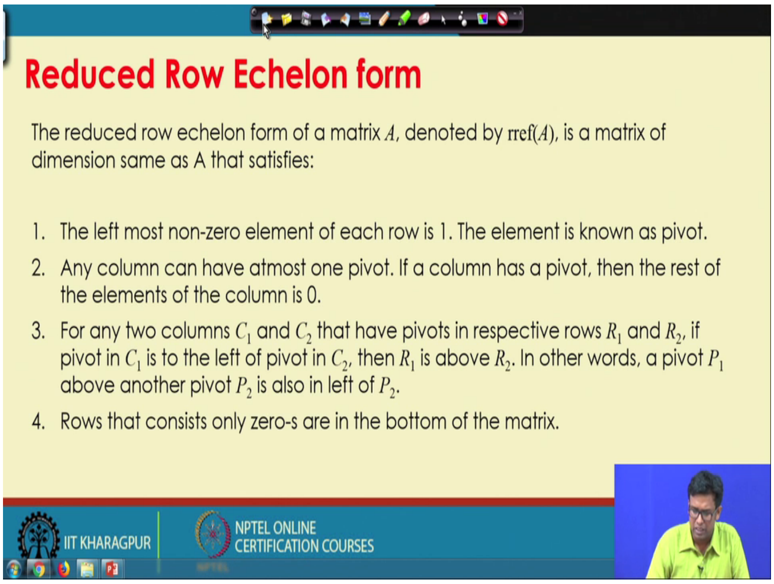
Matrix Solvers
Prof. Somnath Roy
Department of Mechanical Engineering
Indian Institution of Technology, Kharagpur

Lecture - 09
Gauss-Jordon Method

Welcome. In today's class we will discuss about a new method of solving matrix equation which is very similar to Gauss elimination method. In fact, a variant of Gauss elimination method which is called Gauss-Jordon method and through this method, we will see we can directly diagonalize the matrix and get the solution. And this method has is applicability in finding inverse of a matrix. So, we will see how this method can be applied to solve a x is equal to b and also, how we can get a inverse using this method.

(Refer Slide Time: 00:56)



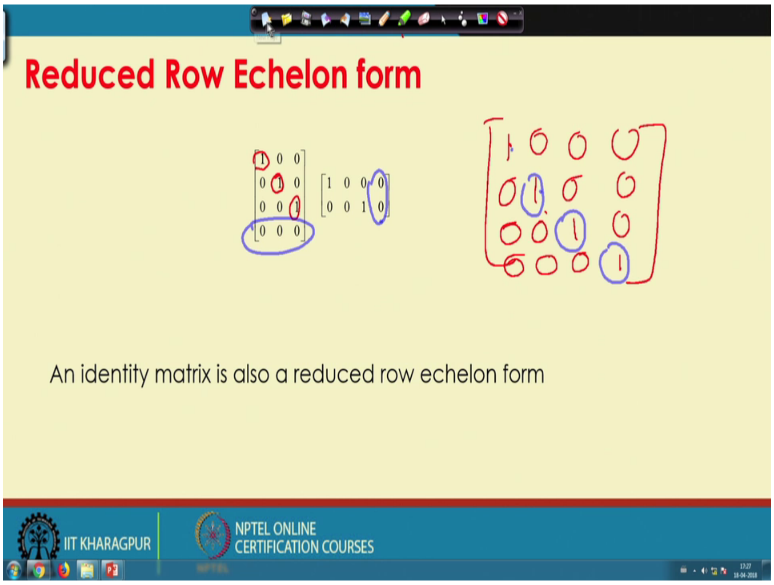
First we start with reduced row echelon form. This is a form of a transform matrix A and it is said that a matrix A will get a reduced row echelon form through matrix transformation and the form is and it is defined by rref A in several computer programs also functions early phase called to get reduced row echelon form of matrix A. And, the reduced row echelon form of matrix A will be a matrix of same dimension of A, which is obtained after transforming A by certain matrix operation and this form will satisfy the following criteria that the left most non zero element in each row is 1 and the these element is known as pivot.

So, there can be zeros and that will first known 0 element coming from the left will be 1, which is the pivot in that particular row. Any column will have at most one pivot and if a column has a pivot, then the rest of the elements on that column will be 0. For any 2 columns C 1 and C 2 that have pivots in respective those R 1 and R 2 if pivot of C 1 is left to the pivot of C 2, then the row R 1 will be above of R 2. That means the pivots will be arranged from left hand corner and come towards the right hand corner in other words pivot P 1 will be above another P pivot P 2 and this P 1 must be also in the left of P 2.

So, we will when we will see the forms we can verify this. When we will see examples of a reduced row echelon form matrix we can verify these terms and rows that consists only zeros will be in the bottom of the matrix. So, there can be rows with no pivot with all zeros, because if the pivot or if the one does not appeared here all the term there cannot be any other non zero term all the other terms will be zeros and that will be in the bottom of the matrix.

So, if we see how does reduced row echelon form matrix look like.

(Refer Slide Time: 03:13)

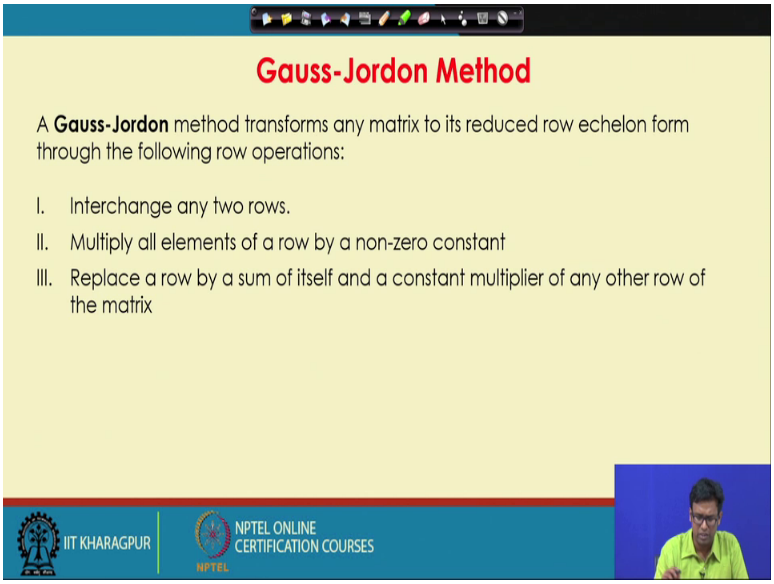


So, these are what is typically called reduced row echelon matrix, this is the pivot term which comes at the left most which is the left most non zero element of one particular row and in one column there is only one pivot term and remaining terms are 0 in any other column. And each of the pivot terms which are above the other pivot term row wise should be column wise in left of the pivot other pivot terms. So, they are kind of arranged from left to right and there can be also zeros in certain rows.

Similarly there can be zeros in certain columns also reduced row echelon form matrices are not necessarily square matrices they can be rectangular matrices also and those will be the rows or columns with zeros only. And this term echelon actually comes from arrangement deployment of croup in an army, how army how in an army soldiers will be arranged in certain form it is like a in the step wise form and the term echelon came from there. And we also see that an identity matrix is also reduced row echelon form, all (Refer Time: 04:36) if we write a matrix like this 1 0 0 0 1 0 0 0 0 1 0 0 0 0 1 this is an identity matrix.

These are the pivot terms which are the comes in the left most term of one particular row and one pivot term which is left to another pivot term must be row wise above the above the other pivot terms also.

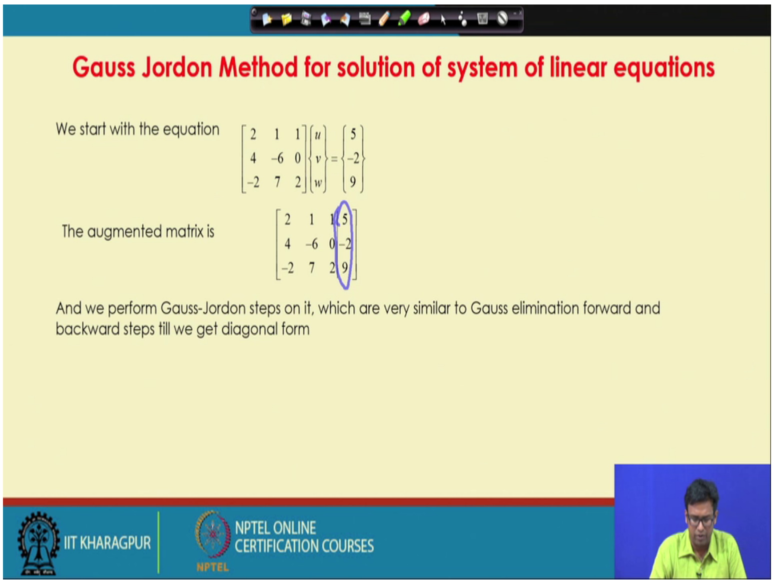
(Refer Slide Time: 05:17)



So now, Gauss-Jordon method states that a method can be transform to its a matrix can be transform to its reduced row echelon form through the following row operations. Interchange any 2 rows multiply all elements of a row by a non zero constant, replace a row by a sum of itself and a constant multiplier of other rows any other row of the matrix. So, these terms like at least first and third one is very much like the processes we following Gauss elimination method. So, through row operations and the third second one is also is probably the last few steps in the back substitution in Gauss elimination method.

So, through these steps we can transform a matrix to a reduced row echelon matrix, and the advantage that that if we can get a reduced row echelon form of a square matrix it will be an identity matrix. So, we can directly solve the equations in one step, because it will be only a diagonalize matrix with all coefficients one. So, the left; so the right hand side vector transformed in that way will give us the actual solutions. So, let us we will look into that form that process in a while.

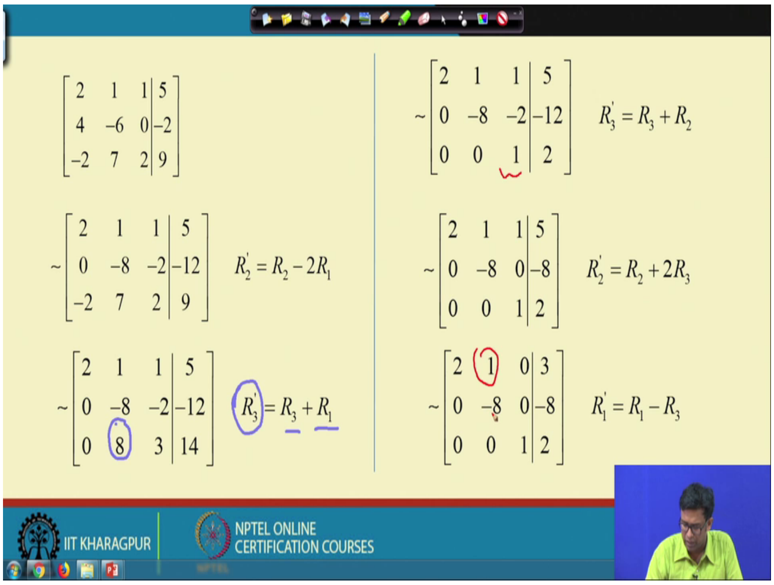
(Refer Slide Time: 06:36)



We start with the so we look how Gauss-Jordon method can be applied for solution of system of linear equation. We start with the equation 211 minus 4 6 0 minus 2 7 2 u v w is equal to 5 minus 2 9. And now we will try to get reduce to echelon form of the left hand side matrix and through the same step we should get solution of the right hand side matrix we should get solution which will be in the right hand side column vector.

So, the augmented matrix, augmented matrix means this matrix augmented with the column vector is a matrix like this and we will see how we get a reduced row echelon form here. So, that the firs non zero element from the left hand side will be 1. And as this is the diagonal triangular matrix the reduced row echelon form will be a identity matrix, in the same way we will get the solution which will come in the in the right most column or the augmented part of the matrix. We perform Gauss-Jordan steps on it which are very similar to Gauss elimination form and including both forward and backward substitution steps and till we get the diagonal form.

(Refer Slide Time: 08:07)



So, this is the augmented matrix we considered in the last step and now we do row operations, first we have to we have to follow same steps as Gauss eliminations.

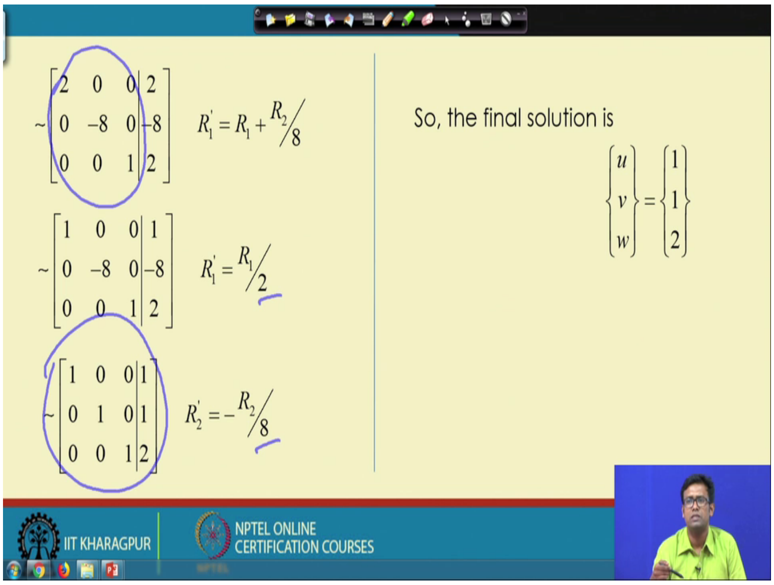
So, we need to make this the second row first term to be 0. So, that the fiverr terms remains only. For the first 2 is the fiverr terms. So, we multiply first row by 2 and subtract from second row the transformed second row is second row minus twice into the first row which gives us 2 1 1 0 minus 8 minus 2 and the right hand side column becomes minus 2 in minus 2 into 5 minus 12. We perform the next step in which I have to eliminate this terms. So, I add first row with the third row I add first row sorry with the third row and get the transform third row which is 2 minus 2 0 1 plus 7 8 1 plus 2 3 5 plus 9 14 0 8 13. Now I have to eliminate this term if we follow a Gauss elimination forward substitution forward elimination steps.

So, I will add the third row with the second row. So, transform third row will be third row plus second row; so 8 minus 8 8 plus minus 8 10 2 plus minus 2 1 to a 14plus minus12 2. So, this we get the third row lastly meant to be 1, which is what we should get also in a reduced row echelon form. So, for third row in order to have to do anything and as this is the last row we now should start the backward substitution step. What do you have to do in the backward substitution step we have to eliminate the minus 2 trans.

So, twice of third row should be added to the second row and second row is equal to second row plus twice of third row. So, 0 minus 8 0 and this becomes minus 8. And now, we have to eliminate this one and the next step will eliminate the other one.

So, first rows first row third column one has to be eliminated; that means, third row has to be subtracted from first row and we get this particular form now we have to eliminate this one. So, we have to subtract add 1 8 times of second row from first row, and we go to the next page.

(Refer Slide Time: 10:54)



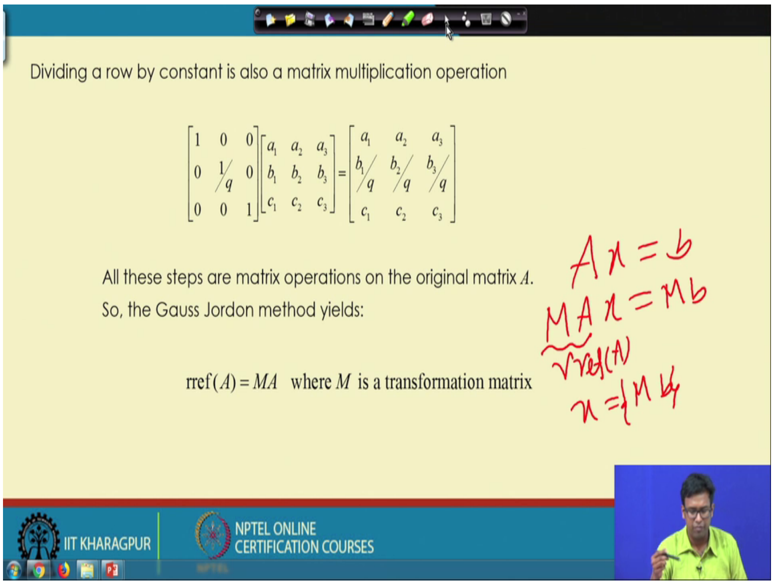
So, 1 8 times of second row is added with the first row and we get a diagonal form, but this is not a reduced row echelon form, because the left hand side the first non zero term coming from the left in any column is a is not 1 in reduced row echelon form they should be 1. So, we have to for third row it is 1, but for second row and first row we have to divide them by 2 and by minus 8 and the steps are like this the first row divided by 2 gives me 1 0 0, which is now this is close to reduced row echelon form except the second row, and then we divided by minus 8 and get 1 0 0.

So, this the left hand side matrix got a reduced to row echelon form therefore, the augmented matrix or the augmented part of it that the fourth column will give me the solution with that and we can see the final solution is u v w is equal to 1 1 2. This is very similar as a Gauss elimination method only thing now this is more formalized as we have to get a reduced row echelon form and we have to do certain matrix operation, which is can be interchanging of rows interchanging of rows are not done here. Because you did not get any 0 pivots, but there these are certain matrix operation like interchanging of rows, adding or subtracting one row multiplied by a constant from another row and dividing all elements of one row by certain number.

So, that from the diagonal form like from this diagonal form from this diagonal form we get to this particular form, because first the first row is first row is divided by 2 and the second row is divided by 8. And in this form the as the main matrix a is diagonalize the right hand side column vector will be directly the solution vector.

Now we know that we have earlier seen that exchanging rows is a permutation multi matrix multiplication by permutation matrix and subtracting one row from other is also matrix multiplication. This step will see is also like dividing all elements of row by echelon is also a matrix multiplication step.

(Refer Slide Time: 13:29)

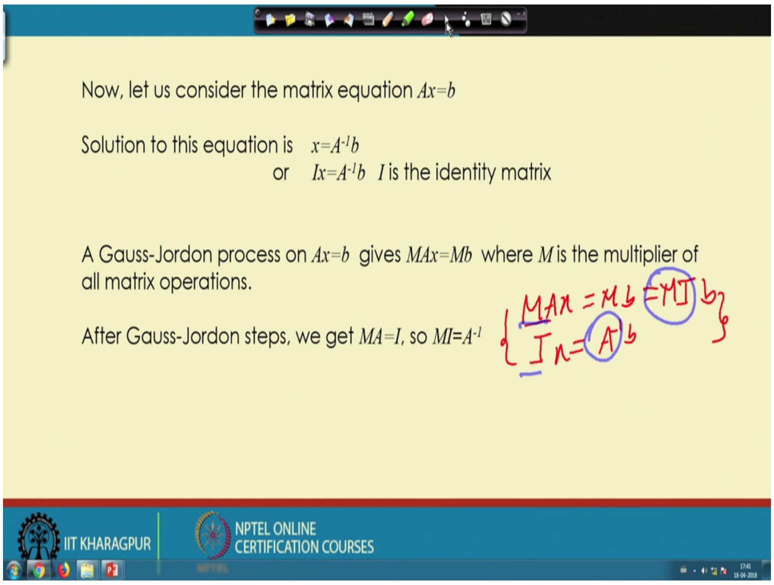


And we see dividing a row by a constant is also matrix multiplication operation. For example: if this row has the second this for this matrix the second row has to be divided by a term q, we multiply this matrix by 1, 1 by q and one and this will be a b 1 by q b 2 by q b 3 by q matrix. So, all the steps in a Gauss elimination in a Gauss-Jordon method that can be replaced by matrix multiplication step multiplication by permutation matrix or multiplication by a matrix like this or multiplication by a subtraction addition matrix. So, the Gauss-Jordon method will yield the reduced row echelon form a matrix A is M into A, where M is a transformation matrix; that means, its product of all the transformation matrices including row operation matrix row division matrix and row permutation matrix and all the series of operation should be multiplied and we will get this matrix A.

So, the Gauss-Jordon method will tell me that as I will I have an equation A x is equal to b, and now I will write M A x is equal to b because M A will be the matrix operation applied on A and which will give me the reduced row form of A and this will give me M b then x is equal to M b should be my solution vector. So, the matrix operations which will transform a to reduced row echelon form of a should be done on the right hand side be effected b to give me the solution vector x.

Now interestingly this method can be further applied to identity matrices and we will see that that gives us the inverse of one particular matrix.

(Refer Slide Time: 15:48)

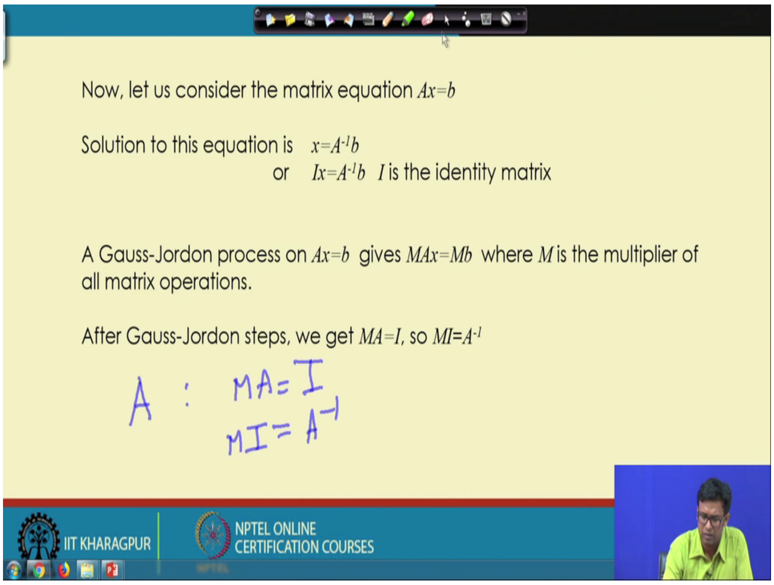


Now, let us consider the matrix equation A x is equal to b, solution to this equation will be x is equal to A inverse b, and or if I multiply identify matrix I with x I x is equal to A inverse b is the solution of the matrix, where I is the identity matrix. Gauss-Jordon step on A x is equal to b will give me MA x is equal to M b and the solution x will be Mb. M is the multiplier of all the matrix operation. So, from here we can relate that the solution is x is equal to M b and that also tells us that this matrix operation M, when applied over A gives us an identity matrix I. Therefore, this matrix operation M when applied over an identity matrix I, should give us A inverse or from here we can write MA x is equal to Mb. Now max our and these directly gives us the solution therefore, this is equivalent to I x is equal to A inverse b.

So, M b can be also written as M into identity matrix into b, and if we compare these 2 equations for a square matrix the reduced row echelon form of MA is nothing, but the identity matrix I. Therefore, M I should be A inverse or the Gauss-Jordon steps for one particular matrix A which will give us the reduced row echelon form of A or for a square matrix which will give us the identity matrix form of A identity matrix transform from A. These Gauss-Jordon steps when will be applied on an identity matrix should give us A inverse. A Gauss-Jordon after Gauss-Jordon steps we will get MA is equal to I or reduced row echelon form of a will get for a square matrix it is in identity matrix therefore, for if the same Gauss-Jordon steps applied on a on an identity matrix M I will give us A inverse.

This is very elegant way of finding A inverse in the sense we do not need to go into the in intricate process of finding cofactors of minors of each element in one particular row and do a series of determinants calculation and then find out A inverse which we initially proposed when we are discussing about determinants. Rather we can see through the processes through which one square matrix can be transform to an identity matrix and the same processes we will apply to an identity matrix and we will get A inverse. And that is basically a Gauss Jordan method of finding inverse we will see it through an example.

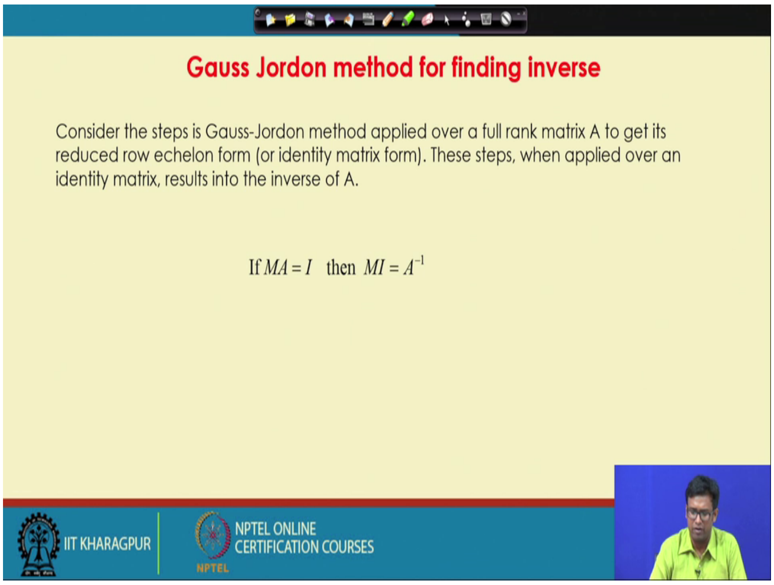
(Refer Slide Time: 19:20)



So, the idea is that we will start with a matrix we will start with a matrix A so, that MA is equal to I am sorry identity matrix M is a set of matrix operation or one when multi multiply we all the all the matrix operations operators and multiplied its a single matrix, which is the matrix operator on A. When we do this set of operations on A we get identity matrix and Gauss-Jordon we will tell that that when we apply the same matrix operation on identity matrix we should get A inverse.

So, we will perform Gauss-Jordon steps on one particular matrix A to transform it to an identity matrix form, we will perform the same steps on matrix I and we will get A inverse.

(Refer Slide Time: 20:30)

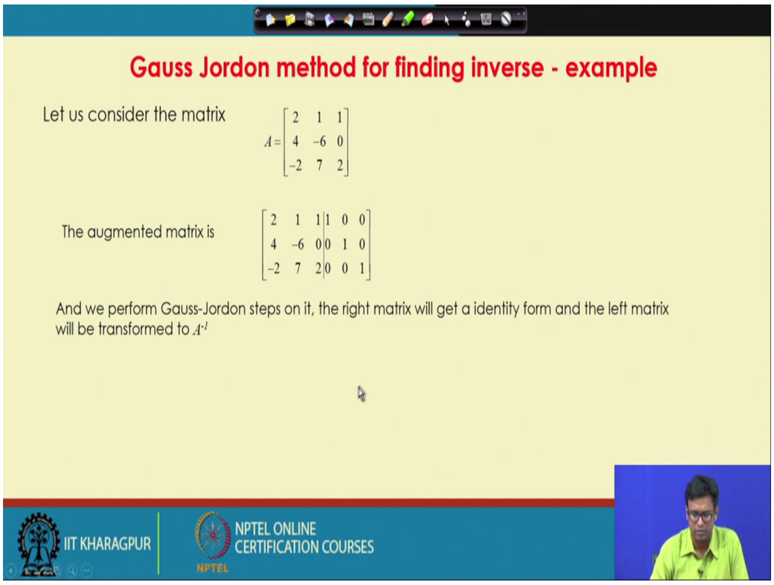


Gauss Jordon method for finding inverse is consider the steps in Gauss-Jordon method applied over a full rank matrix A to get its reduced row echelon form or identity form because it will be same for a full ranks square full rank matrix is always a square matrix for a square matrix A. If it is not a full rank matrix, then this is a singular matrix and inverse cannot be found. However, consider a full rank matrix a square matrix and full rank matrix apply Gauss-Jordon step, to get its reduced row echelon form, which is identity matrix. This steps when applied over an identity matrix we will result into the inverse of A if M into A is equal to I, M into I is equal to A inverse.

So, M is a matrix multiplier l which is basically matrix multiplier of matrix operations or steps of matrix operation. So, different matrix operation step combining exchange of rows, subtracting one row from others, dividing all the elements of one row by one particular scalar apply on the matrix A and apply the same thing on an identity matrix I finally, I will give us fine I will be transform to A inverse.

And we will see it through the same example earlier we solved an ax is equal to b using Gauss-Jordon step we will apply. Now take the same matrix a and saw apply Gauss-Jordon steps over it and same Gauss-Jordon steps we will apply over an identity matrix and we will see that the inverse is coming out.

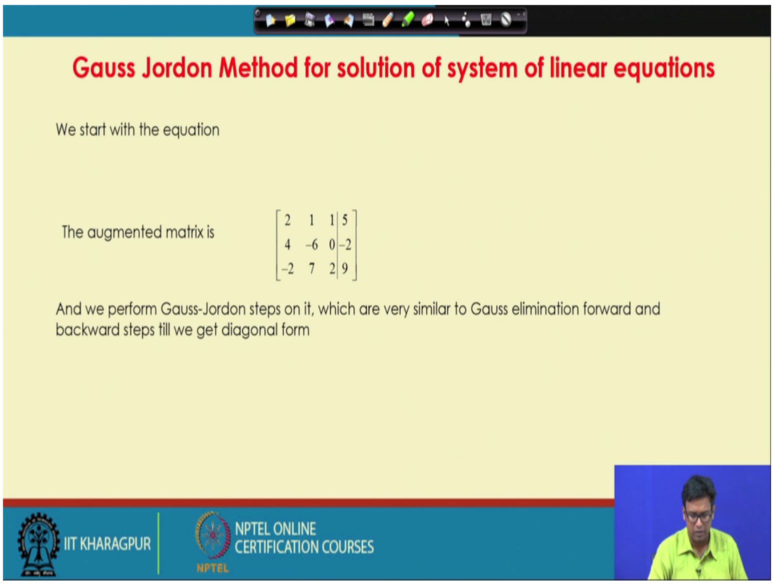
(Refer Slide Time: 22:00)



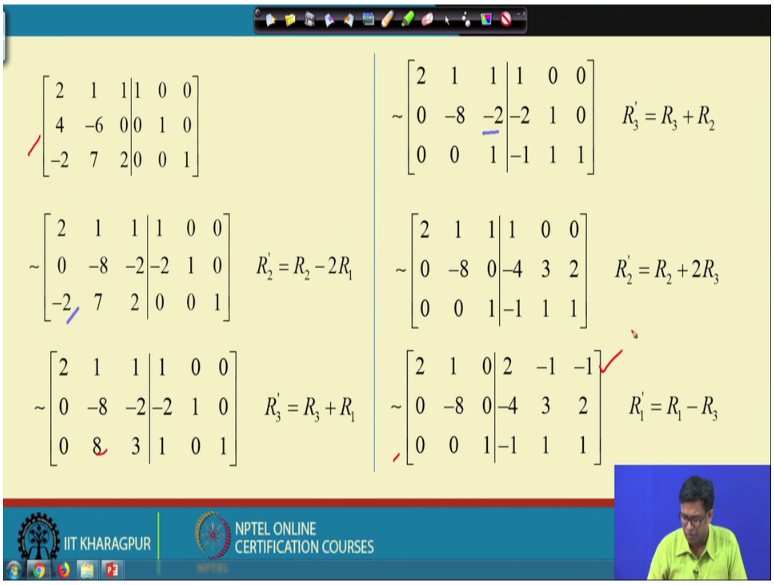
. So, we will consider the same matrix A to 11 4 minus 6 0 minus 2 7 2.

The augmented matrix now we are augmenting it with identity matrix because we want to transform the identity matrix to a inverse also. Same thing with 1 0 0 0 1 0 0 0 1 this 6 by 3 by 6 matrix will try to give it to a reduced row echelon form. So, that the left most first element is 1 and the all other terms in that particular column is 0. We will perform Gauss-Jordan steps on it the right matrix we will get an identity form and the left matrix will be transformed to A inverse. So, the first step is that we start with the equation sorry.

(Refer Slide Time: 22:40)



(Refer Slide Time: 22:50)



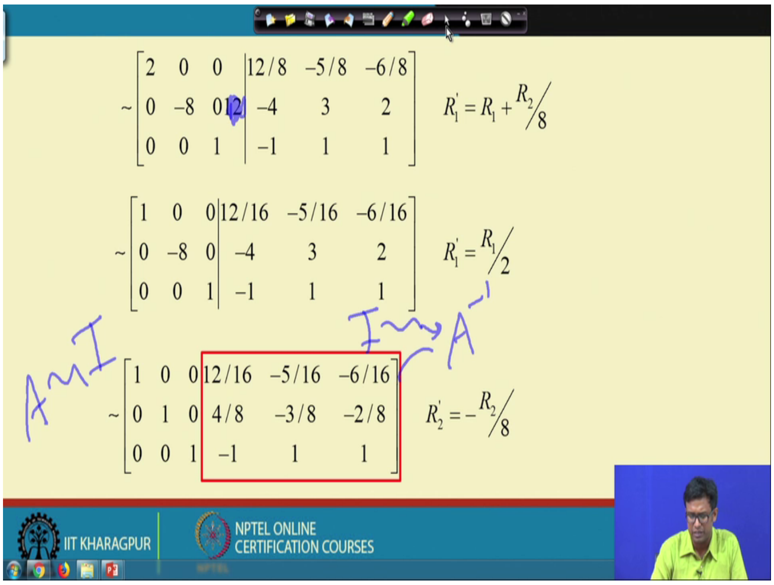
So, the first step is that we start with that augmented matrix and we do exactly the same steps we have done earlier, the second row transform from his second row minus twice of first row, we do the same thing on the identity matrix also, it does not remain an identity matrix anymore this transform.

Now, this term has to be eliminated. Now these terms has to be eliminated. So, we will add first row with the third row and this is eliminated now this term has to be eliminated. So, we will add second row with the third row and the lower the lowest column got a kind of a reduced row echelon forms lowest row got similar from 0 0 1 the first non zero term is 1 here, and we will this is end of the Gauss elimination forward steps. Now we will start the backward substitution step in Gauss elimination. So, this has to be eliminated we will add twice of the third row with the second row, and note that the identity matrix is also getting transformed in interestingly, it got an a upper lower triangular form in the first row of first set of operations through the forward substitution step.

Now, there will be more changes in identity matrix also. So, we go to the step are transforms again row is equal to second row plus twice into transform third row and this term is eliminated, now we have to eliminate this term. So, first row transformed first row will be first row plus minus third group. And now if the identity matrix was a diagonal matrix though all this transformation as we are approaching to a diagonal form of this particular matrix, with approaching to a diagonal form identity matrix from a diagonal from now it is completely populated with non zero terms well.

So now, this 1 has to be eliminated sorry. So, this 1 has to be eliminated; that means, this will be multiplied by 1 by 8 and added with the next term. So, there is some mistake here this will be 0 and this will be 0 not a this is a this there is a typing mistake here. So,.

(Refer Slide Time: 25:30)



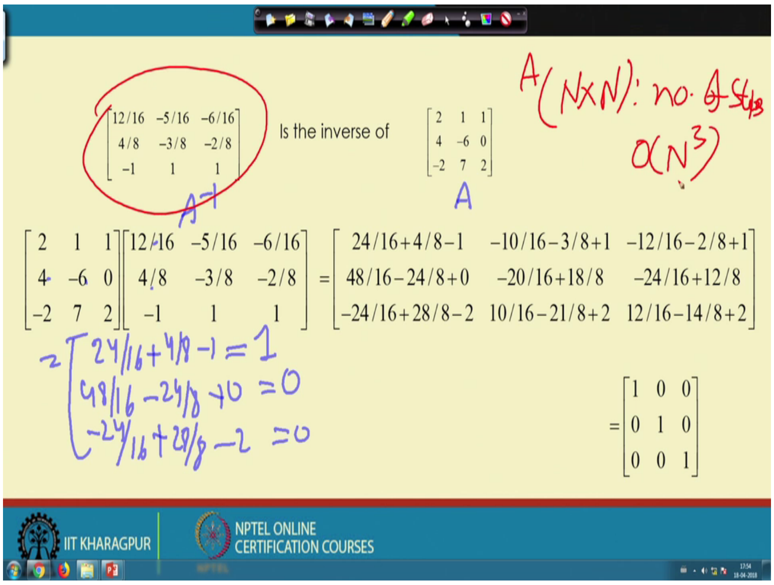
So, first row is first row plus 1 by 8th of second row, and now we have to start giving them all the pivots should be 1.

So, you divide the second row all terms in second row by all terms in first row by 2. So, this pivot is 1 and then all terms of second row by 8 minus 8. So, all this terms are 1. So, finally, we get the identity matrix the right hand side a matrix is transform to an identity matrix. So, in the identity matrix is transform to a different form and this is A inverse.

So, we will check it we will multiply this matrix with the initial A and we should get identity here. However, Gauss-Jordon steps tells me that Gauss-Jordon method for finding inverse tells me that once Gauss-Jordon steps are performed over any matrix a which is a full rank matrix, and the same steps are performed over an identity matrix the identity matrix transform to A inverse or we should say that this is A number inverse. So, A is transform to I and I is transform to A inverse.

Now, we will let us take this matrix and to confirm it take this matrix and multiply with A.

(Refer Slide Time: 27:21)



So, this matrix is inverse of the original matrix A. So, this is A and this is A inverse as per our Gauss elimination steps. So, let us multiply A inverse with A it should be an identity matrix, we will practice this multiplication once more. The first term will be the first row first column term will be 2 into 12 by 16 then 1 into 4 by 8 minus 1. So, 12 by 16 plus 4 by 8 right 12 by 16 into 2 into 12 by 16 24 by 16 plus 4 by 8 minus 1, which is basically 16 by 16 and 1 and the second term the first columns again row term that will come as we multiply this particular row with this columns. So, 48 by 16 minus 24 by 8 plus 0 48 by 16 minus 24 by 8 is 0.

Similarly, we will multiplied the third row third column term here. So, you can probably write I t down. So, the first term will be 24 by 16 24 by 16 plus 4 by 8 minus 1. The first row second column term will be then 48 by 16 and minus first row second columns minus 24 by 8 at this will be this will be multiplied with this is multiplied this plus 0 fine. The first row the third row first column term this multiplied by this will be minus 2 into 12 by 16 which is minus 24 by 16, plus 7 into 4 by 8 plus 28 by 8 and minus 2. So, you will see this is 24 by 16 plus 8 by 16 minus 1, 2 minus 1 this is basically this is 1 this is 48 by 16 minus 24 by 8 which is 0. This is 24 by 16 minus 28 by 8 to if this is 56 by 16 32 by 16.

So, 2 minus 2 is 0. So, we can similarly go an find all the terms and you can calculate it that this becomes an identity matrix 1 0 0 0 0 0 1 00 0 1. So, if we can take a large matrix also and perform the same stuffs and get the same equation and if you get a 0 pivot. So, you have to exchange the rows. For a large matrix the inverse finding using determinant is much involved like you have to find out many determiners find out all the cofactors for that you have to find out number of determinants and it will involve lot of steps even for a 3 by 3 matrix, you could have imagine that you would not have been that easy.

If I go for 4 by 4 matrix the inverse finding using determinant is also complicated; however, this method is straight forward as it follows a simple logic which is same as Gauss elimination steps, only at the last step you have to divide each row by one particular constant to get a identity matrix form and you will end up in getting the inverse of the matrix. And typically for a n into n matrix the number of steps will be same as Gauss elimination steps very similar will be order of n cube, and we can find out that inverse.

So, this is one important method of finding in inverse of a matrix using direct solution techniques. And it has lot of similarity with Gauss elimination method we can do a number of practice problems. I will hand you over this problems in your assignment sheet, where you can use Gauss elimination and find inverse and best thing in finding inverses that you directly multiplying inverse with the right hand side vector and you will find the solution, ok.

So, this is an important step Gauss-Jordon method. And now we will see in next few classes few other direct solution techniques, which are for certain class of methods. These methods are is applicable Gauss elimination and Gauss-Jordon they are applicable or (Refer Time: 32:28) decomposition they are applicable for any full rank matrix A, but for some other matrices which arise from solution of physical systems, there are some more I will say some methods which has which follow we will needs less number of operational steps. Like a tridiagonal diagonal matrix elimination method or using tridiagonal matrix algorithm or using tridiagonal matrix algorithm, we can do alternating direction implicit method. All this methods we will see in the subsequent classes.

Thank you.