Matrix Solvers
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Lecture – 06
Gauss Elimination

Welcome. Today we will discuss one of the most popular methods of Matrix Solver, it is probably little inappropriate to say it most popular method, undoubtedly this is most popular method of matrix solution algorithms; which is Gauss elimination. And this is not only a very popular matrix solution algorithm with use in several applications; this is also the earliest documented methodology for solving system of linear equations.

And this credit goes to Chinese mathematicians who documented it in 2nd century which is 109 AD and later Sir Isaac Newton used this method while solving a system of linear equations. And ah; however, this has been formalized by Gauss in 1910 using the notations and we will use almost similar notations this date and later this is has been heavily used in computer programs.

Even today lot of applications involve solution of matrix equations using Gauss elimination method and also it is probably the first demonstrated method in any matrix algebra class for solving matrix solve equations.

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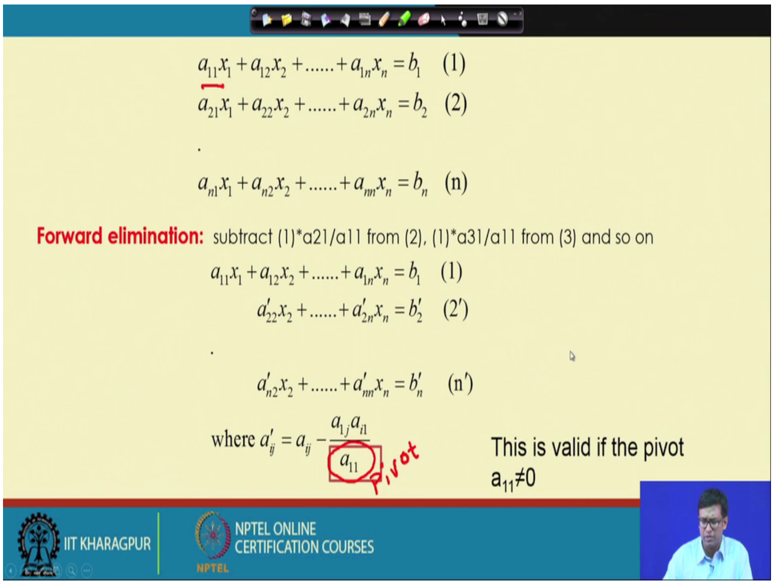

So, this essentially starts with a system of linear equation with multiple variables and then the idea is that you subtract the first equation multiplied with some constant from the second equation and so forth. So, that the first term or the terms containing x 1 is eliminated from the equations and this is county called forward elimination.

The idea is subtract first equation multiplied by a 21 by a 11 from second equation so, the first term of the second equation will be eliminated. Similarly, do it for the third equation and so on. So, a 11 will be the term basically which will divide all the coefficients which will which will be divided by like which will come in denominator for all the equations when subtracted from them and a 11 is called the pivot of this equation.

Now, where if you see what happens when we do this step; the first term containing x 1 is eliminated from all the equations and we get a new sets of equation. However, the solution of both the equations will remain same because they are essentially same equations or similar equations. And coefficients because this has been subtracted from the from any other row the coefficients change in magnitude in any of the rows and this term a 11 is called a pivot here pivot p i v o t.

In a sense that the equation has been transformed based on a 11 and if a 11 is 0 this step is not valid. So, one very important step is that the coefficient a 11 has to be non-zero while doing the forward elimination and this is valid if the pivot a 11 is non-zero.

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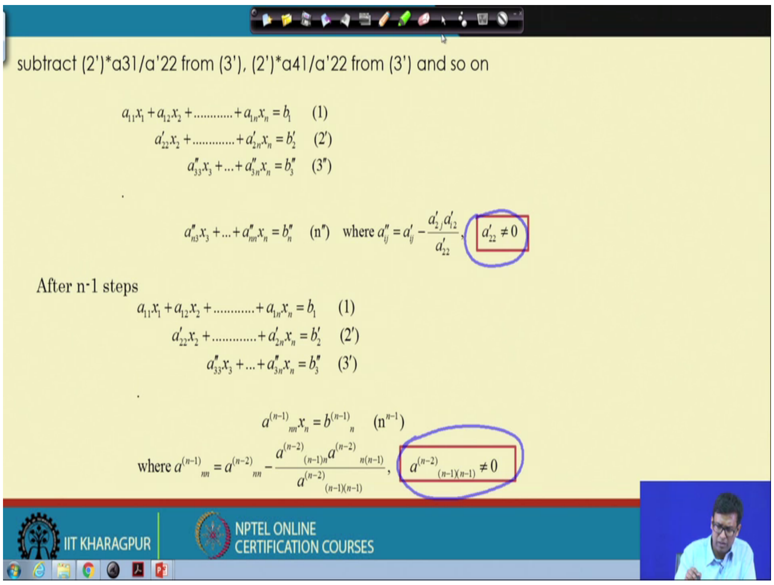

Now, we get a new sets of equation first equation remains essentially same the other equations have been changed and we carry on this steps. The next step is take the previous equations or just 1 second; the next step is you have this set of equation. Now, you choose a 22 as the pivot and divide all the equations by a multiplier of this equation; which is this equation divided by a 22 into say a 33 prime a 44 prime etcetera and eliminate the x 2 term.

So, you while doing this the x 2 term is eliminated from the equation and the equations are re-structured from equation 3 which is a new equation now and each term has been changed accordingly. And if you carry on up to n minus 1 steps at the nth column or at the nth equation after n minus steps operation here, the terms containing x 1 x 2 x 3 x 4 x n minus 1 have been vanished.

Therefore, the new equation which is representing the nth equation has only one unknown and this is one equation. So, we can directly serve for this equation and we essentially get a triangular system of equations. The important thing is that in each equation we using certain pivot terms and the pivot terms is a first term which is left in the in the equation which is being subtracted from all the equations. If the pivot term is 0 this step will fail. So, one important criteria for Gauss elimination is that the pivot cannot be 0.

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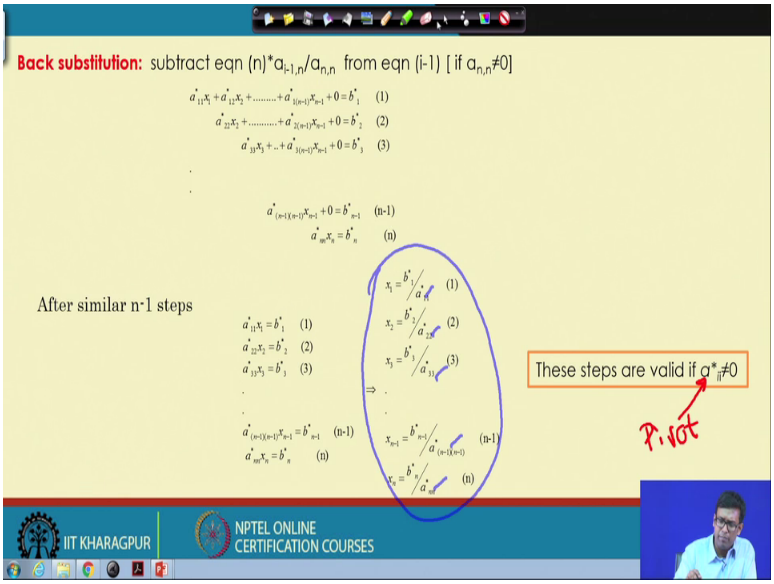

Now, when we did a system of equation like this we do a back substitution and what will we do in the back substitution; the equation we got directly gives me in the last equation directly gives me the value of x n. Now, we will multiply this equation with appropriate factors and subtract it from the previous equation so, that x n term is eliminated from all the equations. And we subtract equation n multiplied by a y minus 1 n a n n from i minus 1 and we get a new sets of equation where nth equation remains same which is one equation one variable.

And in all other equation we when get the x n nth term vanished. So, we keep on so, what happens n minus 1 equation now, only have one component in the left hand side and one component in the right hand side; n minus 2 equation will have two components left inside one in the right hand side and we perform similarly n minus 1 steps like now we will try to eliminate x n minus 1-th term from n minus 2 equation to first equation.

And once we will do that we will do for n minus 2-th term finally, we will do for second term. So, all these terms will be eliminated we will eliminate the so, once all these terms are eliminated, we will eliminate the second term from the first equation and it will also give one equation of one variables. So, after similarly n minus 1 steps, we will get equations which has the coefficients for the particular element on that particular row and the modified right hand side values and now a single step can give me the solutions.

So, and this all these steps will be valid if the diagonal element a ii is 0. For example if a nn is non-zero is non-zero, if a nn is non-zero then only we can perform the first back substitution step. The next back substitution step can be done is the a star n minus 1, the star actually says the present term the present value of the term after all the substitutions.

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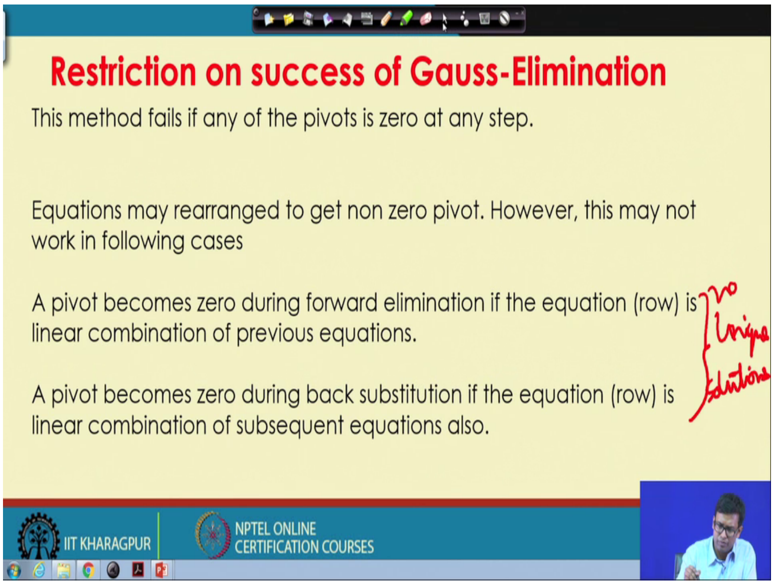

So, a star n minus 1 n minus 1 if sorry if this is non-zero then we can perform elimination of n minus 1-th term from the previous equation.

Therefore, all these steps will be again valid if the pivot here which is the pivot in the back substitution is the diagonal component a ii is non-zero. So, essentially a Gauss elimination we will go for forward substitution and backward substitution and will finally, give me the solution of the equations.

If at any stage the pivot is non-zero, if at any stage is a pivot is 0 we cannot perform the operation. Even when we are finishing the operations this all these terms are divided by the pivot terms only in order to get the x value, the all the right hand side terms are divided by the pivot terms and they also have to be non-zero. Non-zero pivots are very important part of a Gauss elimination procedure.

And that gives us what we call the restriction of the success of Gauss elimination. This method fails if any of the pivots is zero at any of the steps. So, throughout we have to get the diagonal terms non-zero or the terms which will be used in division must be non-zero. In case we get some zero pivot we can will get an example later we can re arrange the equations to get non-zero pivots.

However, there can be certain cases where this rearranging may not work and this cases are a pivot may become zero or become zero during forward elimination if that particular equation is a linear combination of previous equations. And then if this thing happens then we cannot rearrange the equation so, that the pivot becomes non-zero and it can get it can be zero also during back substitution if the row is linear combination of the subsequent rows.

So, in both of the cases there is no unique solution where we have seen this earlier that if a is not a full rank matrix or some rows of a are linear combination of some other rows then we will not get a unique solution of x is equal to b. And if there is no unique solution then Gauss elimination can handle this problem.

So, it is important that we should not get zero pivots, even in case of getting zero pivot we can rearrange the equations; we will see it in one of the examples. But even if the rearrangement does not give us a zero pivot; that means, that there is some linear dependency between the rows, either in the forward elimination procedure; that means, this particular row is linear combination of the previous rows or in the backward substitution process where this particular row is a linear combination of the later rows or subsequent rows.

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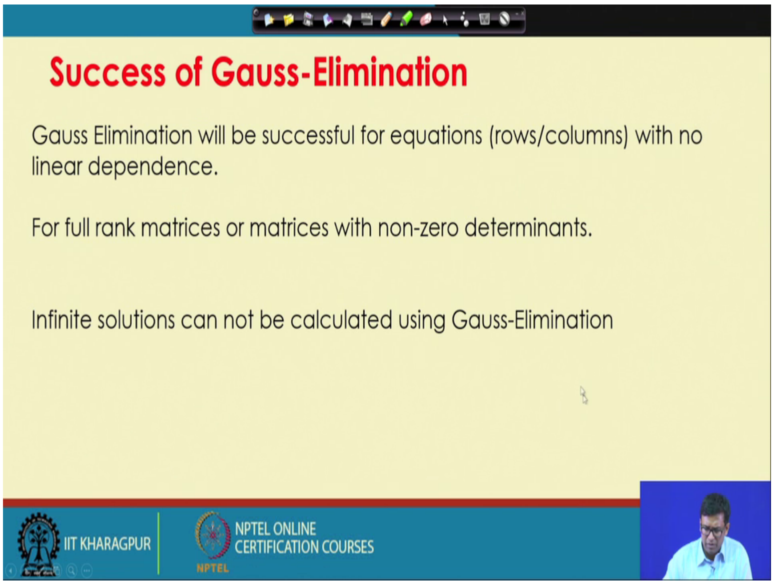

So, the success of Gauss elimination can be defined as Gauss elimination will be successful for equations with no linear independence. Linearly independent rows or columns or equations which means a x is equal to b a will be a full rank matrix Gauss elimination will be successful or for full rank matrices with non-zero determinants Gauss elimination will be successful.

No solution case really we cannot use any method to get exact solution in no solution case ok, but even if we have dependent rows columns or dependent equations we can get infinite solution. However, Gauss eliminations cannot calculate the infinite solution cases. So, this is one restriction of Gauss elimination method that it is only applicable when the equation system has unique solution or a is a full rank matrix or all the equations or all the rows of a or columns of a are linearly independent.

So, you should see some of the examples and also see some of the cases when pivots are being zero and the equations are linearly dependent.

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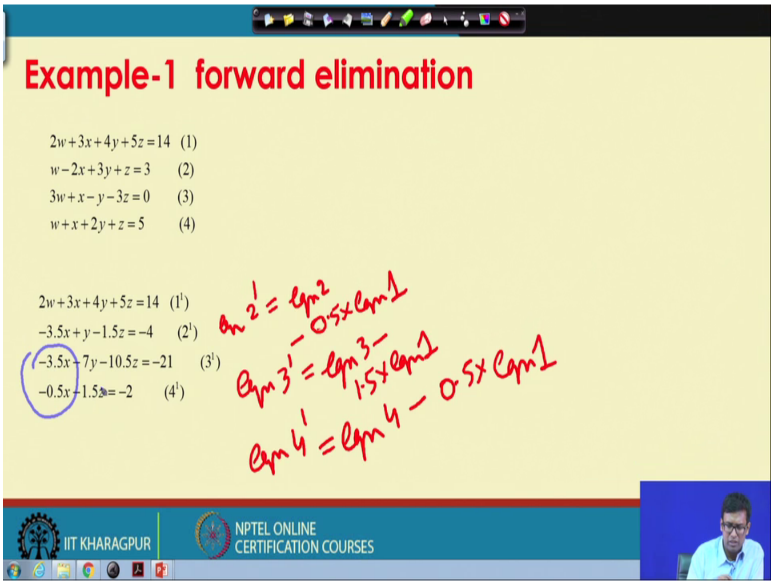

So, the first example is a straight forward case when we have it is a full rank matrix with the equations are linearly independent and we should get unique solution for the matrices. We have a set of equation with four variables (Refer Time: 13:16) that can be several methods to solve it and it is a very small equations, but we are using Gauss elimination for a demonstration.

So, one thing to mention here and we will also come later that Gauss elimination is really not designed for 4 by 4 equation 4 unknown or 10 equation 10 unknown systems. The idea is to solve large matrices for which and that is why Gauss elimination is so popular and also being devised in second century after Christ, still it is in use because Gauss elimination can be programmed for large equation systems. Say 10,000 unknowns 10,000 variables you can solve it using Gauss elimination through a computer programming and it is very simple to write a computer program for Gauss elimination.

However, if you see our first example there are 4 equations and 4 unknowns. So, the first step will be eliminating the first column of this equation. First equation will be untouched second, third, fourth are or this particular column has to be eliminated. So, what we will do we will multiply first equation by half and subtract from second equation; first equation by 3 by 2 and subtract from second equation. Again multiply first equation by half and subtract from second equation and that is how we should get the next equation system; the transformed equation system.

So, it comes as like equation 2 prime is equal to equation 2 minus 0 point sorry equation 2 equation 2 prime comes as equation 2 minus 0.5 into equation 1; now you can check that. Similarly, equation 3 modified version first modified version comes as the old equation 3 minus 1.5. So, this is 2 and 3 3 by 2 into equation 1 and equation 4 prime is equal to equation 4 minus again 0.5 into equation 1. So, we get the present set of equation.

Now, the next target will be in this set of equation I will subtract some multiplier with equation 2 from equation 3 and equation 4. So, that this terms are eliminated and we can check that that equation 3 minus equation 2 1 into equation 2 will give me point 3.5 eliminated, equation 4 minus 1 7th of equation 2 will give me this term eliminated.

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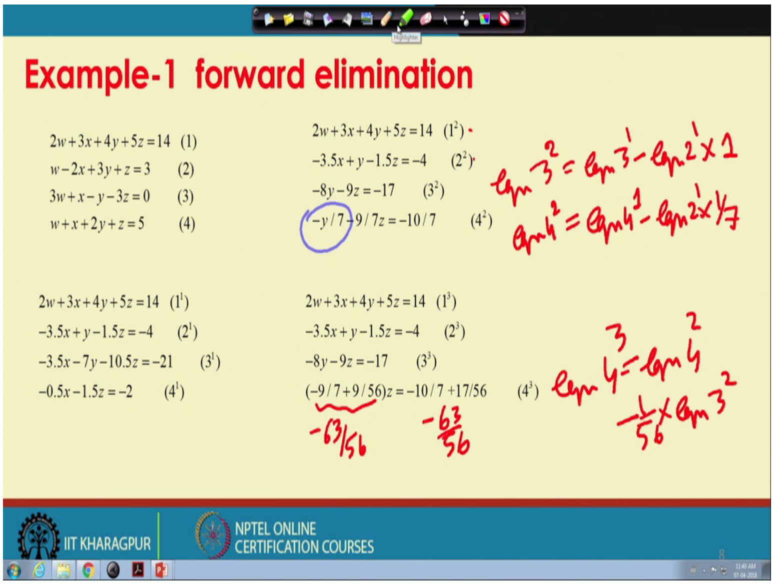

So, so, the next state is that the x terms are eliminated and also we get it like equation 3 at the second stage, equation 2 is untouched, equation 3 at the second stage is equal to equation 3 after the first step minus equation 2, after the first step into simply 1 is just they have the same coefficients. Similarly, equation 4 after the second stage is equal to equation 4, after the first step minus equation 2, after the first step which is after the first and second step these equations remain same basically into 1 by 7.

And then we go for the next step where we had we have to eliminate this particular term. So, we write equation 4 after the third step is equal to equation 4 after the second step minus now 8 and this is 1 1 by 7. So, it has to be divided by 1 by 56, 1 by 56 into equation 3 after the second step ok.

So, here if you see this term is minus 63 by 56 minus 63 by 56. So, one way is that we can directly write z z is equal to 1 and keep on substituting it. However, this is done because Gauss elimination is designed for computer programs, we at least today's version of Gauss elimination we really do not write z is equal to 1 here rather do a backward substitution in the more formalized way.

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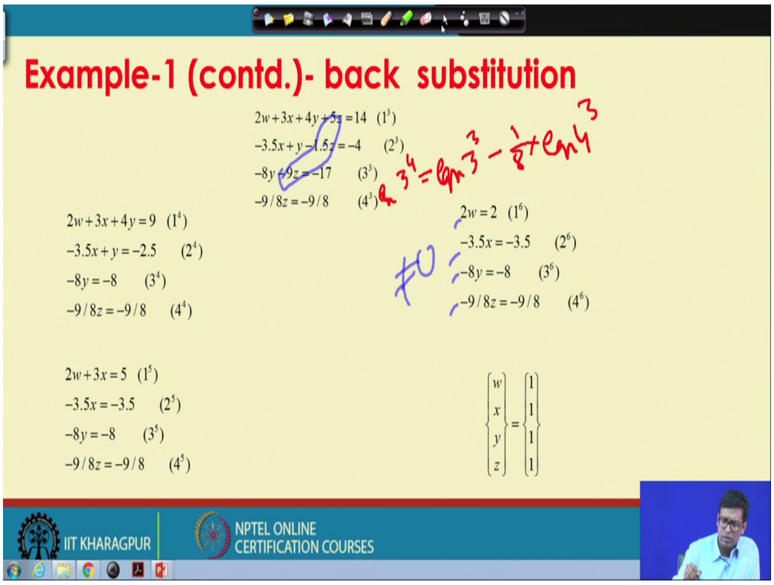

So, what we will do here is that we last we had this set of equation right 63 by 56 is 9 by 8 and now we will start eliminating the z term from the previous equations. And that will be like multiply this by we have to eliminate this z terms from the previous equations. So, multiply fourth equation by 1 by 8 and subtract from the third equation, multiply fourth equation by the relevant required coefficient and subtract from the second equation and so on.

So, so, one step will be like 3 to the power 4 the new equation 3 to the power 4 will be equation 3 fourths after fourth step equation 3 after fourth step not 3 to the power 4, will be equation 3 after third step minus 1 by 8 into equation 4 after the third step and so on we will get the next set of equation from which all the z terms are eliminated.

Now, one thing important is that the during any of this step the coefficient of z cannot be 0 it has to be 9 by 8. Similarly, in the previous step we have also seen the coefficients of x and w and y were not 0, they are non-zero that is why we can divide it for example, as this is this coefficient is 9 by 8 so. So, we can divide this coefficient this by if this is 0 there is no point in doing this substitution. So, it is important to have these terms non-zero.

Similarly, now if y is equal this term is 0 we could not eliminate the y terms from the previous equation as this is non-zero this is possible. So, this is the importance of pivot term. So, we go on a step like this and then we perform another step and then finally, we will see it is only one variable in the left hand side and the right hand side values. And solution is now obtained by w is equal to 2 by 2 x is equal to minus 3.5 by 3.5, y is equal to minus 8 by 8, z is equal to minus 9 by 8 by 9 by 8.

And also at this stage also none of these terms are non-zero, all these terms are non-zero none of these terms are 0. If any of this terms is 0 we cannot solve for this particular equation and we finally, get w is equal to w x y z all are equal to 1.

We will see some other examples when these terms are coming to be 0, but before that it is also important to check how many operations we are doing in or during this particular calculation. And this is again important or we will look into cost of computation. The cost of computation is again important due to the fact that these processes will be followed for a large number of equations. Here we have seen that 6 stage steps are needed and each step we are doing certain multiplication subtraction etcetera. So, there are 4 equation 4 unknowns therefore, 6 steps at least needed to get the 1 equation like de-coupled forms of the equation and then one more step is needed to get x y w x y z by dividing them by the pivots.

So, 7 steps are needed and in each step we are doing lot of calculation computation multiplication subtraction etcetera. As these methods are to be designed for a computer program it is important to know how many steps there following in total, how many arithmetic calculation steps there following or what we call that floating point operations.

How many floating point operations are being carried away by a Gauss elimination process when we are trying to solve a large equation system like 10 to the power 5 equations. What is the total number of equation arithmetic steps?

Because we know that computers has certain floating point per floating point operation per second speed. So, we can estimate that what will be the time taken by the computer for solving this problems.

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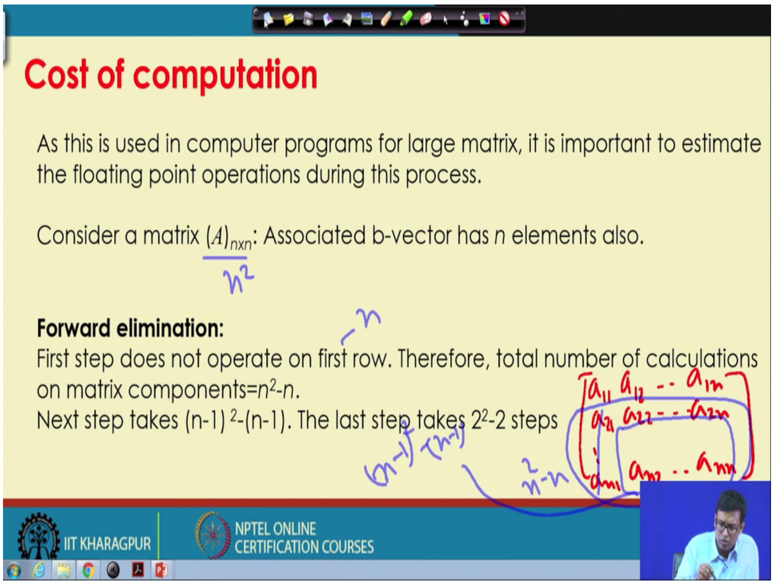

So, we go to the next slide which is the cost of computation as this is used in computer programs for large matrix, it is important to see the floating point operations during this processes.

So, I we consider a matrix A matrix and associated b vector. Now, there are when we are doing certain operation on each rows of A matrix, the same operation is also done on the rows of b vector. So, if you see in forward elimination the first step does not operate on first row, first step operates on the subsequent rows only.

So, if there are n by n matrix first step will do to, the total number of calculation in first step will be each row element will be operated. There will be some subtraction addition multiplication on each row element. So, total number of calculations will be total n square elements in n into n matrix, total number of elements is n square and first row will have n element. So, n square minus n n square will be the operations in the first row.

The next step will be in a in n minus 1 because first column has been eliminated, we only have n minus 1 n minus 1 sub matrix on which this operation will takes place. Like if I see this is a 11 a 12 up to a 1n a 21 a 22 up to a 2n a n1 a n2 up to a nn. At the first step of Gauss elimination we will do operation only on these elements, which is n square minus n. At the next step of Gauss elimination we will do operation only on not this row will be eliminated after the first step, after the first step this row is eliminated.

So, we will do operations only on this elements which is n minus, I will say n minus 1 whole square minus n minus 1.

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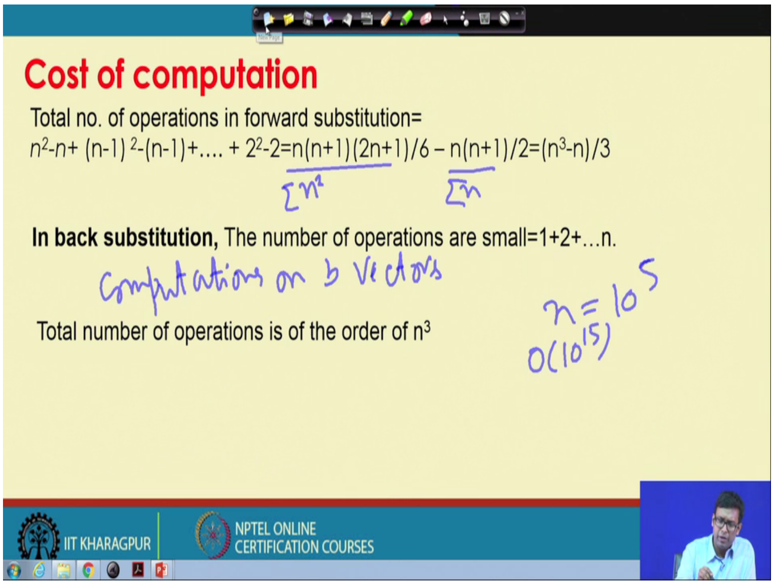

So, in total we will the total number of operations will be n square minus n plus n minus 1 whole square minus n minus 1 up to plus row there will be 2 squared minus 2 operation. And if we sum it up this is n square sum of n square plus n minus 1 whole square plus n minus 2 whole square plus 2 square plus 1 minus n minus n minus 1 up to minus 2 minus 1. So, this is sum of this is basically sum of n square and then this is sum of n first n natural numbers and this is sum of n natural numbers which is n cube minus n by 3.

And in back substitution the number of operations are actually small and there will be only n operations in the back substitutions, also there will be computations on b vectors. However, if I see the total number of operation if n is large enough the total number of operation will be scaled by the term n cube. For example, if I have n is equal to 10 to the power 5, the total number of operation will be of the order of 10 to the power 15, may be 10 to the power 15 by 3 plus something like that.

So, we get in a sense a very large number of operations and the total number of operations will be again function of the order of n cube.

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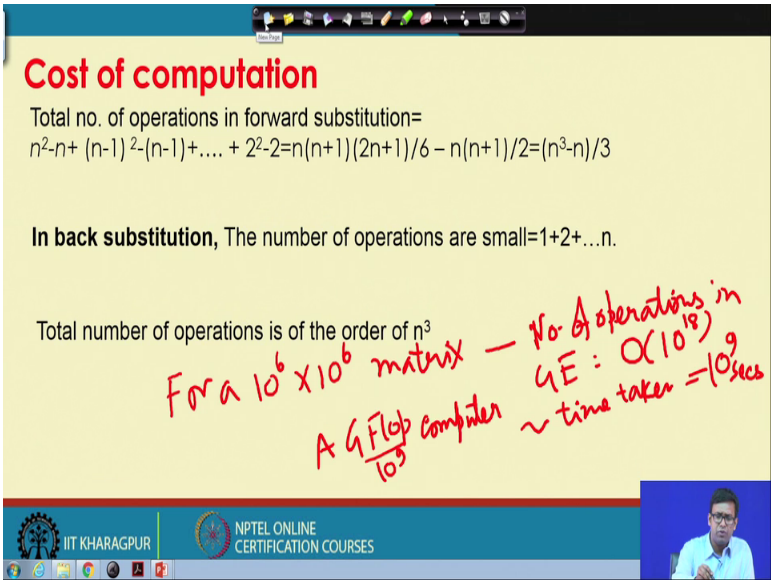

So, for a for a 10 to the power 6 into 10 to the power 6 matrix which is a large matrix and in many cases especially when we are trying to solve some physical equations; like we are trying to do a weather forecast this things used something like matrix equations also.

I want to see how a hurricane hits our east coast and how does it move, what will be the amount of destruction I want to estimate from the path prediction of the hurricane. So, that needs a large matrix equation to be solved and we will do some examples or we will see how physical equations can be translated into matrix equation. So, that will be a large matrix equation for example, it is a 10 to the power 6 into 10 to the power 6 matrix equations. Then number of operations in Gauss elimination. In Gauss elimination will be the will be of the order of 10 to the power 6 whole cube which is 10 to the power 18.

For example now, if we take a gigaflop computer which is a reasonably good computer which keeps a gigaflop operations per second and that will take time taken. So, that there are also 10 to the power 18 floating point operations by a computer which does 10 to the power 9 operations per second. So, the total time will be 10 to the power 18 by 10 to the power 9 will be 10 to the power 9 seconds just to solve a 10 to the power 6 into 10 to the power 6 matrix, which is a very high time.

And that is why for large systems Gauss elimination is kind of restricted due to the computer time taken for execution of the entire program because it is associated with a large number of calculation steps or floating point steps and that can restrict the calculations also.

However, for smaller system like 1000 by 1000 system; Gauss elimination gives can be done in a reasonably first aid time. In a gigaflop computer it will be a it will be only 1 second in which we will get the solution.

So, in next class we will see few other examples of Gauss elimination and specially when the pivots the cases in which the pivots become 0 and how can we tackle these cases for certain matrices, when the pivot is 0. And what are the cases when there is actually no unique solution and pivot and we will see the pivot is being 0 and there is no remedy for that.

Thanks.