Matrix Solvers
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Lecture – 04
Determinant of a Matrix

So, in last class, we have discussed about system of linear equations which are expressed in form of matrix equations. And we have seen that and row and row permutation, column permutation, linearly combining different rows or different columns; keeps the answer or the solution vector nearly same.

There is some change of the order of the solution vectors in solutions or solution vectors are probab 1 solution element of solution vector is may probably linear combination of few other elements something like that that. And we can see it from the physical perspective that is if we write down the equation in form of vector equation, we can see what is the happening when we change do row permutation or column permutation or do linear combinations of row etcetera.

But, what happens to the matrix when we do this operations so that there the solution remains essential same or similar. So, we will discussed will try to look into that that thing and these discussion will be on determinant of a matrix and its properties and applications.

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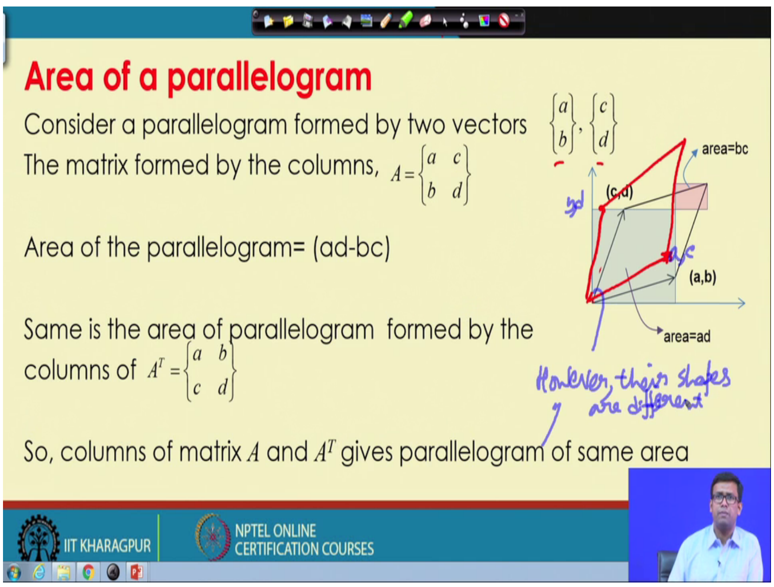

So, if we start looking into area of a parallelogram. For example, there is a parallelogram which is generated using 2 vectors a b and c d. It is in a 2 D space and the matrix form by the columns, in last class we have seen how the column vectors can form a matrix is given as a b c d and the area is essentially given as, this is the parallelogram which will come out of this and the area is essentially given as ad minus bc.

So, in a simple geometrical exercise, we can find that if I find out area of a rectangle with a base and b height and subtract the area of the rectangle of c base and d height, will get essentially same area. This area ad minus bc.

And so, if we take a transpose of A or we form a parallelogram using columns say instead of a b c d, if we use the columns ac and bd; a transpose, this will be again having same area, only thing is that the shape of this will probably change because this line will be something like this. This line will be sorry this will be the.

So, this will be the new parallelogram, but it will essentially have same area; the shape will probably change. So, this is write it a c and b d and we can find out from the formula that the area will be essentially same. So, the columns of matrix a and a transpose gives parallelogram of same area.

However, what we can write is that they are; however, their shapes are different; different. Thus, this parallelograms are not the same parallelogram, but their area is same. Now, if we look into the same problem in 3 D; a parallelepiped, I considered a parallelepiped.

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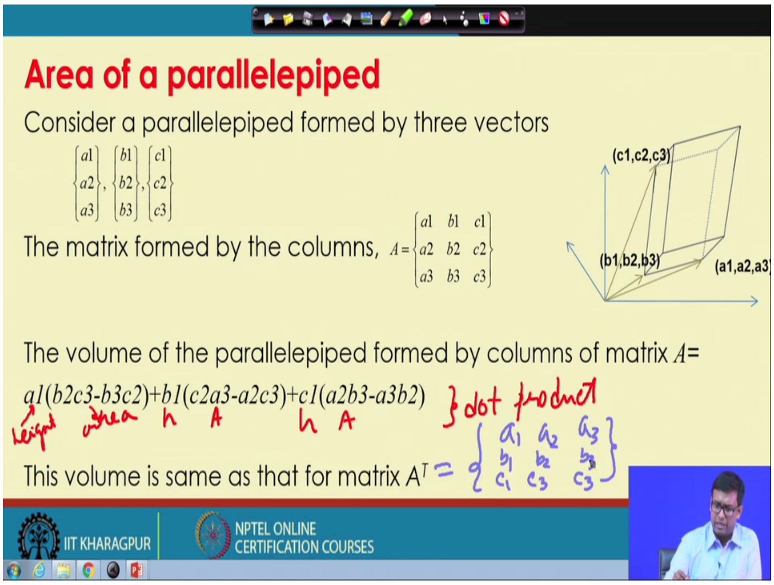

Now, and this parallelepiped is formed by 3 vectors; a1 a2 a3, b1 b2 b3, c1 c2 c3 and if we look into that, this is this is the particular shape and the volume of this parallelepiped is; how do you calculate the volume? Basically, we calculate area contain by these 2 planes; take a cross product and take dot product area of this, these 2 columns.

First we calculate area between these 2 columns and then they take a dot product with third one and this is obtained as a1 into b2 c3 minus b3 c2 plus b1 into c2 a3 minus a2 c3 plus c1 into a2 b3 minus a3 b2. So, this is basically one particular height; a1 is height in the first direction and this is the area in the perpendi of the perpendicular plane.

Similarly, there is a height into area h into area and height into area or this is this comes as a dot product between height into area. And if we have a matrix form by the transpose column, columns of the transpose or A transpose for example, if we have the matrix A transpose like this which is the columns are a1 b1 c1 a2 b2 c2 a3 b3 c3. It will have same volume, but the shape of the parallelepiped will be different in that case.

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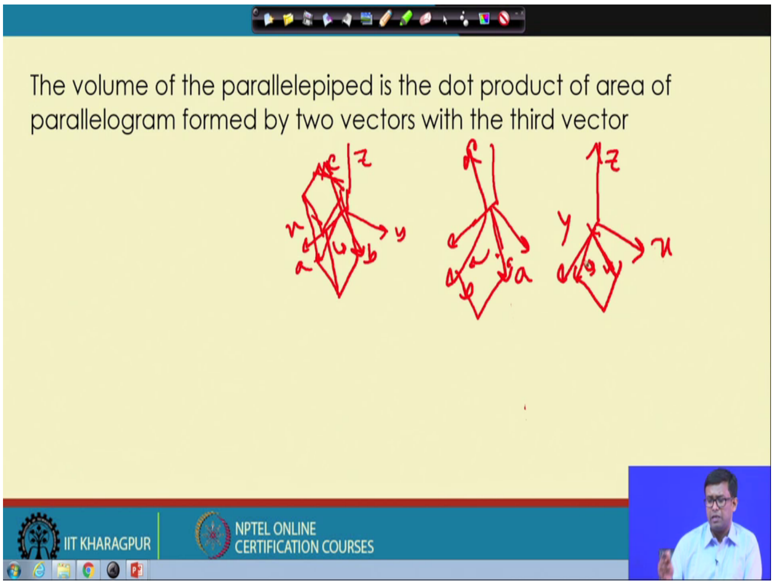

Sorry. Now, the volume is the dot product of the area of parallelogram formed by two vectors with the third vector and from a simple example, we can see that that for example, I have if these are my axis x y z; I have an area formed by two vectors a and b and I take a third vector which has some component in z direction, the area will be come as it is dot with the volume will be come with these area and its dot product and this shape will look like something like this.

This shape will look something like this drawing is very clumsy on; however, it is probably clear. And now, if we do a column permutation or row permutation, what will happen if we do a column permutation? The vect the order of the vectors will change. Its first column is replaced with the second column.

So, will have this will be my for example, this will be a and this will be b and remaining this as the this is the c; this is the c, this as the c. So, the area will have a different since or different direction for the area if this is perpendicular to this plane or if I calculate the area vector clockwise; here it will be anti if I calculate it anticlockwise, here it will be clockwise or the area will be inside the plane.

Therefore, the volume will have a negative sense in terms of the first volume and if you do the row permutation it will be also. So, if we do a row permutation, the x and y axis will be changed. So, this is z axis. This will be y axis. This is this is initial y axis. So, this will be x axis. If I do a row permutation, this is y axis.

Therefore, again the area will have in calculated in the same direction, will have a opposite meaning. So, the value of the area will be opposite; therefore, the magnitude of the volume will be the opposite one.

So, we can say that the if rows of A are or columns of A are permuted, the value will be same, but this value will change the sign.

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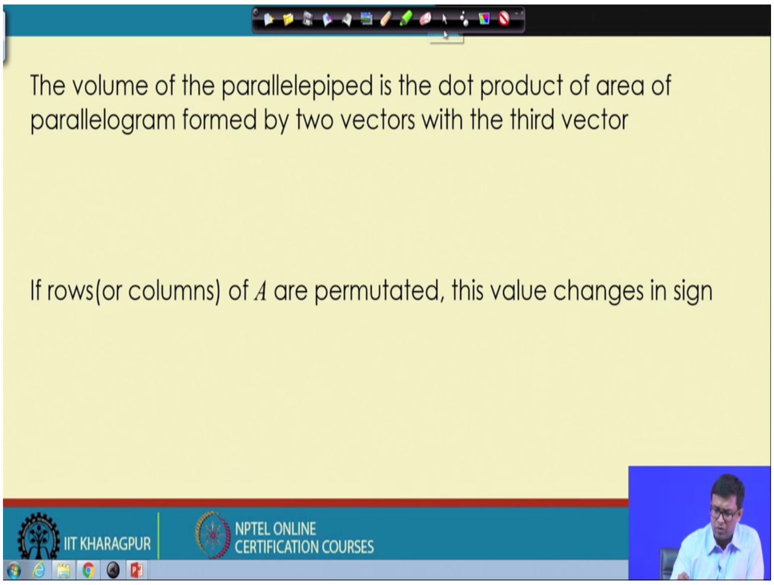

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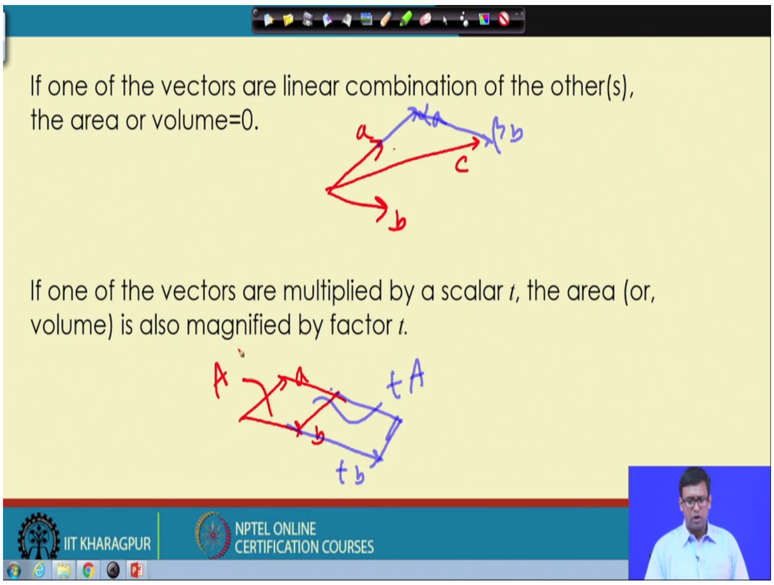

If we have a case, where sorry where one of the vectors and linear combination of other vectors like I have two vectors; this is a and this is b and c is something like sorry linear combination of a. So, probably a magnified by some factor, this is this is alpha a and a linear combination of b. So, we go along a parallel to be so maybe beta b here and now this becomes the c. So, what happened c essentially lies in the same plane of a and b and therefore, the volume has to be 0.

So, if one of the vectors either the low vectors are linear combination of other 2 row vectors or column vectors are linear combinations of other 2 column vectors; this is the volume will be 0. So, one question comes, we are discussing on volume and area only and it will see it in a while.

If any of the vectors are multiplied by a scalar t, the volume is magnified by the function t. For example, I have a and b and this is the area. Now, instead of b, I multiply b by tb. So, the area becomes t times more than the actual area. So, this becomes t times t A if this is if this area is a this is again very simple using simple geometry you can find it out.

So, if one of the vectors is multiplied by a scalar, the entire area will be multiplied by that magnification factor.

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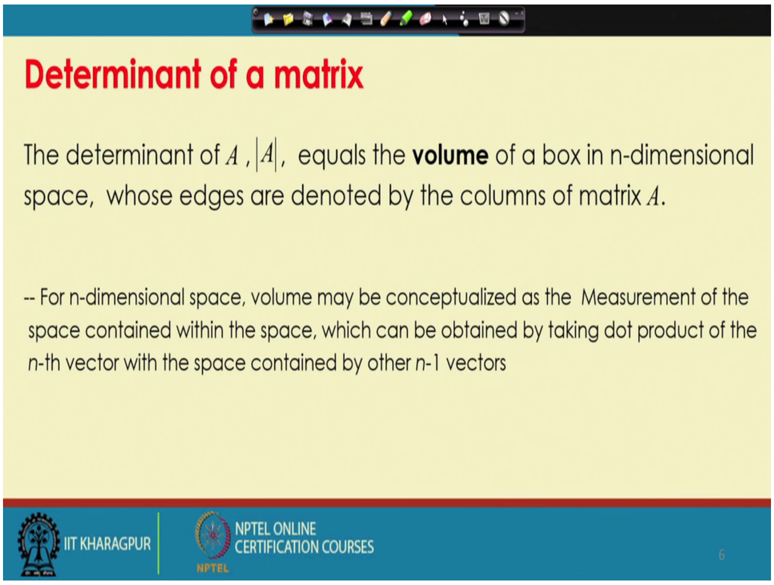

And then, we come to the definition determinant of the matrix. Determinant of A or given by A within 2 vertical lines is equal to the volume of a box in n-dimensional space, whose edges are denoted by columns of matrix A.

So, what is volume in n-dimension that is the question.

At the very first lecturer, we are discussing about multi dimensional vectors because in the matrix equation and we are multi dimensional vectors with a in a matrix equation, we are dealing with many many variables n number of variable equation; we the equation system which has n degrees of freedom. So, if we have 2 variables, we can represent that by a vector in a 2 D space. If we have 3 variables, we can represent by a vector in 3 D space because there are 3 components.

We can say one component is along x; another is along y and the third one is along z. If we have 4 variable system, we can see the fourth one is something like another x is zeta which is difficult to visualize. But I can say that there is another axis which is I am not able to visualize it directly, but it is there. So, it is a 4 dimensional space.

Similarly, we can go to a n-dimensional space. Now, question will be what is volume in n-dimensional space? Ok, let us assume that we are with conceptualizing an n-dimensional vector. What will be its volume? By the term volume will mean that it will be the space, measurement of the space that is contained within the space, which is encompassed by these vectors.

If I try to do a box using vectors and then take parallel of the vector close with parallel of the other vector; once there will be space contain by two vectors and we can find the area, if we increase number of vectors; then, will get and area for two vectors will get a plane. For 3 vectors will get a parallelepiped. For 4 vectors we will get something, but that will be again a enclosed area and we can measure its volume.

And how to measure it's volume? For 2 D, we just take cross product and get the area. For 3 d, we take dot product of the new vector with the area and get the volume. For 4 D, will take dot product of the fourth vector with the volume which has been given by 3 vectors in a 4 dimensional space and get the something live volume of the 4 dimensional space of the of the 4 D shape.

So, n-dimensional space, this volume may be conceptualized as a measurement of space contained within the space, which is obtained by taking dot product. This space, this space is contained by the 4 n vectors and is obtained by taking dot product of nth vector with the space contained by in other n minus 1 vector. So, like we build we built volume based on previous vectors only and dot product with the present vector and move on.

The transpose matrix will have same area or same volume except the shape will be different, we have seen for 2 d the transpose matrix is giving a different shaped parallelogram the area same, but the order of the area is the area is same, order is not order is same.

So, the shape will be different or the orientation will be different; however, it will have same determinant or same volume with same magnitude and same sign. Now, we come to a next thing is determinant is defined for square matrix only. Why square matrix? If we have 2 dimensional space, we need 2 vectors to define a parallelogram.

So, the columns will give a square 2 column; columns of 2 2 into 1 order will give me 2 into 2 matrix; 2 columns of 2 into 1 matrix. So, it will be a square matrix for 3 d to will be again a square matrix. Now, if you have a diagonal matrix, the determinant or the volume measurement is very easy. It is the product of the diagonal, right product of the diagonals.

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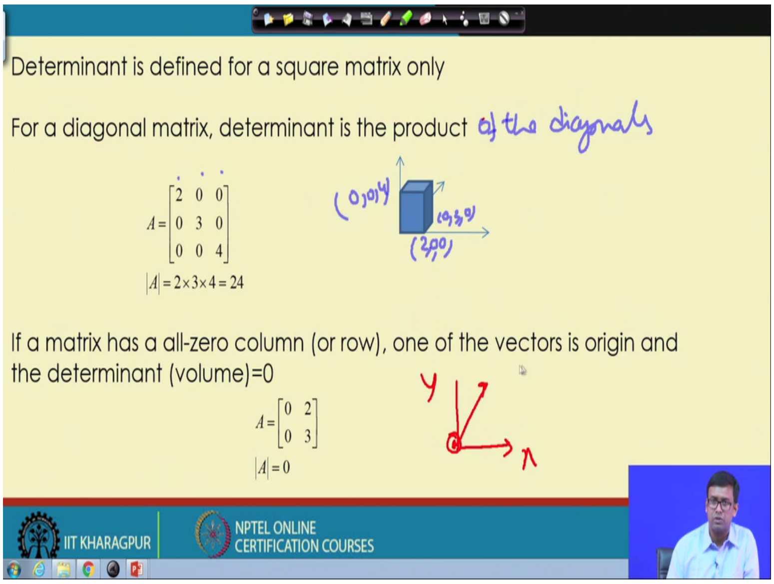

Why? If I have a diagonal matrix, the vectors are 2 0 0 0 3 0 0 0 4. This is probably 2 0 0; this is 0 3 0 and this is 0 0 4. So, each vector is the distance along 1 of the axis. Therefore, total area is the it is a its a it's a brick shaped element now and the total area is the multiplication of the lengths and the total and the determinant comes to be multiplication of the diagonals.

If a matrix has all 0 columns or rows one of the vector is origin and the volume is and determinant or volume is 0 and this is also very easy to say there is an example and if this is also very easy to conceptualize, if we have a 2 D space x y and one matrix is 2 3. This is one vector; the other vector is the origin itself.

So, there is no any it is a line; two vectors will not encompass an area and the area will be 0. So, and this can be extended to n-dimensional space that diagonal matrix will have a determinant which is product of diagonals and for all 0 1 0 column all 0 column or 1 0 row, the determinant will be 0.

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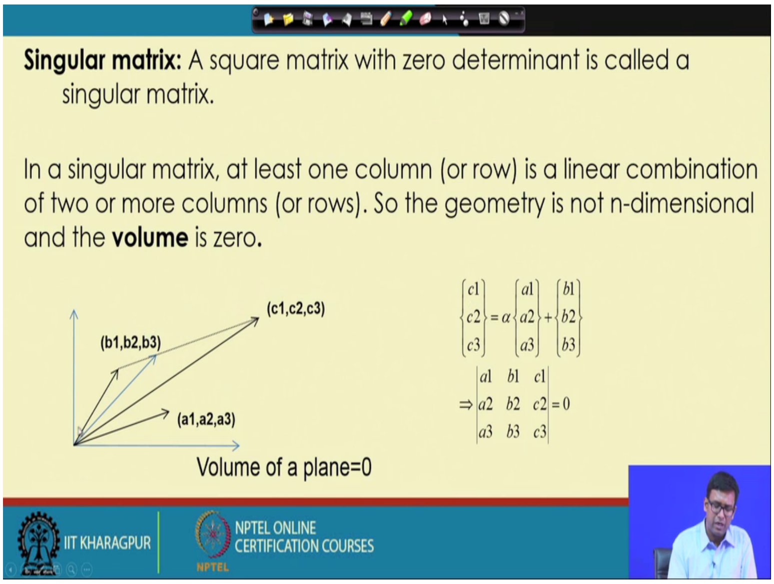

A square matrix with zero determinant is called a Singular matrix. In a singular matrix at least one of the column or row is linear combination of two or more columns.

So, it never encompasses that n-dimensional space. For example, if I have 2 lines; one line is linear combination means a line for 2 D, this line. Another line is a magnification of that. It will never encompass make a parallelogram, encompass any area. We can take parallel of these 2 lines and encompass an area. Therefore, determinant will be 0.

For 3 D, if I have 2 lines and the third line belongs to the plane contain by these 2 lines; it will not encompass any volume and the determinant will be 0. So, for 3 D, it is shown; this is vector a; this is vector b and then, if I the third in third there is a third dimension, but a and b is 1 plane, 2 vectors must been a plane and if I add b with the multiplication of a, with a magnification of a we get the vectors c.

So, a b c if these are 3 vectors, c is combination of a and b; this is we cannot form a parallelepiped using this 3 vectors. Therefore, the area volume of this will be a plane, it is volume is 0 and will see that the determinant is 0. So, the geometry for a n-dimensional matrix, the geometry is not an a n-dimensional geometry and the volume will be 0.

So, one question is that if there is a we have seen if one of the row or column is 0, the determinant will be 0. So, if there is one 0 vector in row and column the determinant will also be 0, that shows that as 0 vector is also a linear combination of other vectors. For example, if I have a and b, I multiply 0 with an a b; I can form c and a c which is 0.

So, a resultant of multiplication of a and multiplication of b can be 0 at any point of time. So, if there is a 0 vector, it will be also singular matrix and the determinant is 0.

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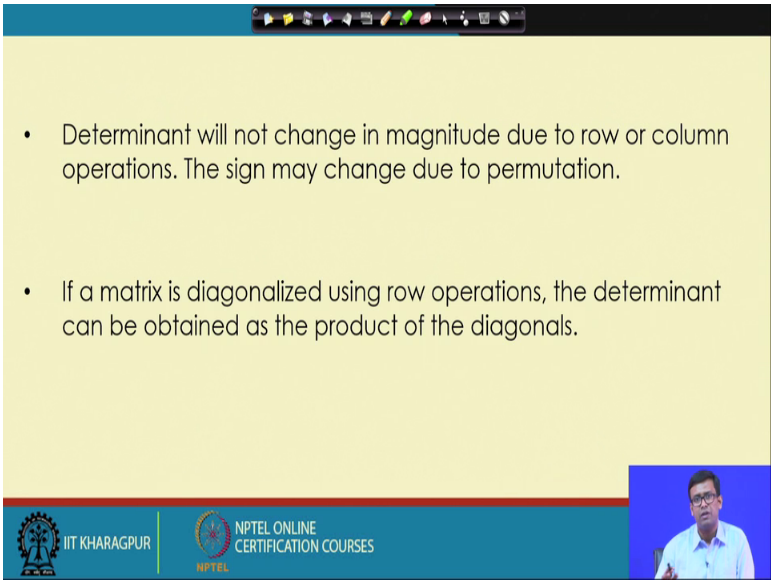

Determinant will not change its magnitude due to row and row or column operations the sign may change due to permutation and we have seen that if we do row transpose, row permutation or column permutation. In case of row permutation, the axis sense of axis will change in case of column permutation the order of the vectors will change and therefore, there is a change of sign.

If a matrix is diagonalized using row operation, we have seen row operations that the matrix can be changed using row operation if by this row operation the matrix is not changing its determinant; if it is a permutation it is its determinant is changing in sign, but all other row operation like combining adding one row with a with a other row.

And then, taking multiple of that adding it with the third row; all these things is will not change the determinant of the matrix and if we can get a diagonal form using this row operations, the determinant can very easily be obtained as the product of the diagonal. So, here are few properties of these determinants. So, we have some of these we have already discussed determinants of an identity matrix is 1.

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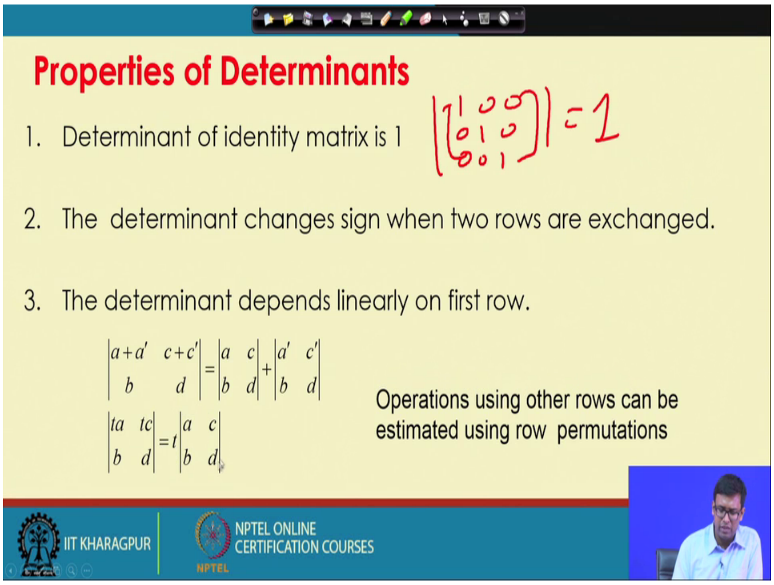

Because, all the diagonal terms are an identity matrix is like this 1 0 0 0 1 0 0 0 1 has all the diagonals are 1. So, its determinant will be 1.

The determinants determinant changes its sign when 2 rows are exchanged or 2 columns are exchanged just by row or column permutation and the determinant depends linearly on first row; that means, if first row can be decomposed sorry if first row can be each element in first row can be decomposed in 2 terms, then we can say they the determinant of this matrix is equal to determinant of these two matrices.

Similarly, if we multiply if each element of first row with one particular term that is like multiplying the entire determinant that by that scalar; however, the as row permutations exist the first row can be permutated to second row.

So, with any row we can do this type of operation, but we doing with only one row and we can see that determinant can be decomposed into determinant of two different matrices or multiplying a scalar with all the elements of one row is multiplying the scalar with a entire determinant and taking determinant.

Multiplying the scalar with all the elements of one row and taking determinant of the matrix is equal to multiplying this scalar with the determinant of the matrix and also the sorry determinant of a is equal to determinant of a transpose.

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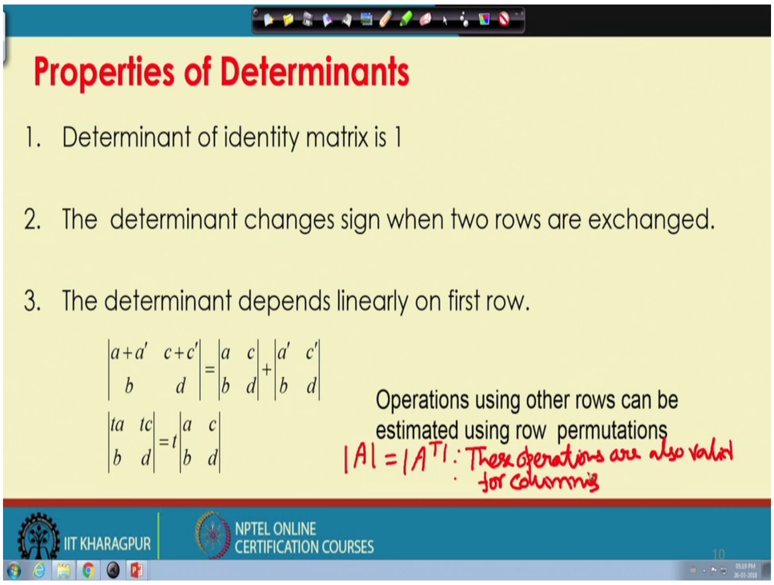

So, will say these operations these operations, operations are also valid for columns because I can change the matrix as a transpose and rows can take the role of columns in that case or columns can we take the rows role of rows in that case.

So, the determinant depends linearly or any row or any column only on that row or that column. If I multiply t with 2 of the like t is multiplied with 1 row only, elements of 1 row only if I multiply t with elements of the next only this will be t square into that.

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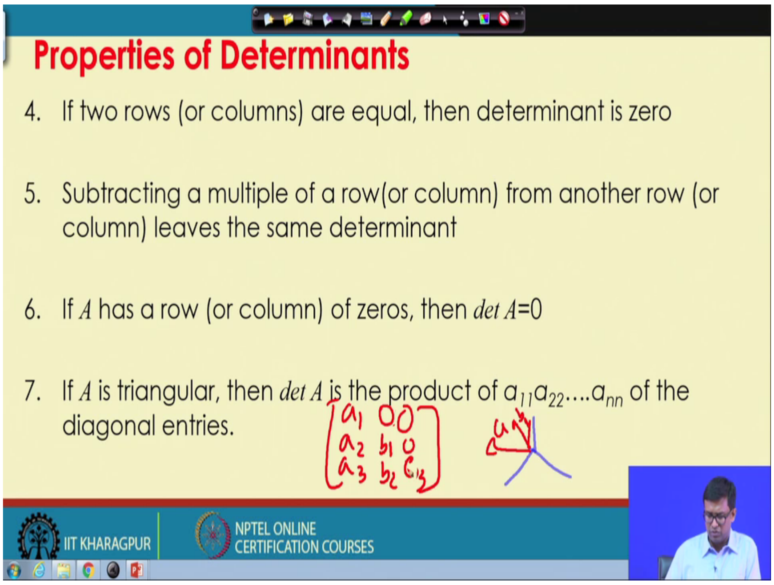

If two rows and columns are equal, the determinant is zero. So, this is like linear combination, subtracting multiple of a row from another row leaves the same determinants. So, it is like taking a linear combination take like of one row or multiple rows and adding it to another row or the column gives the same determinant.

If A has a row of zeros, then determinant of 0, if A is triangular then, the determinant is product of the diagonal entries also in this. If we if we write down the matrix, we can see if you think as a volume calculation it is a we can directly say it. You can probably take an example, if I have a matrix a1 a2 a3; a1 sorry it has to be diagonal shape. So, 0 b1 2 0 0 b3, this is a lower triangular matrix.

So, if you look into the columns the first one is any vector a; the second one has a 0 along x. So, second one is only co b is only contain here. So, a dot b is a first is a cross b and c has both 0 along x and y. So, c is only along this, may be this is x this is y; this is a this is x this is y.

So, c is only along this and if you take the multiplication this is at this area into the direction is intercept in the c sorry this is c 3, this would be c 3 and we can find out what is the determinant here or if you just take the determinant will see as this terms are 0, this will be b 1 into c 3. This is the previous formula used for the volume.

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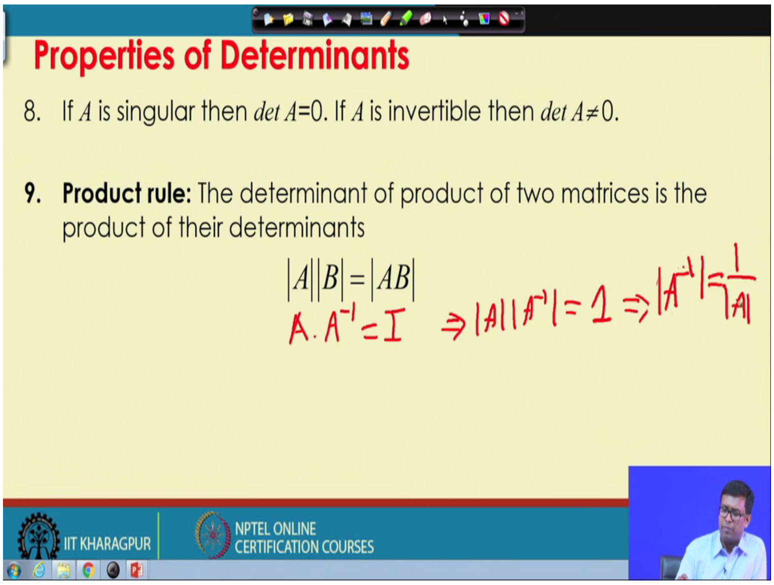

If a is singular, then determinant of a is 0 and we have seen earlier that for a singular matrix is a matrix where rows are linear combination of rows or columns are linear combination one column is linear combination of other columns or one row is linear combination of other rows and in that case the matrix x is equal to b does not have a solution, does not have unique solution.

Because number of equations and number of independent variables, they will not made. Therefore, we cannot find inverse of a we cannot write x is equal to a inverse b because a inverse cannot be found out.

So, if the matrix is singular then determinant is 0 and a inverse also cannot be found out and we can say that if a is invertible, if we can write a inverse then determinant should be non 0. Product rule, the determinant of product of two matrices is product of their determinant and this is very important rule used in several calculations in linear algebra.

And now, if we can write A into A inverse is equal to I; remember this is due able when a inverse exists.

And we can write this implies that mod of a into mod of a inverse and I is an identity matrix which determinant is A.

Determinant of A into determinant of A inverse is equal to 1 and which tells us determinant of A is equal to or rather and determinant of A inverse this tells me that determinant of A inverse is equal to 1 by determinant of A and this is again a very important property. And this shows this particular case that A inverse is equal to 1 by determinant of A inverse is determinant of 1 by A.

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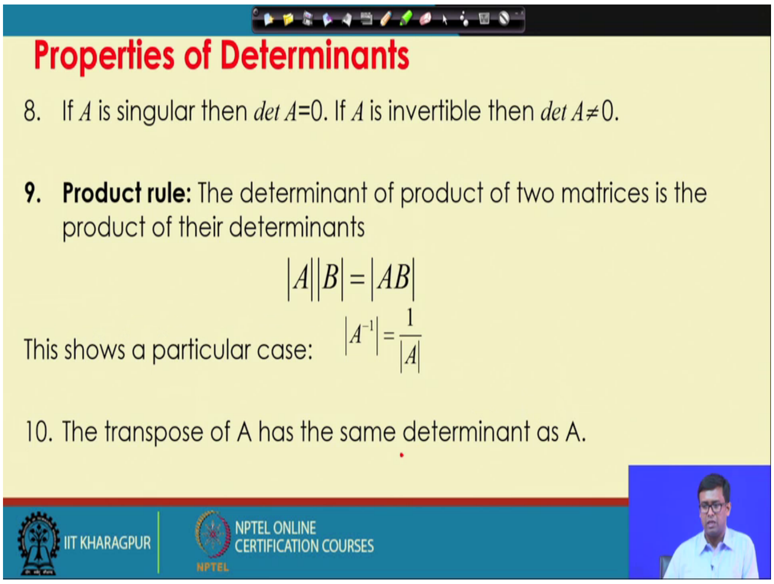

And of course, we have seen transpose of a matrix the tenth properties transpose of a matrix has the same determinant. Transpose of A has the determinant of A. Now, we have seen it earlier that the shape of the geometric will change, but the volume will be essentially same.

So, now, we have to see, how to calculate determinant for a large matrix and what are the applications of determinants specially while looking into the matrix equations. So, it will do in the next class.

Thanks.