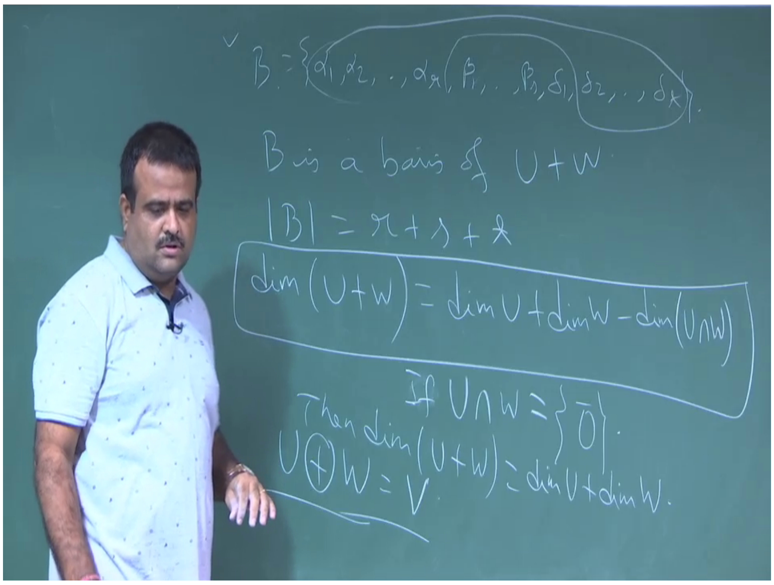

So, B is a basis of U plus B. So, since B is a basis. So, what is the dimension of this? So, dimension of size of B is basically dimension. So, r plus s plus t so, this is nothing, but dimension of U plus B, U plus W which is basically dimension of U plus dimension of W minus dimension of U intersection W. So, this is the general theorem for any 2 vector space. For any 2 vector space V and W, we have this result dimension of U plus this is equal to this ok.

So, now if we have if this is empty if U intersects and W is if this is contained onto zero vector. Then this is 0 then we have dimension of U plus W is B basically dimension of U plus dimension of W and this is called direct sum. This we denoted by U W direction and then one is called complement of others, U is called if this is true then U is called complement of other.

So the then U plus W is, this is basically called complete sum direct sum you said to be direct sum of this. And if it is generating the whole space like if V is equal to this then it is called U is called complement of W and W is called complement of U.

So, this is just the definition of the complement. Now, if it is the having the direct sum; that means, if this is empty then any. So, if you take any vector any basis of U and any basis of V then that together will give us a basis of U plus W. Any basis of U and any basis of W together will give us a basis of U plus W.

Thank you.