Introduction to Abstract and Linear Algebra
Prof. Sourav Mukhopadhyay
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture – 15
Subgroup Operations

Ok. So, good day everybody. So, you are talking about subgroup and some of the properties of the subgroups. We will discuss some more properties of the subgroups especially will take two subgroups, then we will see whether their union is a subgroup, intersection is a subgroup, their product will define how what do you mean they have a product of a subgroup two, two sets. So, these are the properties we will discuss now.

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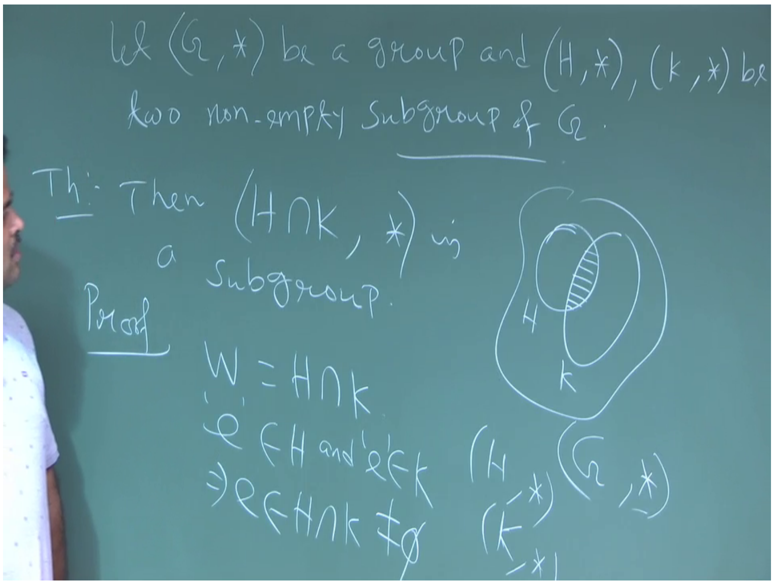

So, for that so, let us take a group let G be a group under this operation star group and H and K the two subgroups of G non empty subgroup be two non empty subgroup of G then we will discuss some of the properties we want to know the their intersection is again a subgroup or not their union is a subgroup their product this two sets.

So, G is a set which is a group under this operator and we have a two subset H and K where H and K are both subgroup under this induce operator of this. So, H is a H is also a group and K is also a group capital K is the set, ok. Now, the first theorem is then intersection is also a group H intersection K is a subgroup ok. So, this we have to prove this we have to. So, first how to prove this. So, first of all you need to show this intersection is non empty.

So, suppose W is the intersection this means this region. So, W is the intersection. So, why this is non empty because the identity element of the group e, now this H and K are both subgroup. So, identify e must belongs to H and e must belongs to K, the identity element of the group then this implies e must belongs to their intersection. So, this implies this is non empty subset of G, this W. So, W is non empty.

Now, we need to show that W is a subgroup of G. So, for that we will use one of the necessary and sufficient condition to show W is a subgroup of G.

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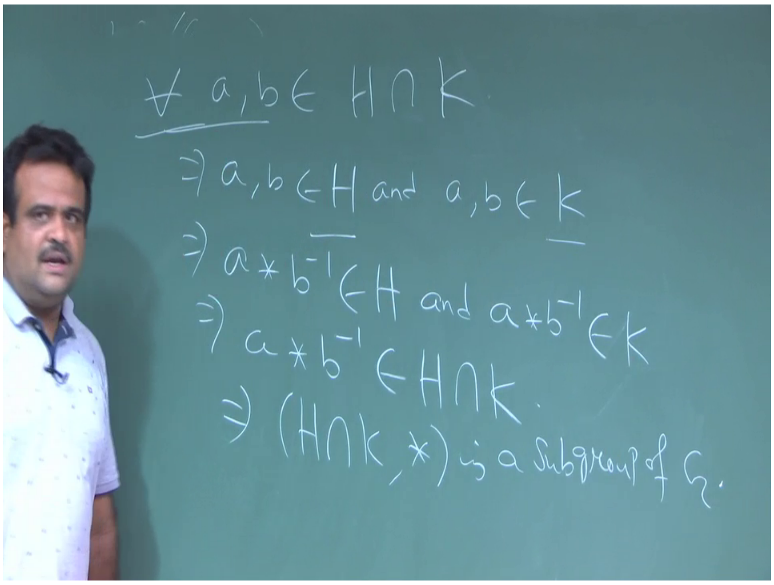

So, for that what we take. So, this is our W this. So, we taken two element from W. So, W is basically and then we need to show for all a, b from W you need to show that a star b inverse is in the W then we are done. Then we remember this is one of the necessary and sufficient condition to be a set subset is a group. So, you want to show this is not yet done,.

Now, this implies a, b belongs to H and a, b belongs to K both because they both belongs to their intersection. So, they both belongs to this now this is a group this is a group. So, this implies and a star b inverse belongs to K because this is a group. So, we know this property this is necessary condition of a group because b is an element. So, b inverse also exists in a group. So, now, a is element b inverse is an element. So, closure property a star b inverse must be this. So, now this imply a star b inverse belongs to W.

So, hence this implies that under these that is was a sufficient this is a sufficient condition and this is true for all a, b. So, these imply this is a subgroup of G subgroup of G, ok. So, the intersection is a subgroup, now we want to know the union whether if we have two subgroup H and K; whether their union is also a subgroup. So, we will see union is it is need not be a subgroup.

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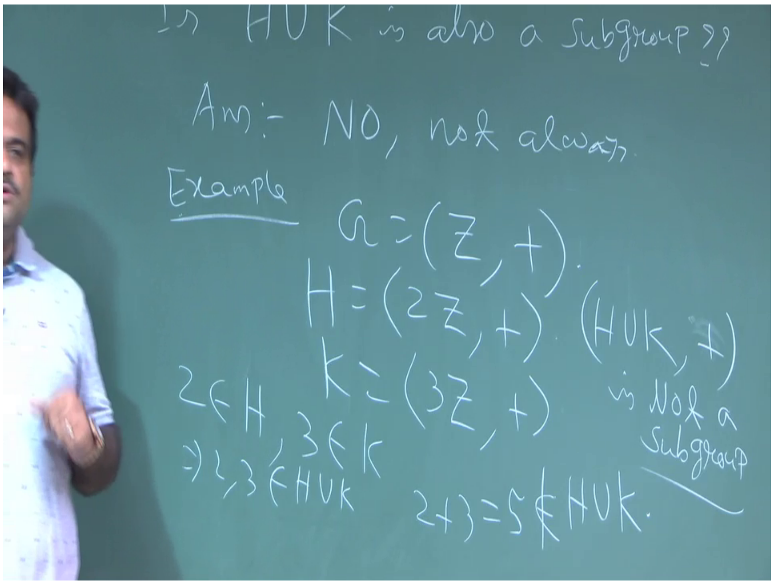

So, the question is the union also a subgroup. So, answer is no not always not always then we have to have a counter example for where it is not happening to be a subgroup where H and K both are subgroup, but their union is not a subgroup, ok. So, what is the example? So, if you take a group G to be the Z class, ok. So, this is the set we consider these along with this operation is G is a group this is the set of all integer, ok.

Now, if you consider two subgroup 2 Z plus and 3 Z plus, this two are the subgroup of G. Now, the union is basically all the elements which are either multiple of 2 or multiple of 3 all the integer which are multiple of 2 multiples of 3. Now, the union is not a subgroup, why? Because 2 belongs to H and 3 belongs to K. Now, this implies 2 and 3 both belongs to H union K, but what is 2 plus 3.

So, if it is a group then the closure property must satisfy, but 2 plus 3 is 5. So, 5 is neither belongs to H nor belongs to k. So, 5 does not belongs to H union K. So, this implies closure properties is failure. So, that means, this is not a subgroup. So, this is an example where H union K is plus under this plus is not a subgroup so, the intersection. Intersection mean there is a subgroup we just prove it, but union need not be a subgroup, ok.

Now, we defined the product of two subset and we are interested to see whether this is a again a subgroup.

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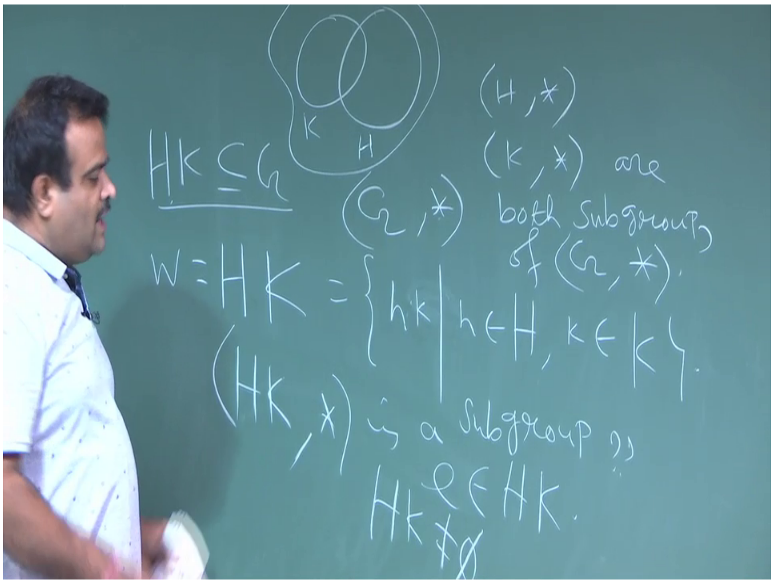

So, again G is a group and we have to G is a group under this binary or composition we have two subset H and K, and suppose they are both subgroup are both subgroup of G non empty subgroup. So, subgroup so, every group with the, if G is a group then G must has G must be non empty.

So, now, because at least identity element will be there. So, the singleton group is the id one I mean identity element from a singleton group that is also a group. So, every group must be non empty in the definition of the group we have we will discuss that, ok. Now, we want to define a set this is called product set this is nothing, but H star K or simplicity we can just write we can omit star we can just write h k small k this is small k. So, this means H operate with K under this operation if it is multiplicative sense this is otherwise if it is additive sense then this will be written as h plus k and this will be h plus k small K, but anyway this is just a operation operator notation.

So, now H is coming from H and K is coming from K. Now, this is the way how we define this product of two set this is this is not a Cartesian product. Cartesian product is the ordered pair, but this is an element in the group itself in G, ok.

Now, we want to see whether this is a question is whether this is a this is a set W whether this W is a subgroup or not subgroup or not because this is this H and K is again a subset of G and it is a non empty subset why because e must belongs to this because e belongs to e belongs to H K always. So, H K this is non empty non empty subset.

Now, the question is whether this is a subgroup or not, ok. So, again it is need not be a subgroup always. So, you have to take an example where it is not a subgroup. So, we take the example of polynomial group that S 3. So, we consider set of all polynomial of degree 3.

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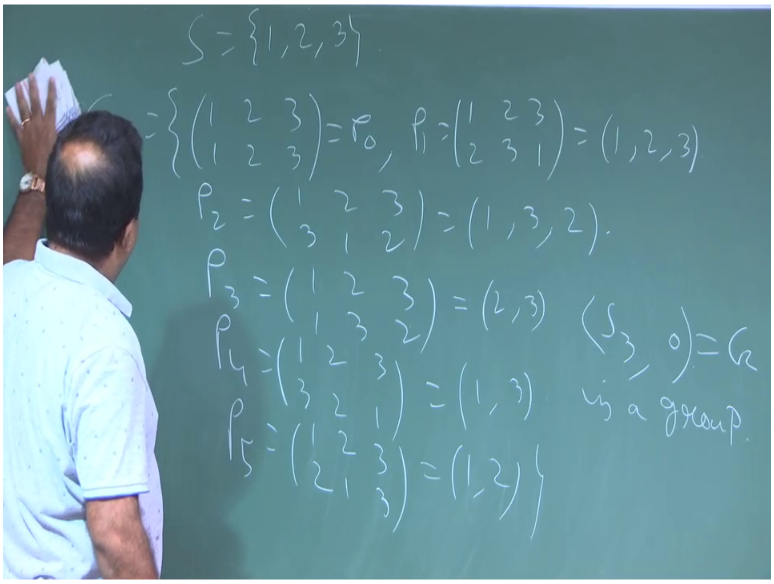

So, this is our S 1 2 3 and if we consider set of all polynomial of degree 3 this will denoted by say this is identity polynomial this will denoted by P 0 and then P 1 is basically 1 2 3 2 3 1 this is P 1 this is nothing, but 1 is going 2, 2 is going to 3, 3 is going to 1, this is a even permutation, this is a cycle.

So, P 2 is nothing, but 1 2 3 3 2 3 1 2 3 1 2; that means, 1 is going to 3, 3 is going to 2, again 2 is going to 1. So, P 3 1 2 3 then again 1 3 2 is basically 2 2 3 this is a even permutation then P 4 and P 5 1 2 3 P 4 is 3 2 1 3 2 1 is basically 1 is going to 3 3 is going to 2 and P 5 we have P 5 1 2 3 and we have 2 1 3. So, this is 1 2 this 3 3 are even permutation and we have identity permutation ok, now this is our set.

Now, we know this is a group this is for symmetric group and where this composition because permutation is a bijective mapping. So, if you take two bijective mapping if you compose this will you can give as a bijective mapping. So, in the permutation sense it is the product of two permutations is again the permutation ok. So, product of cycle basically. So, now this is a group this is our G this is a group. Now, we want to take two subgroup of this H and K. So, you can just you can just erase this part we now we know the P 0 to P 5.

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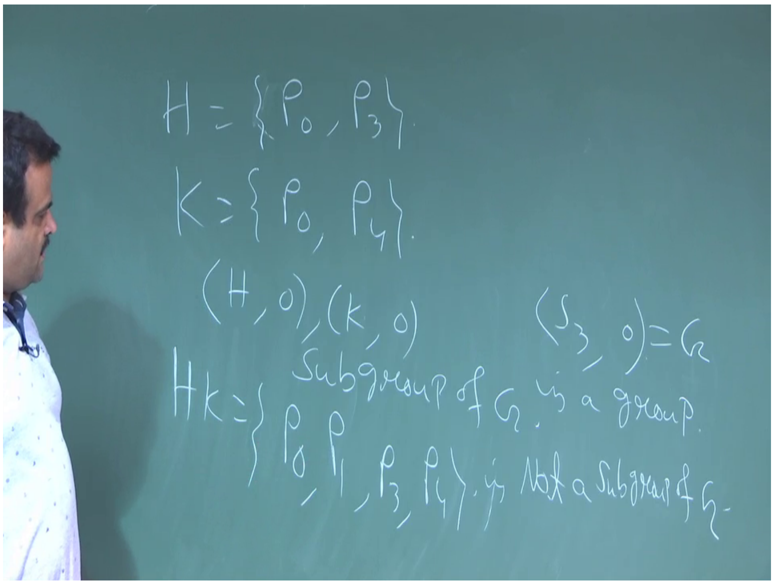

Now, if you take H to be say P 0 and P 3 and K to be P 0 and say P 4. Now, you can verify that these two are the subgroup of G under this operation. So, we can easily verify that these both of these are subgroup of G. Now, what is their product is basically H K. So, all the possibilities of this. So, this will give us P 0, P 1, P 3, P 4.

Now, this is not a subgroup we can verify that this is not a subgroup not a subgroup of G. This is not a subgroup of G because P 1 is not having inverse in this group and even we can verify the composition may not be satisfying. So, this is an example where product is not a subgroup.

Now, if so, so, we want to know that is there any necessary is there any sufficient condition which can say that this will be subgroup if this condition is satisfy. So, that you want to explore. So, we want to know a necessary and sufficient condition under which this product of two sub subsets is a again a group.

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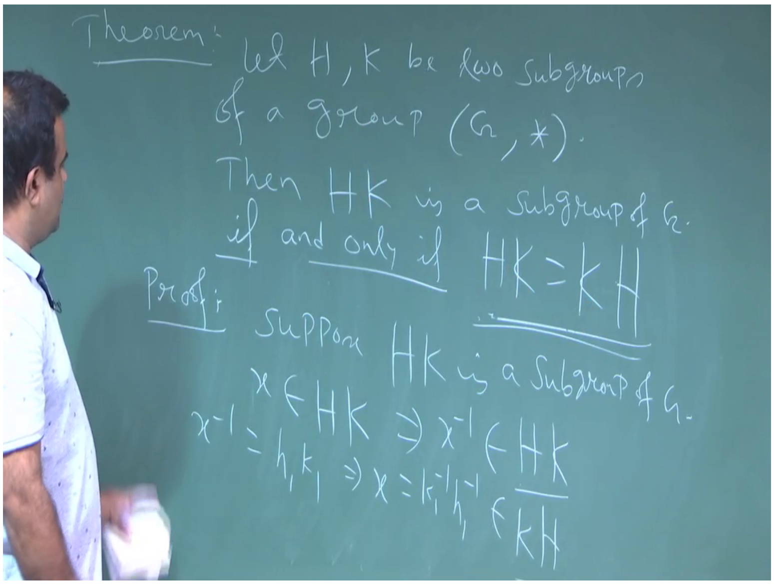

So, this will state by a theorem. So, this is a theorem ok. So, this is telling let H and K be two subgroup subgroups of a group G. So, we can define the operator also then H into K is a subgroup of G if and only if G if and only if HK is equal to KH if they are commutative in the sense, if HK is equal to KH, ok.

So, there is two part; if part and converse part. So, necessary part and sufficient part. So, we just going to us try to prove this, ok. First of all we assume this is a subgroup. Suppose, H and K is a subgroup of G then we want to show these to say set are same. So, for that what we need to show we need to take an element form this set and we need to show that this is an element, in this is the element in H and k. So, if you take an element from this set then this implies. So, this is a group this imply inverse exists inverse is also a group, ok. Now, if we take now inverse exists because this is a group. So, now, we can take this to be x inverse to be this is belongs to HK. So, H 1 and sums K 1.

So, now this gives us x to be if you take the inverse of this K inverse K 1 inverse H 1 inverse. So, this is a member of KH. So, if you take an element from here we can see this is an element of here. So, reversely also we can do that we can take a y from here with the same trick we can. So, this is an element of here. So, this is a subset of this is a subset of this for this two set are same.

Now, conversely you want to see the, this condition this is a, this is an insufficient condition. So, you want to see if we have this condition then we want to know whether this is a, this H and K is a subgroup or not. So, that part you have to convince, ok.

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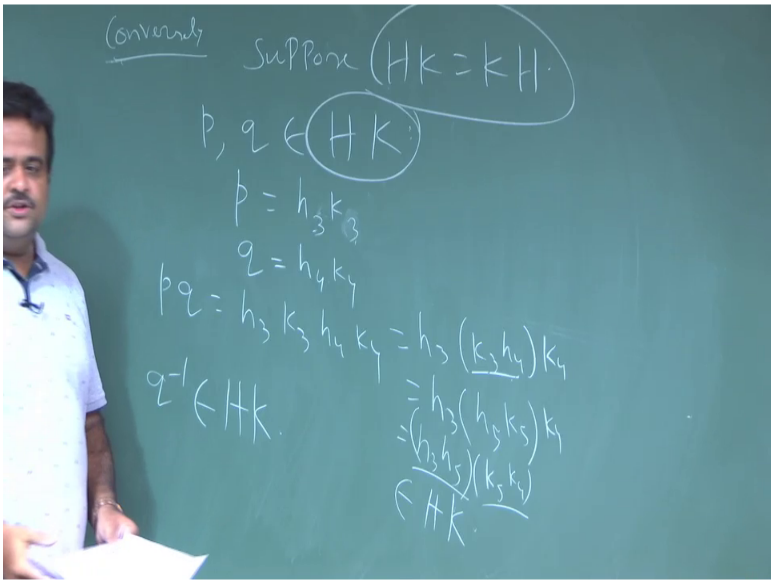

So, to do so, we just take conversely this is the other part of the prove suppose these two set are same. Suppose these two set are same, now we want to see whether HK is a group or not. So, for that we will use one of the necessary and sufficient condition. So, we take let us take two element from these and then we are going to show that two properties are there know p star q is also belongs to this and if you take an element q inverse belongs to this. So, if you can prove these two then we are done.

So, we will try to prove the first one and second one I will leave you for the exercise. So, now, if p, q is this. So, p will be something like h 2 k 2 and q will be you have taken k 1 k 2 anyway you can take k 3 k 4 and q will be sorry k 4 h 4 something like that. So, if you take the product p q so, it is basically h 3, k 3, h 4, k 4 now this is basically we can combine this h 3 k 3 k this is why the associativity rule of the operation star these are basically star operation,.

Now, this is a this is an element form KH now KH is same as this. So, there exists another element from this which is same, need not be this is commutative. We are not claiming that, but there has to be some this there has to be some element like k 3. So, there has to be some h 5 k 5 which will be in this ok, because these two set are same.

So, now this will give us h 3 k 5 and you find h 4. So, this is another h this another k. So, this belongs to HK, this belongs to HK, ok. This is the first part and the second part we have to solve that q inverse is also belongs to HK. So, this you can easily by this taking this type of term we can easily verify that ok. So, this is the necessary and sufficient condition to be a to this to be a subgroup under G.

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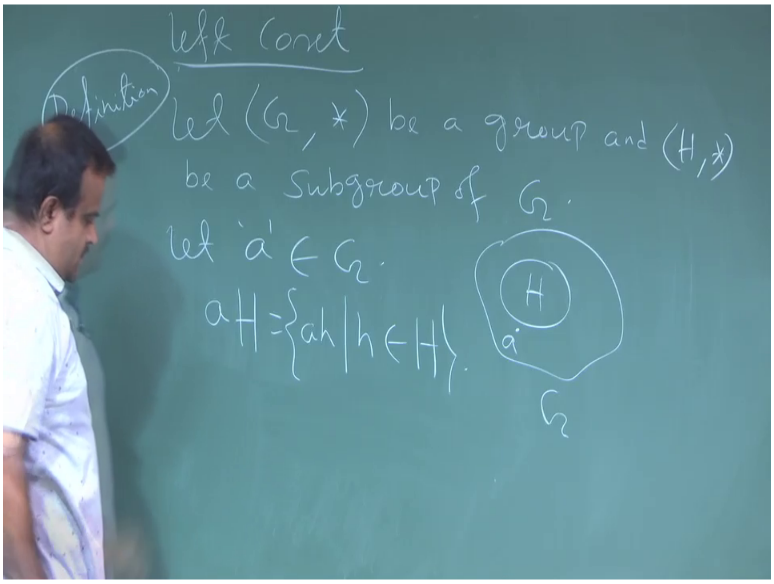

So, now we will define coset; coset of a subgroup, ok. So, first we define the left coset. So, for that let us take H G be a group G be a group and H be a subgroup of G subgroup of G, ok. So, then we define then let a be an element in G any element in G. So, this is our set G this is H which is a subgroup now we take an element from G we this may be from here this may be from here does not matter. So, we take an element from G, anywhere it could be in H also. Now, we define this left coset. So, H this is basically all the sets all the elements like h is coming from H ok, h is coming from H.

So, this is this is in multiplicative sense if star is multiplicative sense now if star is additive sense this will be a plus H, if star is the star is a additive sense ok, but anyway this is the operation. So, we will just take this notation and we assume the star is in multiplicative say anyway this is just a binary operator it could be addition, it could be anything. So, now. So, this is the definition of the left coset this is the definition of the left coset, ok.

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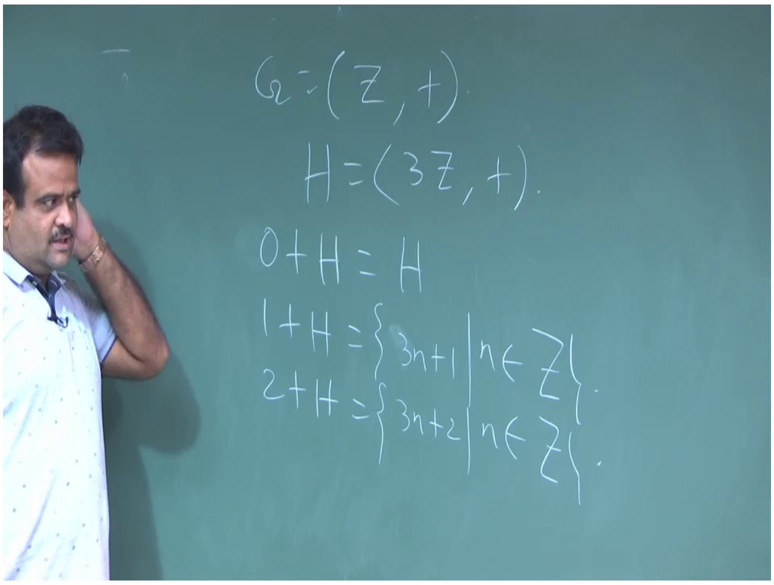

For example, if we take say Z. So, if you take the group to be set up integer, ok. Now, if you take a sub subgroup of G now what are the left coset. So, left coset will be of the form say we take an element a from G 0 is an element. So, 0 plus H this is a left coset this is same as H. Now, 1 plus H which is basically all the elements of the form 3 n plus 1, n is coming from Z like this 2 plus H 3n plus 2 like this ok. So, this is this is in additive sense. So, that is why it is in this form otherwise if it is multiplicative sense it is 2 H, 1 H like this. So, this is one example of this.

Now, we will know some properties of this coset, ok.

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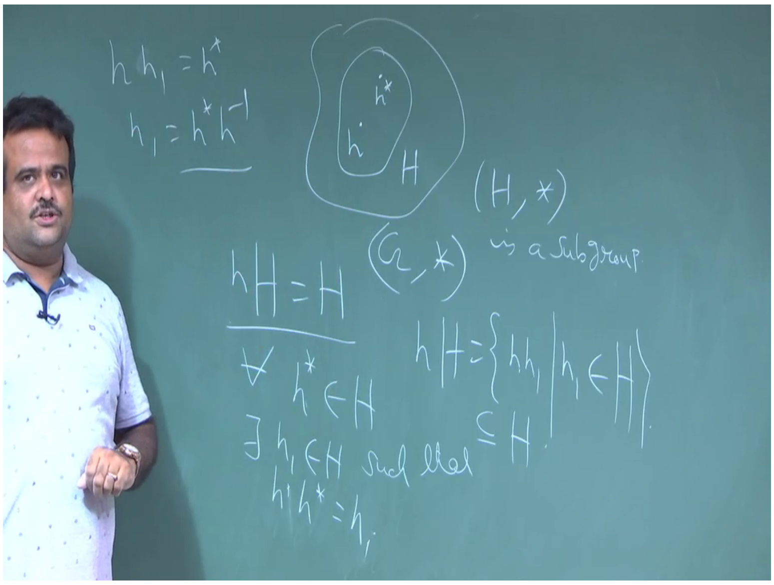

So, now, this is a group G under this operator and this is our H. Now, the first property is telling us if we take the element a from H itself, so, then it is telling us a H H is basically same as H the if a the because this is H is a you have to prove this H is a subgroup H is a subgroup. So, if we take the element from H itself then it will give us a same set H. So, how to show that to show that this is basically we need to show this is a subset of this. Now, H this is this is a subset of this is no no, this is no doubt because hH we defined as this is the h some h 1 and h 1 is coming from h and h is a subgroup. So, this is also belongs to H. So, this is a subset of this.

Now, only thing we need to show if you take an element from H, say we take an element from H say some h star any element for all element we need to show that there exist some h 1 such that such that h 1 h star sorry, h h star equal to h h 1 ok. So, now, how to show this? Now, if you take an element from here h this is basically h star now, that means, ; that means, h star such h 1 exist. So, h only exists because this is this is basically h 1 is basically product of this. So, how to show this?

So, let us take an element from here. So, this is basically our h star now we want to show there exists an element here h 1 such that h compose with h 1 so, h compose with h 1 will give us h star. So, that means, h is h 1 is nothing, but h star h inverse and that inverse exists here because h is a element here. So, this is this is the existence of such h 1. So, that means, this is a this is the subset of this also. So, if you take the element from H, then this coset is same as H. Now, if you take a element outside this then what is the what is happening that you will see in the next class.

Thank you.