Matrix Solvers
Prof. Somnath Roy
Department of Mechanical Engineering
Indian Institute of Technology, Kharagpur

Lecture - 40
Residue Norm and Minimum Residual Algorithm

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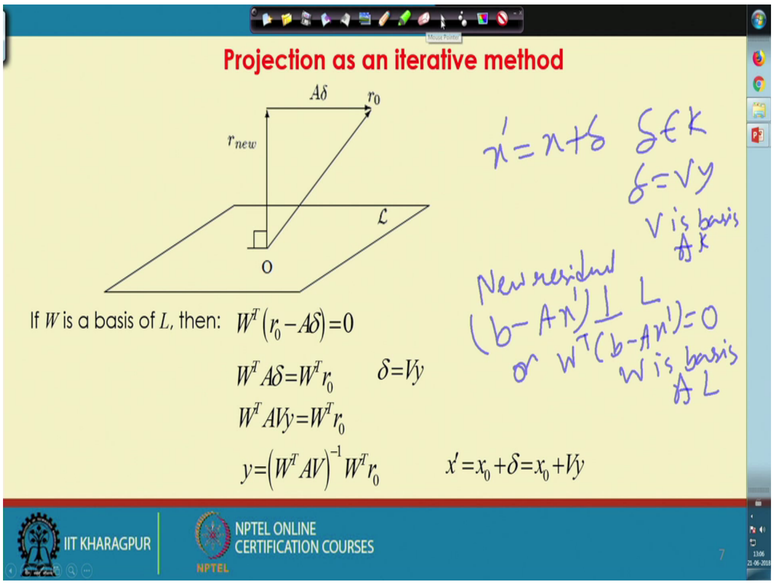

Hi. So, in last class, we was we were discussing stiff general projection methods. The idea is that that there is a projection method for solving the equation A x is equal to b. In that scheme, we are updating x as x is equal to x x prime is equal to x plus delta, so that delta belongs to certain vector space k or delta is equal to V y, where V is basis of. And new residual b minus A x prime is orthogonal to L or W transpose b minus A x prime is equal to 0, where W is basis of L.

So, we have two subspaces K and L, x is being updated in a one particular subspace. The condition for this update is that that the new residual after the update the residual will obtained is orthogonal to another space L. Now, what we will do, we will make few selections of W and V or K and L, how these spaces are, and see what type of iterative methods we are getting, and we will also check whether this iterative methods converge to the right result.

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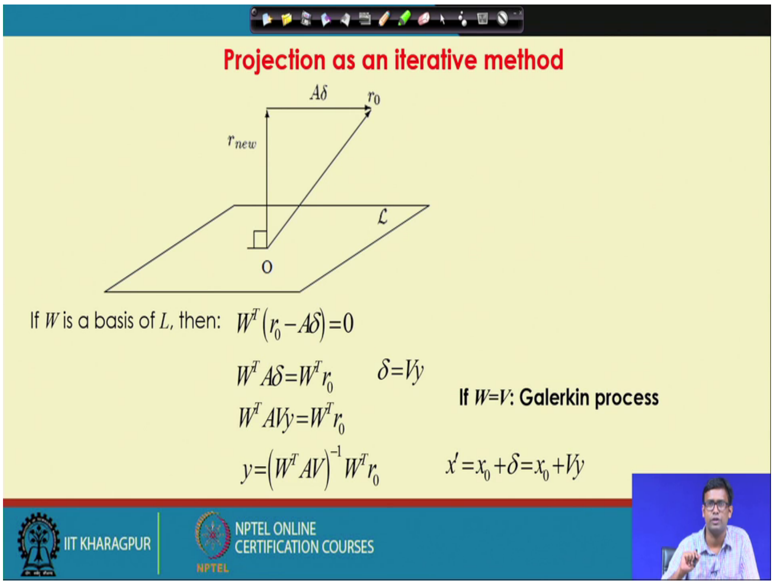

If W is equal to V, remember here if W is equal to V, we call this as a Galerkin process. And this is also an orthogonal projection, when both these spaces are same.

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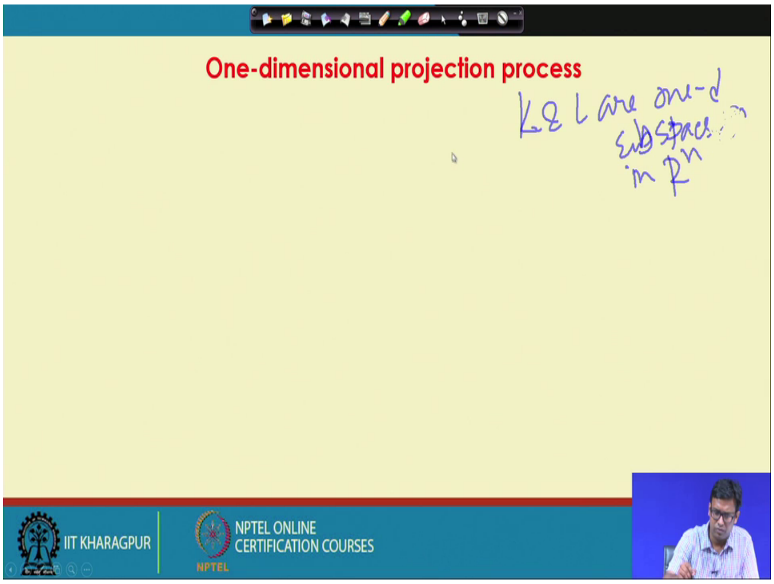

Now, we look into one-dimensional projection process. In a sense that K and L K and L are one-d spaces subspaces K and L are one-d subspaces in R n. So, what we call a one-dimensional projection process, along the by meaning K and L will consider only one vector; K is one particular vector, L is one particular vector. So, x is updated along one particular vector, it is not along in a vector space, it is just only one vector along this vector, x is updated. So, update of x will be like Steepest Descent algorithm. Update of x is equal to x 0 plus alpha into minus grad J. So, minus grad J is one particular vector.

So, here we will think that the space K is one particular vector, it consists only one vector, and update will be certain this vector multiplied with certain magnitude. And L is also a one-d space that means, L was also one particular vector, they can be one-d space in a in R n with multiple component with number of components with 2, 3, 4, n n components, they are subspaces of R n, however they are single vectors.

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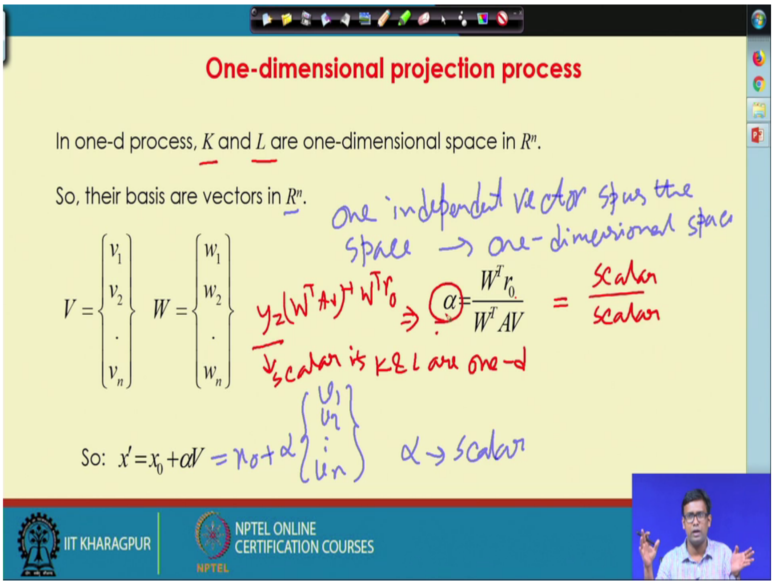

So, K and L are one-dimensional space in R n. So, their basis what is the basis for a one-dimensional space, the dimension of one-dimensional space is one, so single vector in that space is a basis for that. A line in 3D a straight line is a one-d vector; in 2D also straight line is a one-d vector, so it is only a single vector, which is needed to describe that space, which is in linearly independent and spanning that spans over that space.

So, the basis vectors in R n will be single vector v 1, v 2, v n, for example, W is equal to W 1 to W n that is one independent vector spans the space, and then it is called a one-dimensional space. However, they can be in multi-dimensional real coordinate space R n, so they have multiple components, but the basis is a single vector of these spaces.

So, we can write x prime is equal to x 0 plus alpha V, where this is will be x 0 plus alpha into v 1, v 2. So, alpha is a scalar, earlier y was coming to be a vector, but alpha is only one component, because it is a single vector, this is only a magnification or skewing of the stretching of a single vector. So, alpha is a single is a single scalar quantity, x prime is equal to x 0 plus alpha into a single vector.

And we can write that alpha is equal to W transpose r 0 by W transpose AV. If we can see the previous example, where y is equal to if we go back to the previous slide y is equal to again check from the previous slide, y is equal to W transpose AV inverse W transpose r 0, so that will give me alpha is W transpose r 0 by W transpose AV.

Now, what is r 0, r 0 is a vector single vector is a column vector. What is W transpose, W transpose will be now a row vector, and what is so, this is the W transpose r 0 will be again a scalar right. So, this will have 1 into n, and this is n into 1, so this will be a scalar. Similarly, V is a vector, so V is a single vector. So, AV will be a single vector W transpose AV will again be a single vector. So, this is basically scalar by scalar.

So, if we consider the vector spaces K and L to be one-dimensional spaces, then y comes out to be a scalar, if K and L are one-d. And we represent y by the term alpha. If we start with consider or if we assume one-dimensional spaces of K and L, K and L we dis discuss they are m-dimensional subspace of R n. So, each vector in K and L has n components, but there can be less than n less than equal to n independent vectors, which is or m independent vectors, which is spending over K and L.

Now, this choice of K and L is in my hand, this is the way and to define the iterative process. The theorem is that there is an there is an iterative process by which we can find an approximate solution using Petrov using Petrov-Galerkin condition, or if W is equal to V that is Galerkin condition, using this condition that is the solution vector x is updated along one particular space K, and it is updated one that using the condition that the new residual b minus A x updated, new residual is orthogonal to another space L.

Now, if I have chosen K and L and k what is K and L, we have not mentioned in the theorem, we said that it is possible to have iterative method like that, so K and L is depends on our choices. So, we choose we have chosen K and L to be one-dimensional vectors here, the exact form we have not chosen, we have just seen that K has a single vector. The entire base K is along a particular is a basically a line along a particular vector direction, so it has a single basis vector. L is also along a particular line, L has a single basis vector, and that gives us that y or the update of x will be a scalar multiplied by the basis vector along K, L. So, x will be x x tilde will be x 0 plus alpha V, and this alpha comes out as W transpose r 0 by W transpose AV.

Now, with choice of W and V, I can get different values of alpha and can define different convert iterative processes. There is a beauty of this method, it started with a very abstract thing. And then, we came out little too little substantial idea that we can choose few one-d vectors as W and V, and then try to see, if we can get a iterative method out of it.

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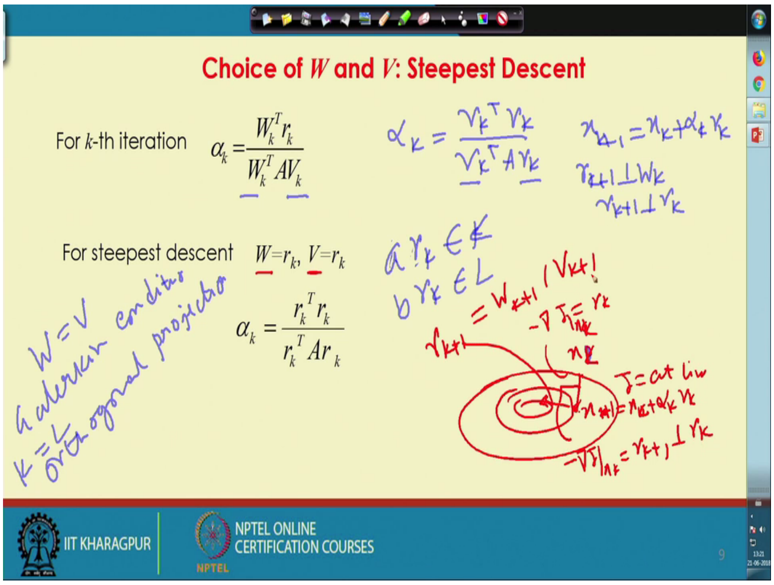

So, for k-th iteration, alpha k is equal to W k transpose r k by W k transpose AV AV k. Let us quickly check with the Steepest Descent method right, and that gave us alpha k is equal to r k transpose r k by r k transpose A r k. So, if we compare them, V k is r k along r k that means, I can write x k plus 1 is equal to x k plus alpha k r k. And W k is along r k, so the new residual is perpendicular to W k or we can write r k plus 1 is perpendicular to r k, and that is exactly what do you get in Steepest Descent method.

If I draw pictorially, this is J is equal to constant line, and I take a this is my x 0, so I take a minus gradient of J is equal to r k, and I get somewhere here x k is equal to x 0 plus alpha k r k, and I get the new search direction or new residual here, which is again minus grad J at so, this is at x 0 minus grad J at x k is equal to r k plus 1. And this direction so, these two are perpendicular to each other, which is perpendicular to r k.

So, by choosing W k is equal to r k by choosing W k is equal to r k and V V is equal to W is equal to r k and V is equal to r k, we can get the Steepest Descent algorithm. So, started with something very abstract, but with right choices we had choice over W and k, k and m. So, this basically K any vector any vector that belongs to K you should you should write it in a different way any vector say a into r k belongs to K, also any vector b into r k belongs to L.

So, I have and as K and L are same W and V are same, so we call it to be W is equal to V, this satisfies Galerkin condition K is equal to L that means, this is orthogonal projection. So, by choice of W and V or K and L both to be r k, it is a Galerkin satisfies Galerkin condition or it is an orthogonal projection method, we can find out the Steepest Descent method.

What is the Galerkin condition here, x will be updated along a particular line. So, we can write along a particular line, which r k particular vector r k residual at that state. And so, this is not I have give, this as this is x k is (Refer Time: 14:50) this should be changed here. This is x k, this is x k and this is x x k, this is x k plus 1; x is updated along a particular line x k, so that the new residual r k plus 1 x is updated to be r k plus 1 is perpendicular to that line only on r k. So, this gives us Steepest Descent algorithm.

Another point, while we reach this is that that if I look into W and V, they are not something constant irrespective of the iterations with different each iteration W and V is changing r k is changing. So, for the new iteration, this will be the r k plus 1 will be the W and V. This is this is r k plus 1, this will be W k plus 1 or V k plus 1 with the and the new iteration, W and V will change, so r k will change in the new iteration. So, each iteration W and V changes, and alpha k also changes, however we get a new such direction, therefore x k changes; x k changes along different direction in each different iteration also.

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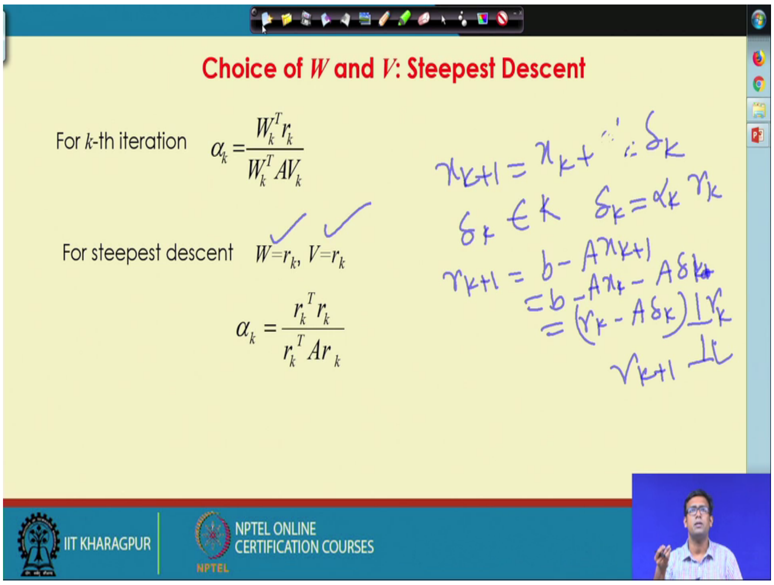

So, again what we get here is that that x k plus 1 is equal to x k plus alpha k delta k, where delta k belongs to the subspace K or delta k is a sorry x k plus delta k I am sorry, x k plus 1 is equal to x k plus delta k or delta k is equal to alpha k r k. And r k plus 1 is equal to b minus a x k plus 1 is equal to b minus A x k minus A delta k plus A delta k, which is r k minus A delta k. This is perpendicular to r k this or this is perpendicular to r k plus 1 is perpendicular to L (Refer Time: 17:32) this gives us what is Steepest Descent algorithm, which is obtained by certain choices of W and V.

Now, we started with the question that can we have methods for non-symmetric matrices, can we have methods for non-positive definite matrices or can we have a method for any general matrix any general non-singular matrix. So, singular matrix solving A x is equal to b by an iterative scheme is not what we are discussing, is not is not very it is not probably proximal, is not very simple, but we are looking only solution of A x is equal to b in a in an iterative method.

Steepest Descent, what we have discussed using geometric constructions is for symmetric positive definite matrix, which is obtained by minimization of a particular functional, J is equal to x transpose A x minus x transpose b. The same method can be obtained from a projection method, because it is kind of we have guns are cooler till now, we develop Steepest Descent method based on minimization of J, which is x transpose A x minus x transpose b for symmetric positive definite matrix.

Now, looking into the minimization and the gradient search algorithm associated with the minimization with summarized it to be a generator to a general projection method. Then we looked into a specific case of general projection method, which is one-dimensional vectors with W and V or K and L is equal to r k, and we obtain Steepest Descent, but can we obtain some other method from the general projection method, which is applicable for wider class of matrices.

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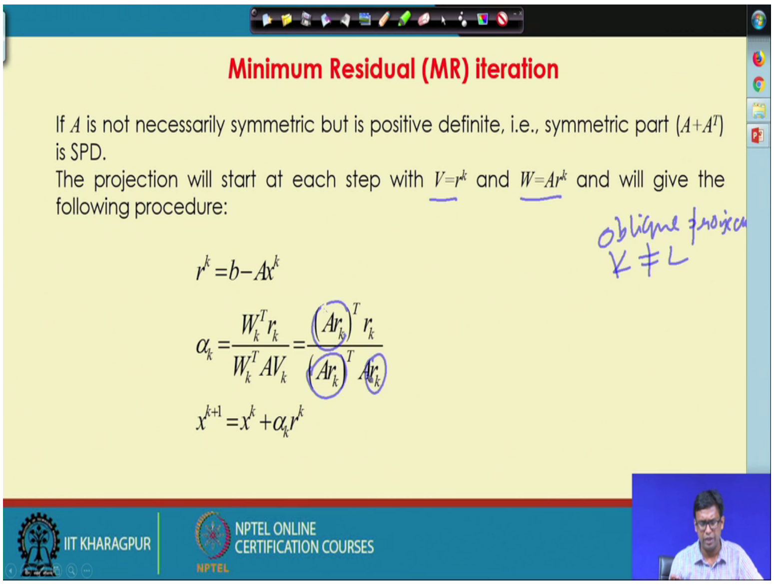

The next method is called minimum residual or MR iteration. A is not necessarily symmetric, but is positive definite that is A plus A transpose is a symmetric positive definite matrix, which is symmetric part of A. A is a positive definite matrix, but A is not symmetric matrix. And we define a projection method at each step with V is equal to r k and W is equal to A r k, so it is not a Galerkin condition, and this is an Oblique projection, K is not equal to L.

And we end up with the following procedure r k is equal to b minus A x k. Alpha k is equal to W k transpose r k W k transpose AV k, where W is replaced by W is replaced by A r W is replaced by A r and W is replaced by A r, and V is replaced by r k. So, we get a new x k plus value of newer value of alpha k, how to update the x k, which is A r k transpose r k A r k transpose A r k, and we update x k plus 1 is equal to x k plus alpha r k.

So, if we look into the algorithm, this is now this is an algorithm, which is for non-symmetric matrix, however the matrix has to be positive definite. If we look into the algorithm, alpha will it is exactly same as Steepest Descent method. And in practice, we will do, so we will take a Steepest Descent method computer program and do little modification, and we will get the minimum residual iteration. And we can see that 1 is apply applying applicable only for symmetric matrix, so if the other is more generally applicable. However, so we start with the Steepest Descent we start with guess x k and update x k as A r k transpose r k r k transpose A r k is alpha k x k plus 1 is equal to x k r k.

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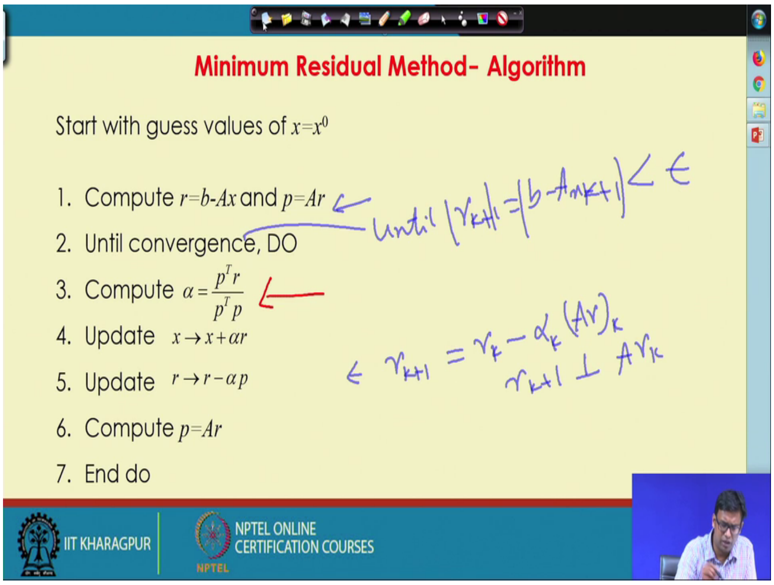

So, if you look into the algorithm, compute b is very same as Steepest Descent only the alpha computing part is different, you start with a guess x is equal to x 0. Compute the initial residual r is equal to b minus A x and p is equal to A r. Then until convergence, check and compute alpha, which is p is equal to A r transpose r by A r transpose A r. So, you have to do matrix multiplication only here, and then only do vector multiplications.

Update x is equal to x plus alpha. Update r is equal to r minus alpha p. Now, in this part, how r k plus 1, so this basically tells us that r k plus 1 is equal to r k minus alpha k A r at k-th level, this is the relationship between r (Refer Time: 22:51). And by this, we are also ensure we also ensure that r k plus 1 is perpendicular to certain plane. What is r k plus 1, is perpendicular to a r k, as a plus 1 is perpendicular to this particular plane also. So, if you take dot product r k plus 1 dot r k with that particular value of alpha, it should come to be 0, fine.

So, update this, and compute p is equal to A r. So, update p, and then count do this loops. And until convergence means until r k plus 1 is equal to b minus A x k plus 1 is less than this its mod sum norm of it is less than a very small number, then repeat this steps.

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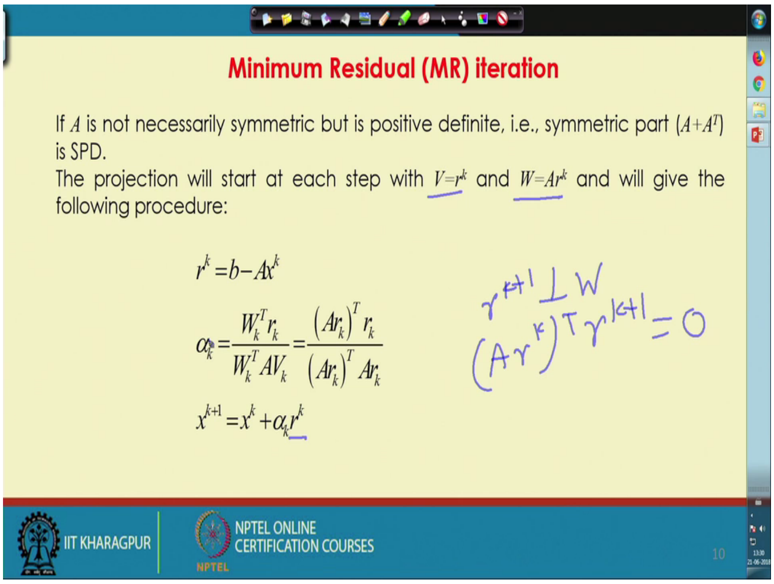

So, what we are doing here that is we are we have taken a update space x k is equal to x k plus alpha r k, update x k is changed along k along V, which is r k. And r k plus 1, so we have also take in that case (Refer Time: 24:35) r k plus 1 is perpendicular to W or A r k transpose r k plus 1 is equal to 0. From there, we have found out what is alpha k and designed a method.

So, again if we think of a general projection method, what it is doing physically. For example, when we talked about a Steepest Descent algorithm, physically it is minimizing a function J, which is x transpose A x minus x transpose b, and searching the updated value along gradient of minus J. Now, what is this algorithm, doing physically.

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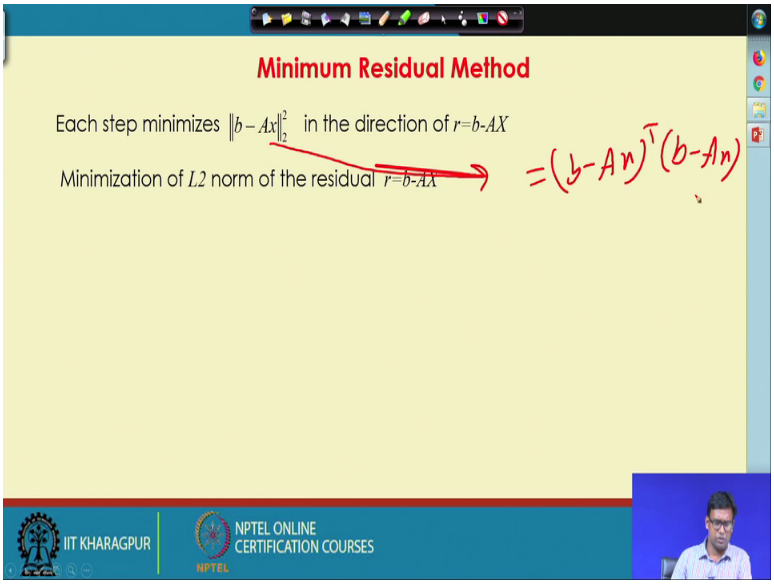

Each step in minimum residual method, minimizes b is equal b minus A x second norm b minus A x square the square norm of b minus A x. And in each direction, it changes b x in a direction b minus AX. So, this is minimization of the L2 norm of the residual b minus AX. So, earlier it was minimizing x transpose A x minus x transpose b for Steepest Descent algorithm.

For minimum residual method, it is minimizing that L2 norm of the residual. So, this is finding, again it is a minimization for finding only. Instead of finding minimization of x transpose A x minus x transpose b, it is finding the minima of L2 norm of the residual, which is b minus x transpose b minus A x transpose b minus AX. So, this is this is nothing but this is equal to b minus A x transpose b minus A x, it is find the minimum residual norm of that. Now, we as we are discussing about in iterative method, we also need to look into the convergence of this method, whether this converges or not.

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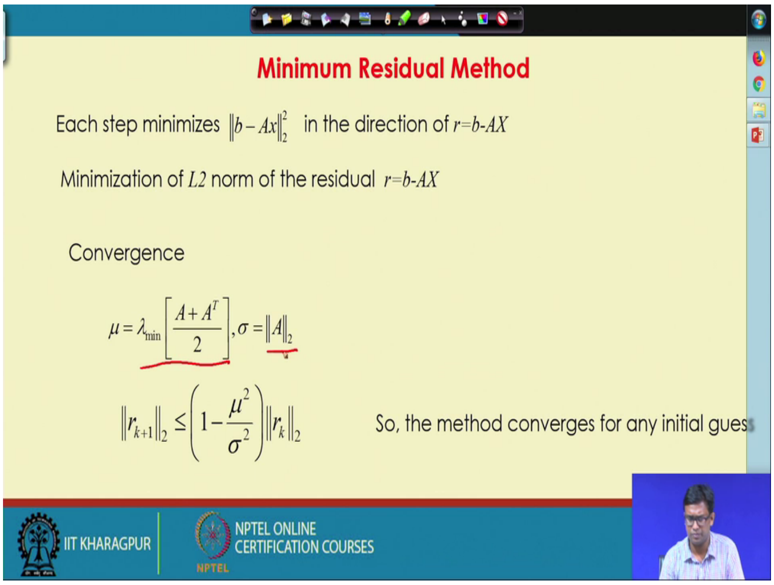

And the convergence theorem gives that if mu is the minimum eigenvalue of A plus A transpose by 2, and sigma is the L2 norm of the matrix A, I mean a matrix norm, so this is the second norm of using L2 norm matrix A. Then the residual r k plus 1 2 is always less than 1 minus mu square by sigma square r k by 2. So, if the matrix is positive definite, then this value this is again a positive number right, then this value is and this has to be because this is a positive number, this is a positive number, this has to also be positive. So, if then this method will actually converge for any initial guess.

And the rate of convergence will depend on the minimum eigenvalue of A plus A transpose by 2, and the second matrix norm second matrix norm of the matrix A. Now, this is again for a matrix, which is positive definite; if we have a matrix, which is not positive definite.

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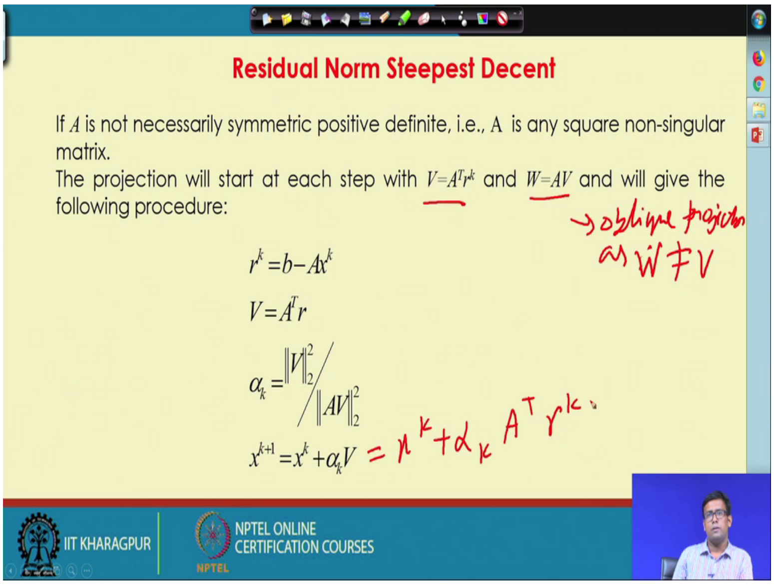

So, if we have a general matrix A is not necessarily symmetric positive definite, it is not necessarily symmetric not necessarily positive definites, if it is a any square non-singular matrix. The projection method will start at each step with V is equal to A transpose r k and W is equal to AV, so this is also an oblique projection as A is as sorry as W is not equal to V, this is also an oblique projection method. And the idea will be r k is equal to b minus A x k, where V is equal to A transpose r.

Alpha k is obtained as Vs second norm of V 2 square square second square norm of V 2 by AV 2 second square norm. And x k plus 1 is equal to x k plus alpha k V k, which is basically this is equal to x k plus alpha k A transpose r k.

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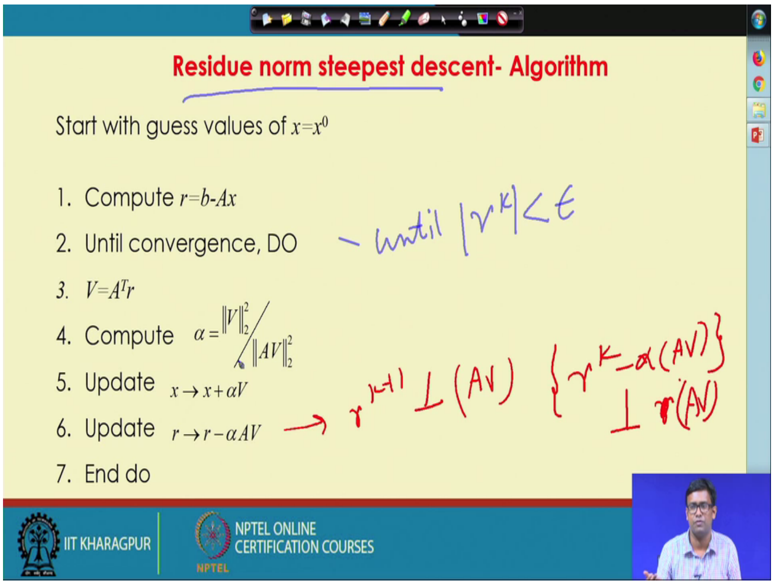

So, if we look into the algorithm, we will start with guess value x is equal to x 0 until convergence V is until convergence means, until r k is mod of r k is less than a small number epsilon. V is equal to A transpose r. Compute alpha, update x is equal to x plus alpha V, r is equal to r minus alpha AV. In why r is equal to r minus alpha AV, we can quickly check that W is equal to AV, so r r k plus 1, so this will tell us that r k plus 1 is perpendicular to AV space. So, r r is equal to r minus alpha AV.

So, we are taking alpha AV from r k, so, it is what is along AV is taken away, so what remains is basically perpendicular to r k plus 1. If we apply the ideas, we obtain during orthogonalization, you can decompose a vector along one particular vector, and it is perpendicular component. So, if we take away the component which is along that particular vector, what will be remaining that means, r k minus alpha into AV, if we take out that particular component should be perpendicular to AV that is the idea. And this gives us the residue norm Steepest Descent algorithm, which is which is applicable for any general matrix any general A matrix.

And (Refer Time: 31:20) and this starts with one matrix vector multiplication, here A transpose r. And again, you have to do a matrix vector multiplication AV here. So, there are two matrix vector multiplication associated with it. This is more expensive compared to the earlier methods. However, as is a general method, the advantage is that the matrices, which cannot be solved using Steepest Descent or for example, the matrix, which is not diagonally dominant and not positive definite using even (Refer Time: 31:54) Gauss-Seidel, Jacobi that can be solved using residue norm. This is for any general matrix, now there is no restriction on the structure of the matrix.

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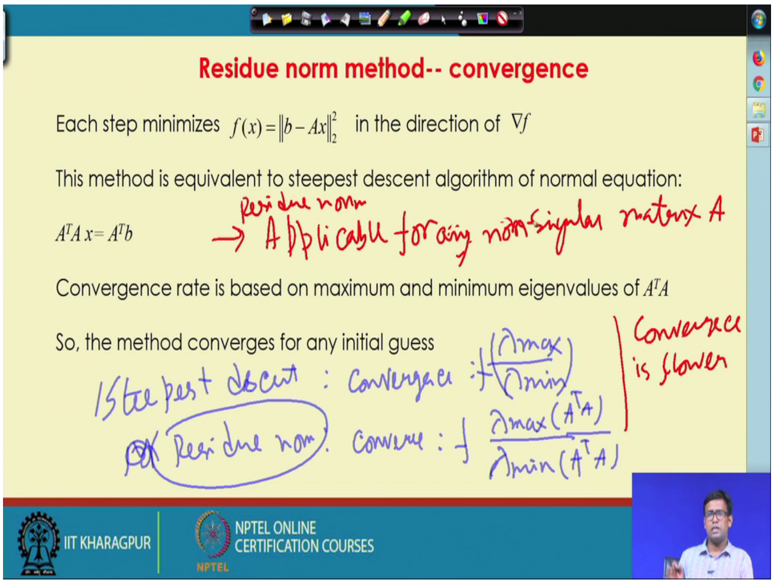

So, we will also check, what this is physically doing, each step minimizes b function b minus A x square. Earlier it was minimizing for minimum residual, it was minimizing in the direction of r, now it is not minimizing in the direction r, rather it minimizes L2 norm square norm of L2 square of r in the direction of grad f.

This method is equivalent to Steepest Descent algorithm or normal equation A transpose A x is equal to A transpose b. So, this is a symmetric matrix, in this is also positive definite matrix A transpose A x, if A is non-singular. So, this is Steepest Descent algorithm on a equivalent to Steepest Descent algorithm on a SPD matrix. However, this is applicable for non-SPD matrices. Convergence rate is based on minimum and maximum and minimum eigenvalues of A transpose A. So, the method converges for any initial guess.

Now, there is a small catch here that is that if we think of Steepest Descent, convergence is function of lambda max and lambda min. And in minimum in residue norm, convergence is a function of lambda max of A transpose A by lambda min of A transpose A. For example, lambda is a A is a symmetric matrix, then A transpose and A are same.

So, lambda max by lambda min will be this case lambda square max by lambda square min of this matrix A.

So, the if the condition number is always greater than 1, the condition number will be square, so condition number of the residue norm case is much higher, condition spectral condition number of A transpose A is much higher than spectral condition number of A. So, the convergence rate will be much smaller convergence rate is in a way inversely related with the spectral condition number. As the spectral condition number is higher for residue norm method, the convergence rate will be small slow.

So, though it will converge, we can probably note it down and we will check it, rather convergence is slower compared to the Steepest Descent or Gauss-Seidel or Jacobi type of method, because the spectral condition numbers are higher in of if we consider A transpose A. However, this is a robust method. As this can take any matrix any non-singular matrix, and give a solution to that. Till now, the matrices we have considered before that we are restricted. The algorithms we have considered, we are restricted for certain class of matrices, but this is a general this is applicable for any general class of matrix.

So, though it is a slower method and it does lot of calculations, it does a two matrix vector multiplications, finds the square, norm, etcetera. In each step, the number of steps are high, but this method converges for any matrix, so this is advantageous. In certain cases, where the other methods will not work, the residue norm method will be of application.

So, this is the best thing is that this is applicable residue norm is applicable for any non-singular matrix A that is the best part of this let me write down for any non-singular matrix A, and that is the best part of this particular method. So, in next class, we will see some do some coding exercise, and how the programs are written for this different methods, and we will also see how what are the performance of these methods for different matrices.

Thank you.