Matrix Solvers

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Lecture - 60
Multigrid Methods – II

Welcome, we are discussing Multigrid Methods for solving iterative matrix equations. What we have observed in last session is that, that convergence rate is slower, if we have a large number of grid points or if we have a large matrix; compared to a smaller matrix in the same geometry we have less number of grid points.

And it is slow due to the fact that large wavelength components of the error which is required to smooth the solution for large number of mesh points is very slow to converge. And what we have calling large wavelength relative to a particular mesh size; if we increase the mesh size that becomes a smaller wavelength.

And therefore, that converges first that part of error is minimized first if we use a larger grid point or if you use a coarser mesh; that larger grid spacing or coarser mesh. So, we came into a concept of geometrically looking into the matrix problem and reducing the computational steps by using multiple levels of grid.

First use a coarse mesh; in which you solve the equations and get faster convergence. And then with that the converge solution in the coarse mesh, come to a finer mesh, where again try to converge the equations so that you get an accurate solution.

So, this particular technique we will we got some discussed about the concept in next session. And the particular techniques we will look into. We will start from the multigrid concept; what is done in a multigrid tech method.

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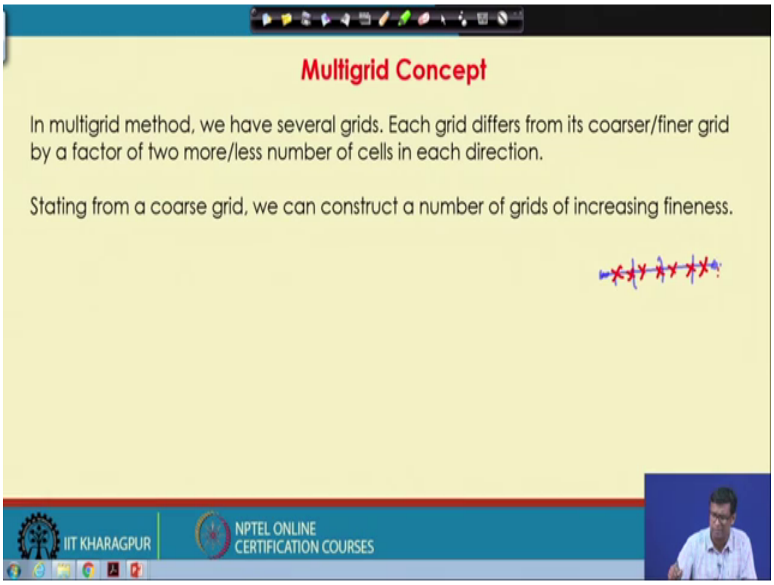

So, in multigrid method; we have several grids. Each grid differs from it is coarser or finer grid by a factor of 2 more or more or less number of cells. Factor of 2 more or less number of cells in each direction.

That means, if I have a mesh and I go to the coarser level the coarser level we will have half number of cells in that direction. And if I go to a final level, the final level we will have double number of cells in one particular direction. Starting from a coarse mesh we can construct number of grids increasing fineness.

So, we will see it, but say I have a geometry where initially we have 4 points, this is the coarse mesh. And then I double the number of internal points. So earlier there are 3 internal points, and then each or rather let us see that this is this is also 1 point so, these are the ends.

So, earlier we have 3 spaces; 1, 2, 3, 4; 4 spaces and 3 internal points. And now we have the older points will also be there. So, 1, 2, 3, 4, 5 6, 7; 7 points. So, we can construct the number of we can have a finer mesh by increasing fine mesh.

We can inject more points inside and make the mesh more defined. So, there it is first you solve in the coarser our mission then with this solution you interpolate it back to the finer mesh, take it as a great solution and again solve it.

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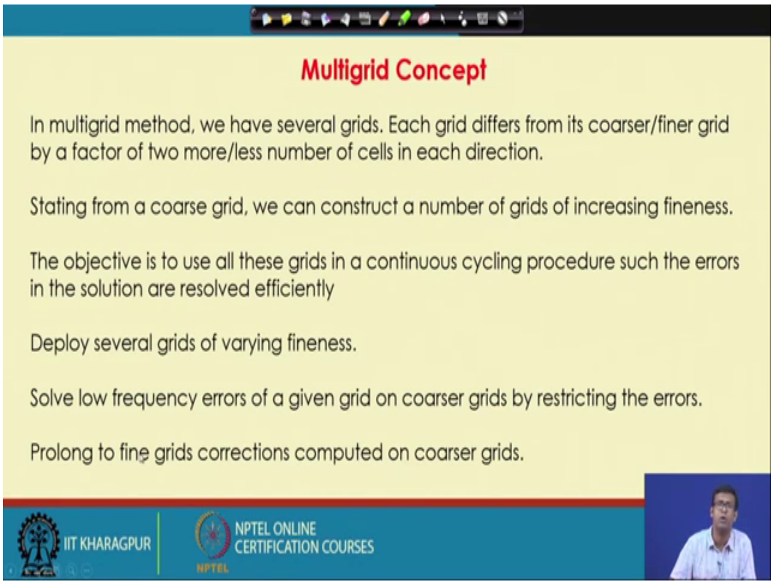

The objective to use all these grids in a continuous cycling procedure the such that the errors in the solution are resolved efficiently. And that is what we are trying to do. So, deploy several grids of varying fineness, solve low frequency errors of a given grid on coarser grid by restricting the error.

Low frequency error in a given grid is high wave length error on that grid. But solve that in a coarser grid by restricting the restrict error in a coarser grid so it becomes a low frequency. High frequency lower number error in the coarser grid and can be solved first.

Prolong the solution to the grids from the coarser grid solution prolong it interpolate it to the finer grid. So, that the corrections so that you can get accurate result on the finer grid. So, first solve it in coarse grid which will be you will get a faster solution, then take the coarser grid solution prolong to find grid.

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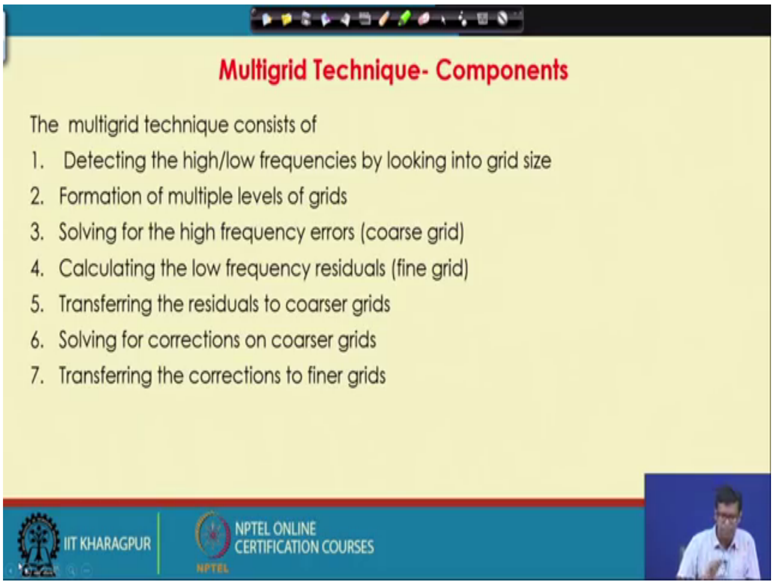

The component the multi grid technique consists of this 4 few steps. Detect what is high frequency and what is low frequency by looking into the grid size. So, you look into the grid size and find out that whether all the errors will be relatively of higher frequency or lower frequency. If the errors are of have if you all most of the errors are a higher frequency.

That means, if the grid is coarse so now we do not need to do any multi gridding. But if it is a mile if it is mile enough you have to do one level of multi multigrid. If it is very fine you have to do multi level of multiple level of fine gridding so that the largest wavelength error becomes somewhat very resolvable by the most coarser grid, it becomes small wave wavelength error compared to the coarse grid.

So, detect that number of grid levels and there are some mathematical prescriptions for that. But however, the application scientists do these things by their expertise and experience. Check whether multi grid is required for this grid level and how many cycles of multi grids will be needed.

Formation of multiple level of grids so once you have looked into the grid size you now you can form that what will be the different grid levels and from multiple levels of grid. Solve for high frequency errors which are low wave. Large wave length high frequency and solve for sorry.

This will be yeah solve for high frequency errors in hole speed. This will be, just a second so for high wavelength error this should be solve for high wavelength error I will change it which is low frequency.

No, this is fine actually I am sorry I will. Let me explain this, solve for high frequency errors in the coarse grid. What is high frequency error in course grid, what is; high frequency error is low wave length error? Small wavelength errors are difficult to reserve.

So, what is small wavelength error in a fine mesh will be large wavelength sorry large wavelength errors are difficult to reserve. What is large wavelength error in a fine mesh will be a small wavelength error in a coarse mesh and therefore, it will be a high frequency error. So, first solve what is high frequency error in coarse mesh and in and this is basically the small wavelength errors in coarse mesh and larger length errors in the fine mesh

This will be quickly solved. And then the low there will be some low frequency residuals will be there. And that you have to take it into the fine grid and solve it. This residual is now low frequency to the fine grid which is coming from the coarse mesh, which was high frequency error in coarse mesh; this again has to be reserved.

So, it cannot be direct, and everything cannot be directly solved in coarse mesh you have to come back to fine grid also. Transfer the residual these residuals are low frequency residuals in coarse mesh. You transfer them to the fine mesh low frequency residuals in fine mesh you transfer them to the coarse mesh where there will be high frequency errors and solve for them.

So, there is a cycling between fine mesh and coarse mesh. Solve for corrections in coarser mesh transfer the corrections to finer grid and do a cycling till you get a convergence. So, the idea is now actually very simple you solve the equations in coarse mesh see what are the corrections in the solution variable.

Transfer these solutions variables into the finer mesh, update the solution at the finer mesh and see whether you have got the residual right it is 0 rather. If residual is non-zero you calculate the residual in the finer mesh and this residual or the required correction for the residual in the finer mesh.

This has to be again transferred in the coarse mesh and there will be a coming down enough from a cycling from fine mesh to coarse mesh. Till you get a converse solution and the converse solution must be available at the finest mesh because we are trying to get the solution at finest mesh where the desired accuracy can be obtained.

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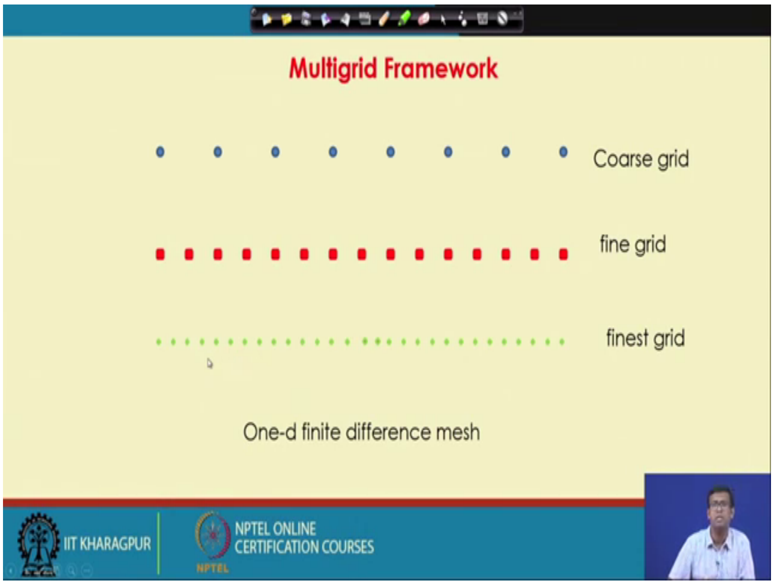

So, how will be a framework? So we take one-d finite difference method this is the core spirit. We inject 1 point in between each so, you get a fine grid. The grid spacing is exactly half number of into if these are boundary points number of internal points are actually doubled. Then you increase one more level of fineness increase one more point here.

So, what was a single spacing here now this is a there are 3 spaces for; sorry, now there are 4 spaces in between. So, each has been reduced by 1 by 4 times you get a finest grid here. This is where I need the solution; I will start getting solution here. And then the solution that has been converged here I will put it here and update it fine.

And now the solutions I will which will be converge here I will put it here and update it fine. And now the update what I am getting here; that this will be the correction parameter say I will pull it back. Because this is a difference matrix and difference matrix earlier it was converged here, the new solutions will not appear to be converged here probably. So, I have to again do these rounds of iteration till I get a convergence.

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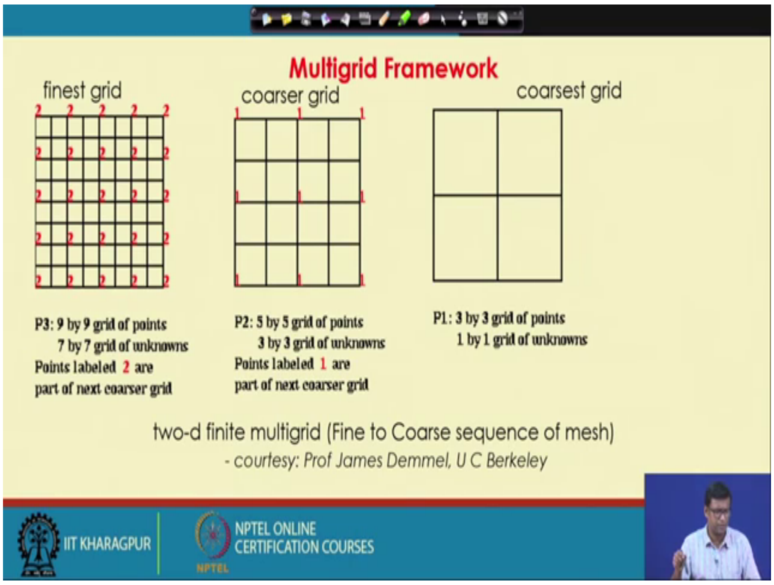

For a 2D mesh point this is taken from Professor Demmels lecture notes in UC Berkeley. This is the finest mesh so from finest mesh we are going to the coarse mesh. The points which were I have level 2 will be the member of the coarser grid. So, when you are solving in a coarser grid we only solve in this points, we will not solve the in the internal points.

When you go to the coarse mesh the points labelled as 1 will be there. Now if this is the geometry I can see the coarsest mesh is just sufficient to reserve the entire boundary. There is one point in this boundary, one point in this boundary, one point in this boundary, one point is this boundary; 4 internal points except the corners in the boundary.

There is sufficient to reserve; there is least number of points which we can deploy to get at least one internal point here. This is the coarsest mesh and this is obtained by removing points from the finest mesh. And this is from fine to coarse sequence of the this is the fine to coarse sequence of the meshes.

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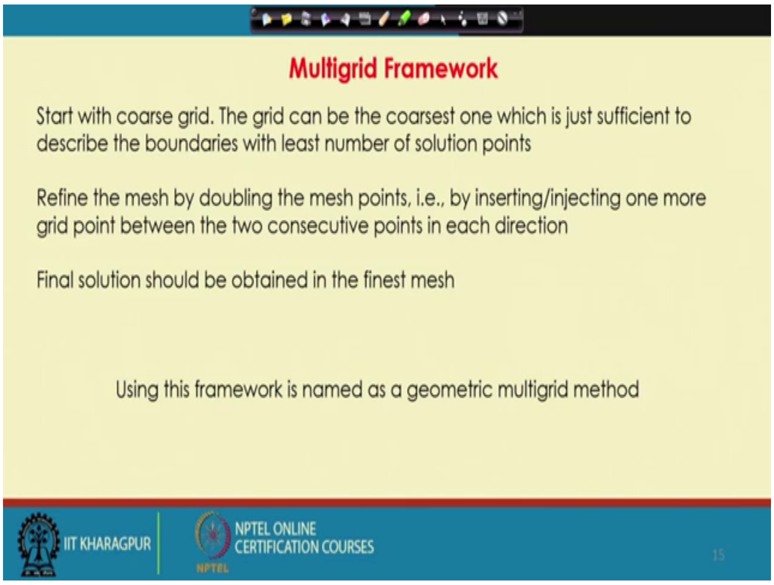

The multigrid framework will be start with the coarse grid; the grid can be the coarsest one which is just sufficient to describe the boundaries with least number of solution points. So, earlier we have shown one of the coarsest mesh, only one internal point except everything is a boundary point. Refine the mesh by doubling the mesh points by inserting rejecting one more grid point between 2 consecutive points in each direction.

And you can refine it to certain to few old levels instead of one more one multigrid level you can use number of multi rate levels. Like last 2 example show we have seen has 2 3 levels grid levels to multigrid level. So, 1 is the finest 1 and 2 more coarse coarser levels. Final solution has to be obtained in the finest mesh only and this is I am asserting it several times because we need the desired accuracy which can be obtained only with the finest mesh. Using this framework is named as geometric multigrid method because due to the fact that we are using a geometric framework to create the grids.

We are not taking the matrix directly and seeing what can be done on it what we have done in preconditioning technique. Rather we are getting the geometric framework, getting a governing equation which is a differential equation discretizing it, knowing the geometric framework and changing this geometric grid system by inserting more special points to make finer mesh or reducing more special points to make a coarser mesh and that is why this is usually termed as a geometric multigrid method.

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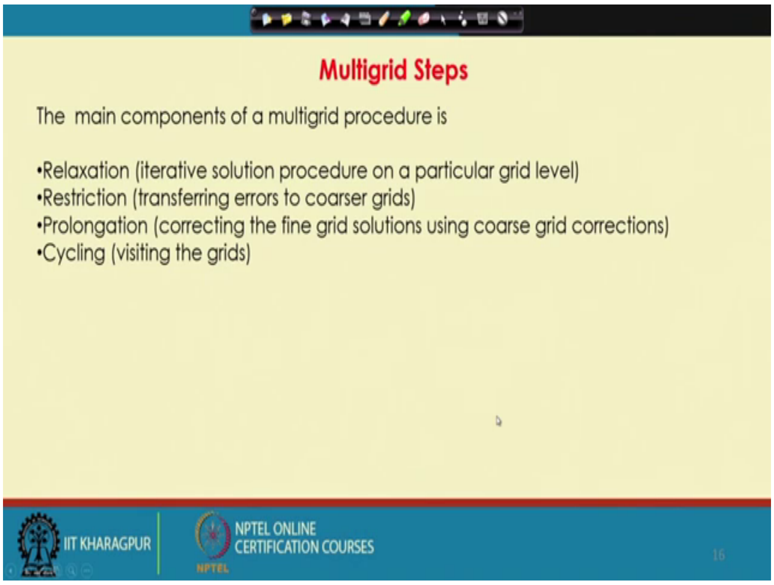

The main components of multi grid procedure is relaxation or iterative solution procedure on a particular grid level; restriction are transferring these errors from the final level to the coarser level; prolongation correcting the final grid solutions to using the coarser grid corrections.

Getting the corrections in coarser grid and transplanting it if the finer grid and cycling That means, that the solution you obtained here is actually not a converge solution. So, you have to go number of steps till you get a converge solution at the finest grid level.

Because if you want to correct the solution at the finest grid level, it is again a matrix solver which we will I have all the low frequency large wavelength components and we will take a lot of time. So, when you do few iterations in the fine grid level only correct the use the coarse grid level corrections and correct some of the fine grid levels. And then go to relaxation where the main iterative solution procedure is run on the coarser grid level.

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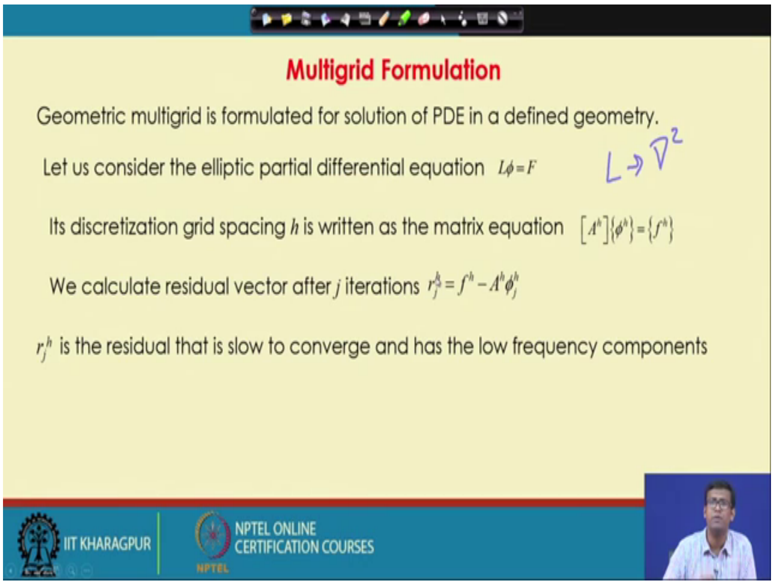

Geometric multigrid solution is formulated for a solution of PDE in different in a different geometry. Let us consider and elliptic partial differential equation L is elliptic equation like L is equal to say Nablus square something like that L is a. So, this is the PDE elliptic PDE, which we are trying to solve. Why elliptic PDE? Because boundary condition in all domains are defined then when you can solve elliptic PDE.

So, like a Laplace equation or a Poision's equation let us solve think we are solving an elliptic PDE. When this equation we convert it to into a matrix equation with digitization h grid spacing h, we get a matrix, if we change because we are all discussing about different grid levels if we change h to 2 h or go to a coarser level the matrix will be different. The solutions will also be a different vector solutions are member of rn that will be member of rn by 2 solutions are also different vectors.

And the right hand side matrix will vector will also be different, but for 1 particular grid spacing let this be the matrix solution which we try to solve using multi grid method. We will calculate the residual vector after say after jth iteration the residual vector is our j f h minus h phi j. This residual is slow to converge if I try to get the residual to be 0by something like a Jacobi or a conjugate gradient method, there is a slow to converge method because it has low frequency high wavelength components.

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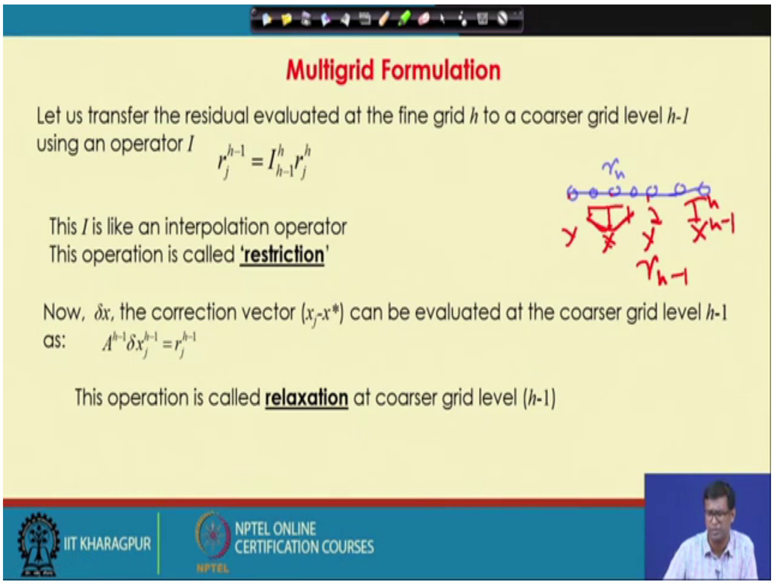

So, we transfer this residual to a grid level coarser grid level which is h minus 1 or twice h one more coarser level of that. That is the new residual is equal to the old residual multiplied with the transfer function. And this transfer function so what we are doing is that the residual is calculated in this mesh so this is rh. We are transferring this residual into a new mesh there one more point is needed. We are transferring this residual into a new mesh which is our h minus 1. So, there is requirement of transferring this to this, this to this, this to this, [noise see one idea is to directly map it, but directly mapping might give us wrong solution because this residual actually contains information from these two residuals also.

So, something like a trapezoidal interpolation, might be required and this transfer function is called I h h minus 1. This might be a trapezoidal interpolation function like, this residual is actually form saying these are the values this residual is actually formed using a trapezoidal interpolation of these something like this can be done.

So, transfer so do an interpolation to transfer the residual at the final level to a coarse level. I is like an interpolation operation, this operation is called restriction. From residual from final level transfer to coarser level is called a restriction operation. Now delta x the correction vector xj minus x star can be evaluated at the coarser level. So, solve the matrix equation at the coarser level; see what is the correction. This is called our relaxation operation at the coarser grid level or h minus 1 grid level.

Now so, once this is done, the solution is in a sense obtained in the coarse grid level; based on the grid solution that is coming from and a residual that is coming from the fine grid level. Again the residual at fine grid level which is being restricted to course grid level is not same as the residual which should have been evaluated, if you are just solving the equation at the coarse grid level. Because it contains information from the nearby points using to say we have used a trapezoidal rule. So, it has some of the high wavelength component of the error also. So, when we will solve it will get a solution which is not a right solution for the coarse grid level. But that that he has some because that has some components from the finer grid levels.

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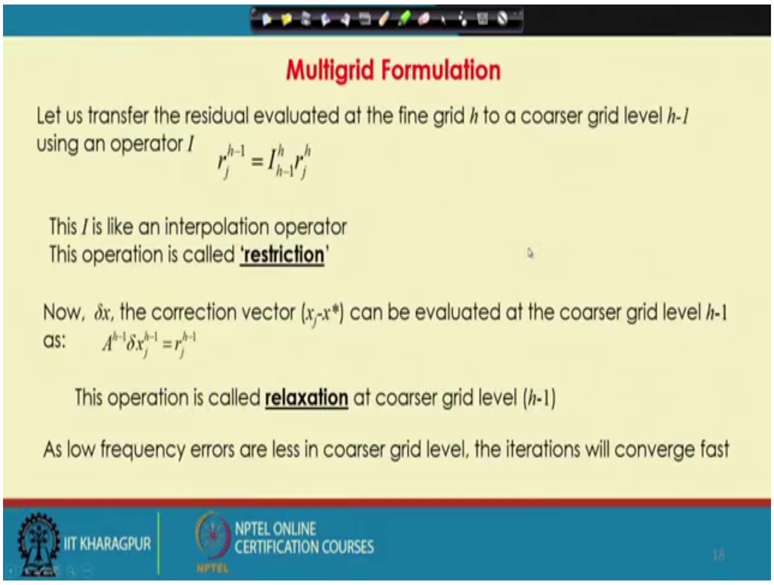

So, low frequency errors are less in coarser grid levels, these iterations will converge first because the low frequency error has been converted into high frequency error by increasing the grid size.

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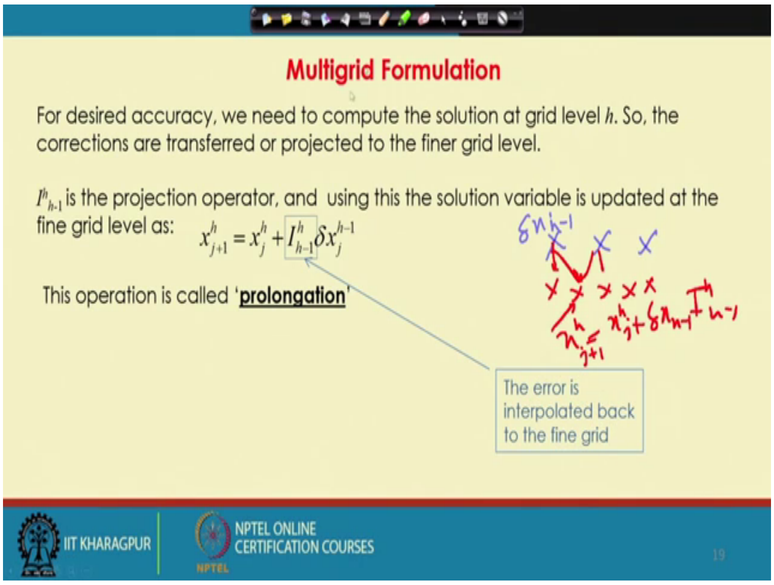

For desired accuracy we this will converge, but it will converge to a different solution and we also need to compute the solution at the finest grid level h. So, the corrections that we are we have computed in coarser grid level are to be transferred to the finer grid level I h h minus 1 is a projection operator, kind of inverse of the previous projection operator but not exactly inverse for numerical stability issues.

Is a projection operator which uses the solution variable which we will be used to update the solution variable at fine grid level, using the coarse grid level solution. So, this is calculated at coarse grid level use this and this projection operator and calculate at fine grid level. So, what will happen like you have these points and here you have calculated delta x h minus 1. And now this is your fine grid level and here you update x h is equal to x j plus 1 is equal to x h j plus delta x h minus 1 into I h h minus 1.

So, you get we get some of the effects from this to this particular correction value. So, the effect from here, here, here we will come to this particular point. And then we will update the solution at this point. So, solution is updated here using the corrections using the changes in the solution vector on the corrections, which is computed in coarse level and then interpolation like operation is done for that. This error is interpolated back to the fine grid level, this is called a prolongation operation. The error calculated at the fine grid, the correction calculated at the coarse grid level is transferred to the fine grid level so that we can update the solution in finer grid level. Will this give me a converged solution? No, because we have calculated something in fine grid level which has looked some into some of the low frequency oscillations in the course grid level, but has not resolved for all the high frequency oscillations here. So, you have to again do some correction at the fine grid level. Then this is again like an interpolation from coarse level to find level this I operator.

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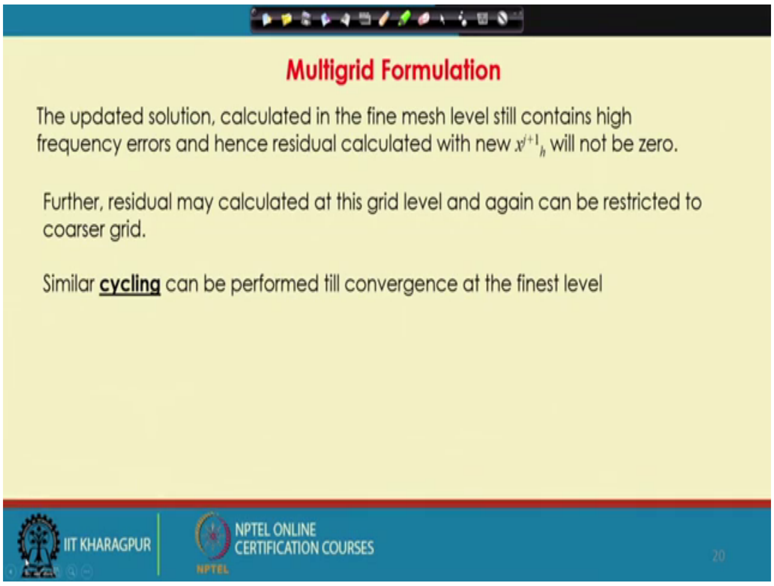

The updated solution calculated in the fine mesh levels still consider some of the high frequency errors. High frequency means in between the 2 grid levels the middle grid level we have used correction from I prolong grid correction factor from coarse grid level and put it here, which has not been resolved here.

So, it still contains some of the high frequency errors and the residual calculated with this will not be zero. So, the residual may be calculated at this grid level. And the same loop that this is transferred to the residual calculated to the fine grid level and transferred to the coarse grid level and you solve the equations to the coarse grid level, this may be followed. So, the residual may be calculated at this grid level and will again be restricted to the fine grid level.

And similar cycling from this is called a Cycling from coarse grid level to fine grid level, again going taking the calculating residual at fine grid level, restricting it to coarse grid level again solving here. This cycling operation can be converge can be performed till we get convergence at the finest grid level. That is residual at the finest grid level is close to zero.

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So, the steps what do we have discussed, that relaxation iterative solution procedure in a coarser grid level restricting. Then transferring so or rather if the first iterative solution some of iterative solution point procedural in a finer grid level, restricting it to coarser grid level. Again relaxing here they the orders may be changed so a converging getting the updated solution at coarser grid levels.

Calculating the correction factors at coarser grid levels and correcting the fine grid solution using then coarse grid corrections. And then cycling; they again going to coarser grid level and doing it the orders may change of relaxation and restriction because in a coarser grid after you do restrict in coarser grid level you have to relax. So, this is the steps of a multi grid procedure.

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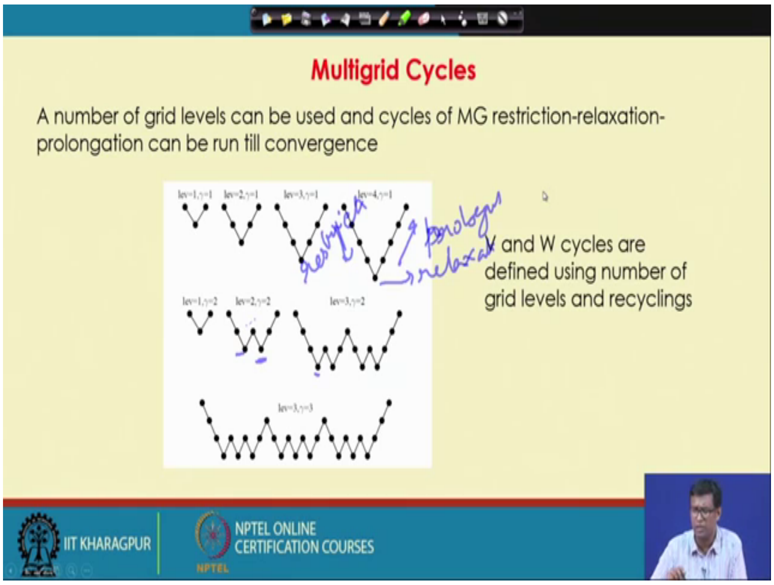

A number of grid levels can be used. And number of cycles of relaxation, restriction, prolongation, can be rather the first restriction, then relaxation, then prolongation, that is the cycle that can be run till convergence. There are very well known V and W cycles.

If we have one fine level you go to the coarse level and again come back to fine level and probably the solution has been already conversed that that is called a V cycle. Or you again go back to the coarse level and follow this. So, coming from here to here again coming from here to here this is called a V cycle.

There can be multiple levels and multiple grid levels and there can be different V cycles, but the cycling pattern is same you directly go from a fine level to the coarser level go to the coarsest level. Relax the solution here and this is the restriction operation. So, this will be the restriction and this will be the prolongation and this is the relaxation. There is another thing called W cycle in which you can do relaxation this is a relaxation this is a relaxation again this is a relaxation these this is not a relaxation.

You go to a come to a coarser mesh. Coarsest mesh do relaxation solve the matrix equation then correct it a to a middle level coarse mesh true prolongation. And then you again restrict the solution to the coarser level to the coarsest level do a relaxation this is the W cycle. There are multiple W cycles with different grid levels and different cycling.

So, these are two commonly known cycles V and W cycles are defined using grid levels and recycling. And the number of grid levels used and number of recycling's.

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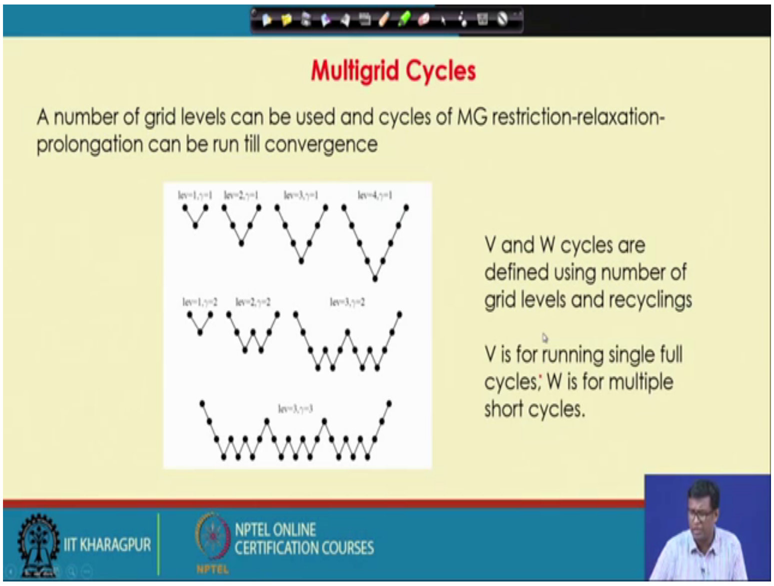

V is for running single full cycle from phi into coarse again phi, again phi to coarse phi. And W is for multiple short cycles from finer to fine to coarser. And again coming back to fine again coming back to little more fine and again coming back to coarse these are called W cycles.

Both, both these cycles are used depending the applications and the type of matrix, the computational requirements, etcetera. Both these cycles can be used for different purpose. If I can define a W cycle or some cycle like this in which going through one particular cycle will give me the converge solution. I do not have to recycle it several times.

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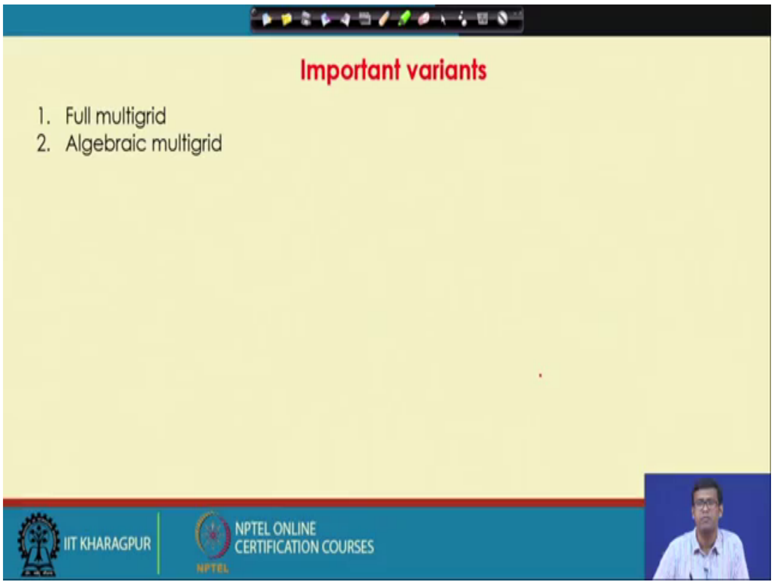

This is called a full multi grid method which is little more complex method, we are not discussing here. And the multigrid method is usually based on the geometric what you have discussed is geometric multigrid method geometric coarsening and fining of the mesh. But especially for unstructured mesh and there can be different matrices also which is not coming out of a Laplace equation or an Elliptic equation in a different geometry there can be different matrices where geometric make multigrid cannot be applied.

Here what can be done is an algebraic multigrid procedure; that means, take the main matrix class few of the solution variables to one particular solution variable. Like make a block and make it one particular solution variable and solve it solve the equations there and then prolong get it to the actual solution variable.

So, there is a idea of algebraic multigrid this is mathematically much more complex. But many solvers there are been lot of developments using algebraic multigrid also. Here you do not need to know about the geometry directly.

There are many commercial and open source software's which already you have coded the algebraic multigrid. And you can use it over your matrix solver over the general matrix solver you are using. So, this is another way to look into the multi grid procedure.

However, we are not discussing the details of the multi grid procedure due to short of time because with this is a course not on multigrid exclusively, this is the course on matrix solver. So, I try to give some introductory concepts on matrix solve multigrid method, which is useful for rethinking matrix solversin which they can be redesigning the matrix solvers in our direction that they can give us first solutions. So, these two techniques we have. reviewed on a matrix solver. So, one is the multigrid method another is preconditioning technique where given a matrix solver given the particular for a particular matrix which is a large matrix and slow to converge.

We can use this techniques over a given matrix solver and get faster solution ok. So, with this I guess we come to the conclusion of the course, we started with discussing the basic properties of a matrix, then discussed about direct solvers like gauss elimination, LU TDMA type of solvers. Looked into the cases where there are infinite solution and null space solutions are required to get a solution.

Where there is no you know solution and a based estimate can be obtained from normal solutions of these. What we looked into the direct solver method. And then we came to iterative solver looked into a group of solvers called basic iterative methods. Then came to projection based methods and Krylov subspace based methods and look also looked into the block matrix solvers.

As well as the parallelization strategies which probably can be craved out of the block matrix solvers. And then we looked into the methods for accelerating a matrix solver given a particular matrix which is slow to converge which is pre conditioner and another method is matrix solvers.

This kind of summarizes the course and it has been a probably it will be of use to a larger community, who will be using matrix solvers for their research and academic purpose. Hope I try to convey the main key points of matrix solvers. And hope this will be of use to this larger scientific computation community.

Thank you all.