Matrix Solvers

Prof. Somnath Roy
Department of Mechanical Engineering
Indian Institute of Technology, Kharagpur

Lecture – 36
Steepest Descent Method: Finding Minima of a Functional

Welcome. In last few classes, we started discussing over iterative methods. And I have given you some introduction to basic iterative methods and also worked on their convergence analysis, which are Gauss-Seidel and Jacobi based methods. And later we have seen that convergence rate of this methods can be improved using certain methodology called successive over relaxation. However, what we have seen there is that these methods are restricted only for diagonally dominant or irreducibly diagonally dominant matrices. That is the diagonal term in its absolute value must be greater than equal to or sum of all the off diagonal terms in their absolute values.

And at least for one row the diagonal terms absolute value is greater than not greater than equal to is greater than sum of the absolute values of all the off diagonal terms. Only for these cases, the matrices Jacobi and Gauss-Seidel or successive over relaxation can give a solution. And we have seen that, if we can take a diagonally dominant matrix, and if we do a low permutation, the solution still remains same. However, the matrix changes, but Jacobi or Gauss-Seidel fails for that. So, there is a particular restriction in Jacobi and Gauss-Seidel.

And the convergence rate depends on what is a maximum eigenvalue or the spectral radius of the iteration matrix G that cannot be greater than 1. If it is less than 1, then only the method will converge, and at that shows that little converge for any diagonally dominant or irreducibility diagonally dominant matrix. However, if the maximum eigenvalue is large is less than 1, but still large, the convergence rate will be slow. And we can only improve the convergence rate using successive over relaxation technique.

Even there is a optimum value of omega or over relaxation factor based on which you can get highest convergence rate. And if we will later discuss about a number of iterative methods, if we compare success like optimum omega successive over relaxation Gauss-Seidel with a faster iterative solution technology, we will see that Gauss-Seidel or a SOR

Gauss-Seidel is still very slower than this faster iterative solvers. Those solvers we will discuss.

So, the basic problem with Gauss-Seidel and Jacobi iterations are in two fold. One is that they are restricted only for diagonally dominant or irreducibly diagonally dominant matrix. And another issue is that their convergence rate is limited by the maximum value of the spectra spectral radius or maximum value of the eigen maximum eigenvalue of the iteration matrix or spectral radius of iteration matrix.

If it is large, even using so successive over relaxation, we cannot increase it. To increase the convergence rates to a very high extent it. There is an optimum omega for, which it will be maximum. So, there is some restriction on the convergence rate or there is some restriction on the number of iterations that is to be performed for solving a matrix using Gauss-Seidel or Jacobi or SOR the Gauss-Seidel. So, now we think of looking into faster solvers that is that is the very importance of this particular course is that how can we solve large equation system using iterative methods, which give us first solution.

So, when we start looking into faster solution, we have seen that beyond SOR Gauss-Seidel is restricted. So, we will look for some other solution techniques. And these solutions techniques will not be direct solution techniques, so that means that I will start with any arbitrary x 0. And only use that particular equation x is equal to b substitute guess value x 0 and update x. This directs direct iterative techniques are restricted in terms of their applicability for diagonally dominant matrix as well as in terms of the convergence rate or speed of the solution.

So, you look for some other iterative techniques. And what we start to discussion of this other from here we will start discussion on other iterative techniques. And a class of techniques named three rough space, we will follow later from this discussion only. So, we will start discussion with projection based iterative methods. And this with the particular method, I will try to discuss in this one or two sessions is steepest descent method.

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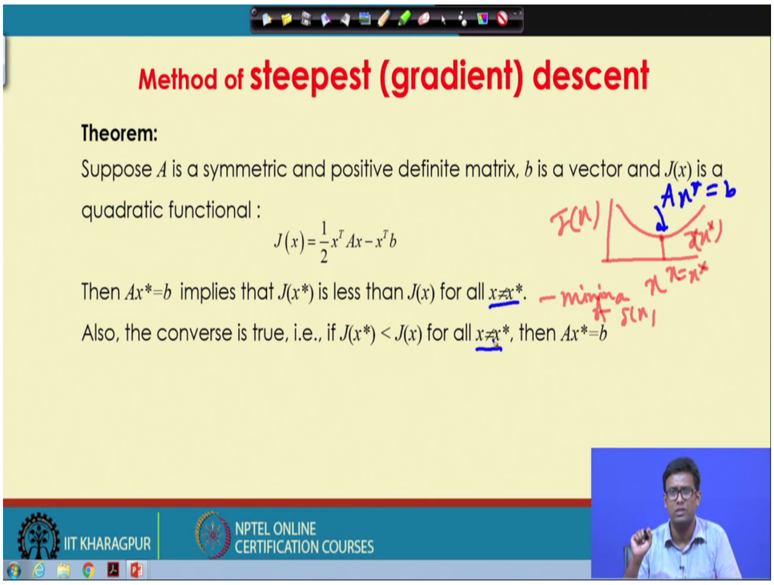

So, this is also known as gradient descent or a gradient search method. So, if we look into this method, it starts with the theorem. Suppose, there is a symmetric and positive definite matrix a, and b is a vector. And now we define a quadratic functional J, a quadratic functional means it is basically a quadratic function in which all the components of the vector x is associated; x is a vector of same order as b. So, this quadratic function is defined as J x is half x transpose A x minus x transpose b.

If we can define this function J x, then A x star is equal to b will imply that J x star for any value of x star J x star is less than J x for all x, which is not is equal to x star. So, this is basically says that if we can differ define a functional half x transpose A x minus x transpose b, we can say that A x star is equal to b, where we will get A x star is equal to b for that particular x value of x x star J x star is less than all J x or J x star is the minima.

So, if we think x to be a single valued a vector of dimension one, and we write that this is x, and this is J x, so this is the J x functional. And here at x is equal to x star, J x star is less than any other J x. This is the minima or we can say that this is minima of J x. And at this particular x is equal to x star location A x star is equal to b or A x star is equal to b is a solution A x star is equal to b gives us the solution is x star is only where J x is minima instead of solving A x star is equal to b, now we can try to find out minima of J x that that is the main philosophy behind this method.

The converse is true that means, if we find out a minima of J x that is J x star is equal to J x for all x, which is not x is equal to x star. So, for a one particular x star, I get a value J x star, which is minima. Then at that A x star A x star is equal to b. So, when we write for all x not is equal to x star, all x not is equal to x star that says that there is one particular x star for which this is solvable. So, this equation system obviously has a unique solution.

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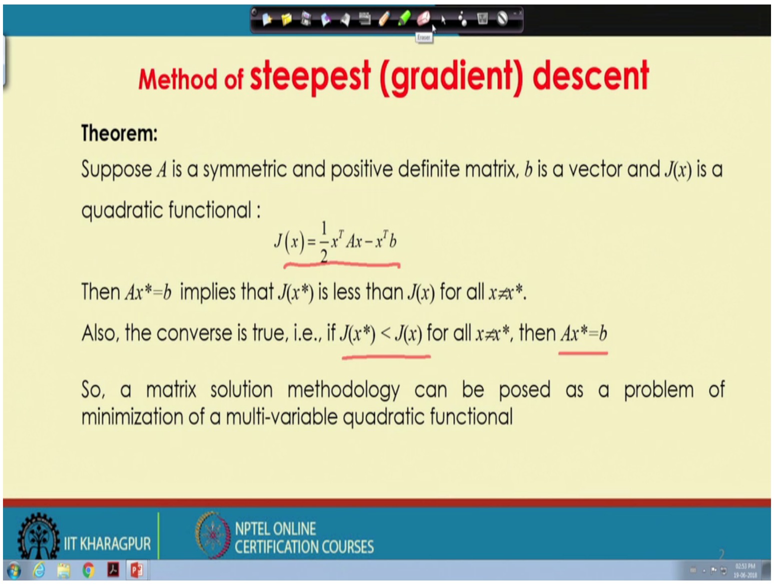

So, what this theorem mean that a matrix solution methodology A x is equal to b, I have to solve this. This can be replaced by finding out minima of J x. If we can define a function J x is equal to half of x transpose A x minus x transpose b, and try find minima of J x. The minima is a point, where A x star is equal to b is satisfied. So, matrix solution methodology can be posed as a problem of minimization of a multi-variable quadratic function.

Instead of solving A x star is equal to b, we will x is equal to b, we will solve, we will find the minima of J x. And that is why this is sometime called also search algorithm. Min gradient search algorithm, we will we will look into the gradient method later, because minima is associated with gradient in a sense. So, so the idea is that instead of solving A x is equal to b try to find out minima of J x. And then, there will be an iterative method for doing this.

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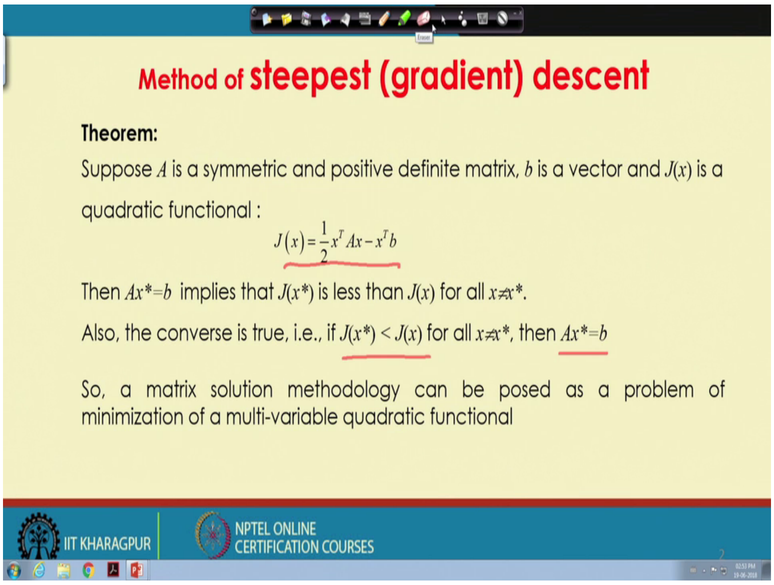

So, now I we got the theorem that J x is equal to x trans half of x transpose A x minus x transpose b. Minima of J x, so theorem basically says minima of J x is at a point where A x is equal to b. So, if we can find out the location, where J x is minima that will point to a vector for which A x is equal to b.

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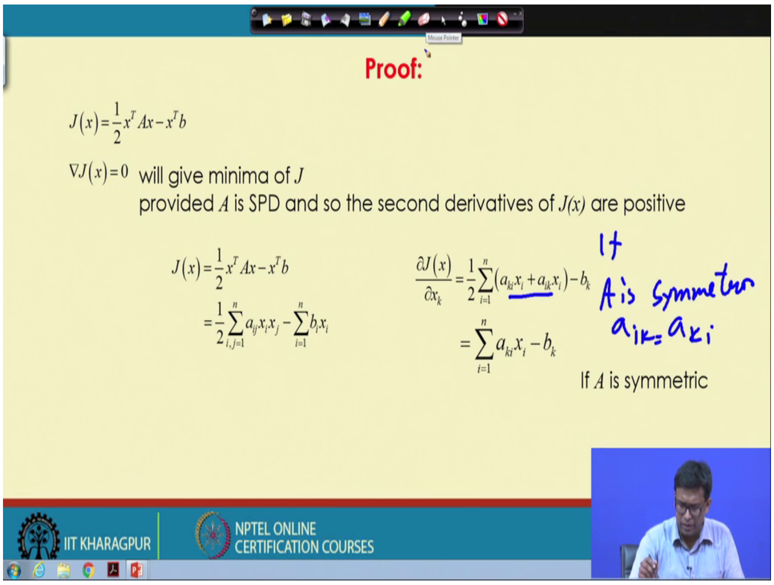

So, we will look into that proof of the theorem. So, its starts with J x is equal to half of x transpose x A x minus x transpose b. And for minima, we must have the gradient of because there is a multivariable function gradient of J is equal to 0. The first order derivative is 0 and provided A is symmetric positive definite matrix, so that at least if A is positive definite, the second derivative of J x. So, nabla squared J will be positive and we will see it later.

So, if minima gradient of J is equal to 0, and A is positive definite matrix, then this will indicate the minima of J x. So, now we write J x is equal to half of x transpose A x minus x transpose b. If we expand it, this term is particular row and of a, and it is multiplied with x i and x j, and then all these terms are summed up. So, this is a scalar, so this is half of i is equal to 1 to n, J is equal to 1 to n sum of a i J x i x J minus sum of i is equal to 1 to n b i x i.

And if I take its derivative with respect to k th component of J, we get half of i is equal to 1 to n a k i x i plus a i k x k minus b k. Now, if A is symmetric, this has to be noted carefully. A is if A is symmetric, a i k is equal to a k i. So, this becomes i is equal to 1 to n twice a k i half of twice a k i, so a k i x i minus b i ok.

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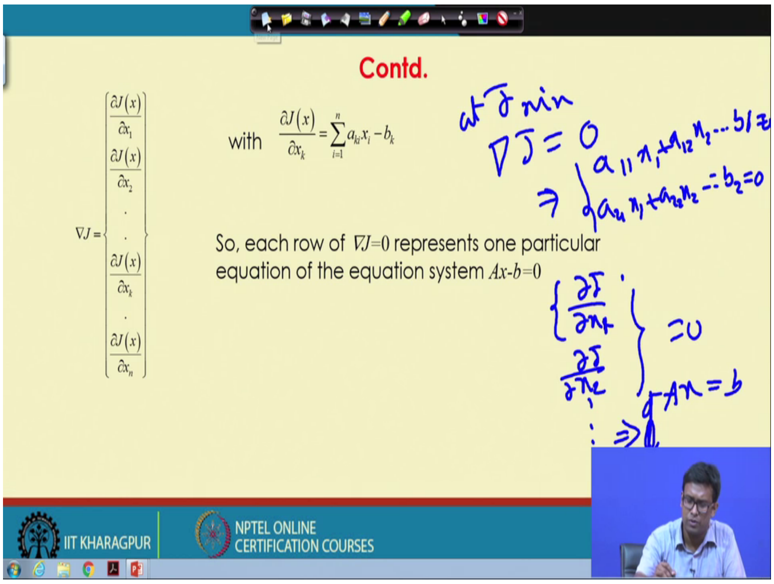

So, for if I write gradient of J that will contain all the vector del J del x 1 del J del x 2 to del J del x n, and each of this del J del x term is basically i is equal to 1 to k a k i x i minus b k one particular value. So, the when we will find the for minima our requirement was grad J is equal to 0 at J minima, so grad J is equal to 0 each term will be del J del x 1 is equal to 0, del J del x 2 is equal to 0 that up to del J del x n is equal to 0, which is sum of a 1 i x i minus b 1 is equal to 0. The next term is a 2 i sum of a 2 i x i minu[s]- minus b 2 is equal to 0, so on.

So, each of that each row of del J is equal to 0, each row of this matrix del J is equal to 0 represents one particular equation, which is belongs to the equation system A x minus b is equal to 0. So, this will finally give me that a 11 x 1 plus a 1 2 x 2 minus b 1 is equal to 0. Then a 2 1 x 1 plus a 2 2 x 2 up to minus b 2 is equal to 0, so on. So, this del J del x is equal to 0 or this equation system del J del x k or del J del x 1 del J del x 2 is equal to 0. This equation system will give me nothing but A x is equal to b or b minus A x is equal to 0, which is A x is equal to b.

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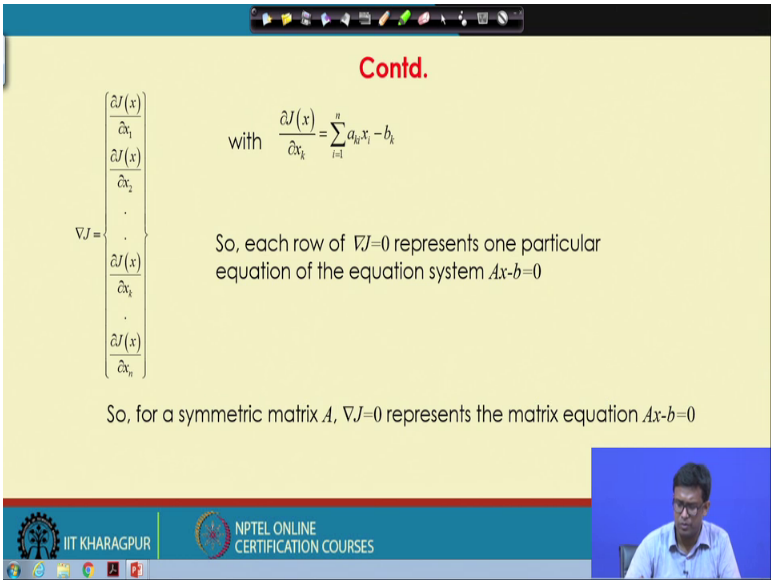

So, for us and this is only for symmetric matrix, because if it is a non-symmetric matrix, then I have to write half of a k i plus half of a i k x x i is equal to 0. So, for a symmetric matrix A gradient of J is equal to 0 actually represents the matrix equation A x minus b is equal to 0.

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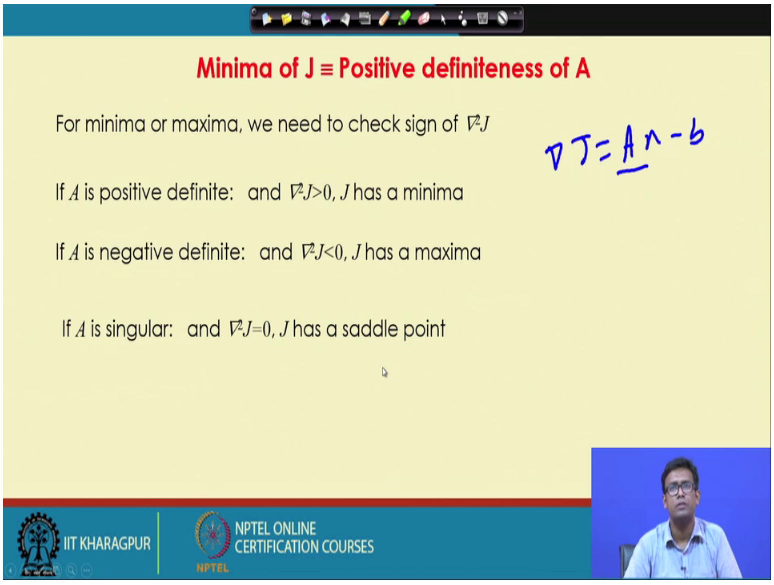

So, for minima or maxima that has to be decided from the sign of nabla square J is equal to 0. And J is equal to so we got gradient of J is equal to A x minus b right. So, second derivative of J or Laplacian of J will be the matrix norm of matrix A only. So, if A is positive definite, then nabla square of J is greater than 0, the matrix norm of a is greater than 0, and J has a minima. If J is negative definite, J has a maxima. And in case J is singular, then nabla square J is equal to 0 or J has a saddle point.

So, you can ponder upon the question that how a saddle point will be in a multidimensional space. However, like we call it and in a 2D space, we get an inflection point, if second derivative is 0, so it is something like that. So, if A is symmetric, then we got gradient of J is equal to 0, replay is similar to x is equal to b. And if A is singular positive A is positive definite, then nabla square J is greater than 0.

So, a gradient of J is equal to 0, nabla square J is also greater than 0 that means, J is minima, and their A x is equal to b. So, we have to look into the case, where A is a positive definite matrix, and nabla square J is greater than 0 that means, is where J has a minima. So, is a positive definite matrix and that means, J has a minima at that particular location, where del J del x is equal to 0. And if a is a symmetric matrix, then this particular location where del gradient of J is equal to 0 represents the matrix equation A x minus b is equal to 0. So, if A is symmetric and positive definite, then A x minus b is at a particular location, where J has a J is x transpose A x minus x transpose b as the minima.

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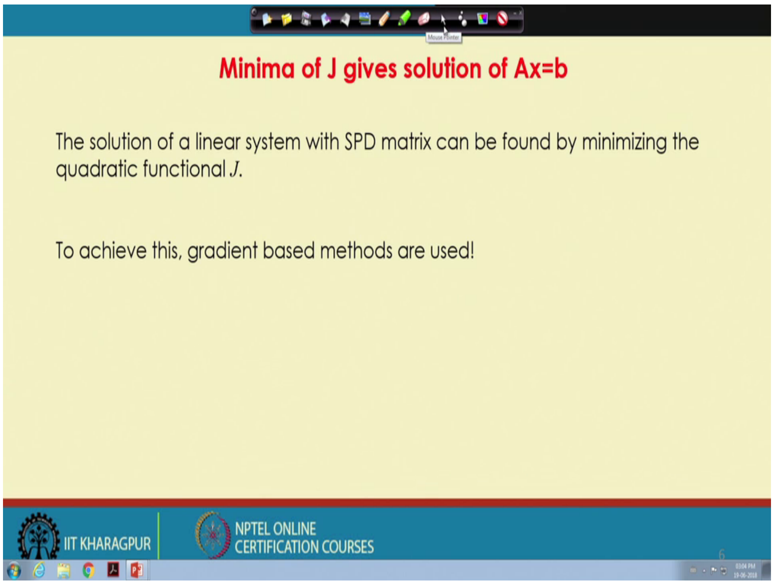

And that proves the theorem what we start started with and further tells us that minima of J will give us a solution of A x minus b. The solution of a linear system with symmetric positive definite matrix can be found by minimizing the quadratic functional J or we can write that J is equal to x transpose maybe I will write it here J is equal to x transpose A x minus x transpose b. At x star J x star is equal to J x min, then A x star is equal to b.

So, if I can find out an x star, for which J is minimal that particular a x star will give me the solution of A x star is equal to b. So, instead of solving A x star is equal to b, we can now think of finding minima of J x star. And this is this is, if and only if A is symmetric. So, this particular methodology, we are discussing, which we are only discussing this for symmetric positive definite. So, later we should also focus, how we can improve this to a general matrix, so which may not be symmetric, may not be positive definite, how we can improve this methodology for that particular case.

However, at this stage we are only considering symmetric positive definite matrix. And we observe that solving the matrix equation can be substituted by finding out minima of a quadratic functional. So, our idea will be, now because we have see seen solving matrix equation by direct iterative methods. And at the beginning of the class, I tried to list out the issues with the direct iterative methods, what where are they restricted.

Now, our goal will be instead of solving these using direct iterative methods, we will use an iterative method for finding out minima of J. Instead of solving A x star is equal to b, we will try to find out, how we can achieve minima of J. And the location, where we will achieve minima of J, we will say that this is the location, where A x star is a A x is equal to b is also satisfied.

So, we will say that the solution now is posed as a problem of finding minima. And to achieve this in an iterative methods, we will use something called a gradient based method. The method, we will use called as gradient base based method. And we will explore this method in the subsequent slides.

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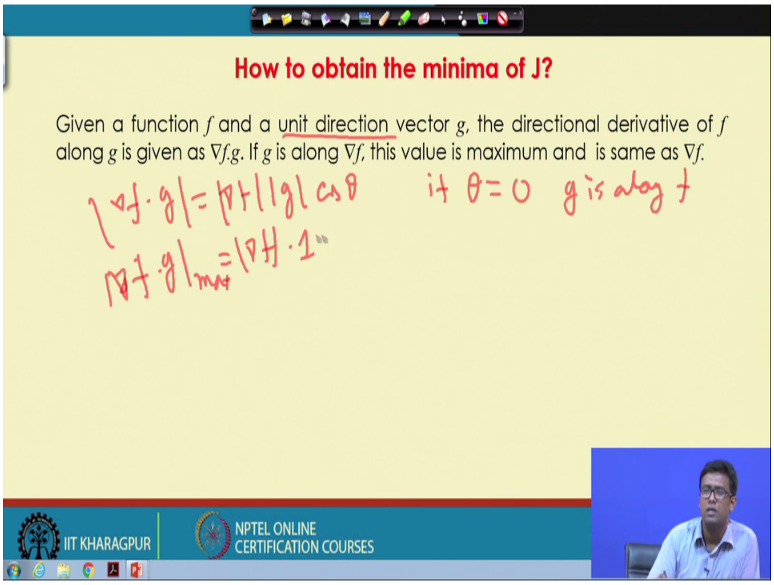

Now, the question is how to find the minima of J. In this class at least in this class, we have always discussed about matrices how to do work with matrices, how to find out solution of matrix, when we can have solutions etcetera. But, now we are in our little different paradigm, where we have to find out minima of a functional is little different than what we are doing in the matrix equations.

However, if you can remember the idea of symmetric positive definite matrix, when we discussed that we have looked into minima of a function and etcetera. So, now our question is how to find this minima in practical application. So, given when for example, when I am trying to find out a minima, I have a function and I have to see, where the function is minimum. So, I will start with the value of the function, if I am going in the right direction, I am approaching towards minimum of the function, because it is a continuous function, which you are discussing with. I will see that the value of the function and is reducing, it will reduce until it reaches a minima, and the then it will increase.

So, if I am going in an iterative method, say I am going in a trial and error method, I found out a value a particular value of J. And next it, I find out another value of J. If this is reducing, then I am approaching towards the minima. When I have approach the minima del J del x is equal to 0, so at that particular location, if I change my primary variable little bit, there is no change in the value. But, later I will see that the value of J is increasing with change in the primary variable. So, I will start with a value, and I will see if the value is reducing, then I am approaching towards minima. And in order to reach minima first, I will should see that the value of the functional is also reducing in the first test manner. So, we need to see, when the value is reducing and how we can reduce the value as fast as possible.

So, look little bit into vector calculus. Given a function f and a unit direction with a g, the directional derivative of f along g is given by gradient of f dot g. And this is directional derivative means, rate of change of f along the direction g. If g is along gradient of f, then g is an unit vector. So, gradient of way f dot and unit vector in that direction, and that is the maximum value of gradient of f dot g. And this value is same as gradient of f. So, this is probably straightforward that gradient of f dot G is basically with its magnitude is gradient of f into g into cos theta. If theta is equal to 0 that means, g is along f. And mod g, g is a unit direction vector. So, mod g is equal to 1. So, this is a gradient of f into 1. And this is the maximum value of gradient f dot g, because if theta is non-zero, then there is this is theta is our cos theta is always less than 1, and this is a smaller value. So, given a function f, and a unit direction vector g. the directional derivative of f along g will is grad f dot g. And grad f is the grad f is the maximum value of reduction of change of f along any or any of the direction. It is maximum along the direction of grad f, and the maximum value is grad f.

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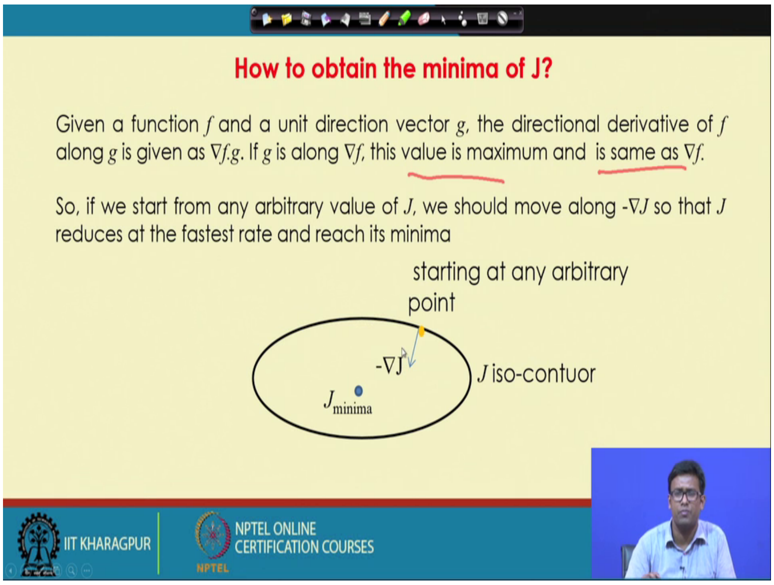

So, if we start from an arbitrary value of J, we should move along minus grad J, so that the reduction is if we move go along grad J, the rate is maximum. So, if we go along minus grad J, the rate is minimal or the reduction is maximum. So, if we start we will start from an arbitrary value of J, and we will move along gradient of minus of gradient J, so that J reduces at the fastest rate, and it will reach its minima. So, if there is a J minima in the in this 2D space anywhere, and this is a J ISO-contuor.

Now, I start from one particular location here, because there is an iterative method, I have to start with something, I start from this method. And I have to read J minima. So, how should we go, I will see that in which direction J is reducing fastest. And the reduction is along gradient of minus J. So, I will move along from starting from some value x here, I will change the values along this line and see, where the value is minima, because it will reduce fastest in this particular direction.

Now, when I am in this location, it is reducing fastest. But, when I came to here, then J has been changed. So, minus grad J is probably different, it is not reducing in the fastest way. So, it might here it is not reducing in the fastest way, it might not ever reach the minima, we will see later. So, we have to use an iterative method for that. However, if we start from one particular point, I have to grow along minus grad J, so the reduction is fastest.

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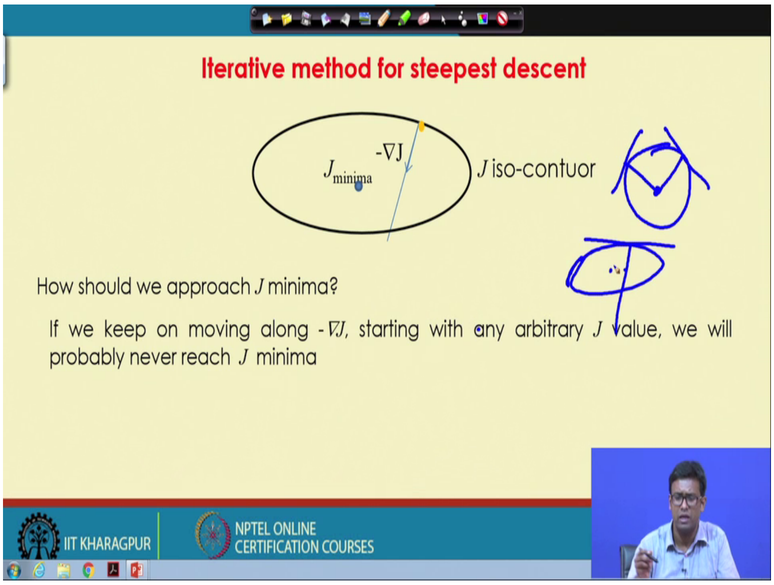

So, and the question is so I know that I started from an arbitrary point, I am going along minus grad J, and I have to come to J minima, how should we reach that. So, if we move along this particular line, we might not be ever able to reach the J minima, because along this line J is first reducing. And then, we can see, if we keep on minus grad J, we can see that it is clearly missing J minima, starting from any arbitrary J value, we will possibly never reach J minima. We will reach J minima in only in one case that is the functional of J is a perfect circle.

So, if we start from any point, the normal should take me to the center. But, if it is not a circle, we if it is an ellipse, and I start from somewhere here, I will the minima is here, I will never reach the minima, I will go somewhere else. So, you will probably never reach the minima. However, we can see this we will start we if we draw this is the J i, so contour that means, J is constant here, so minus grad J is normal to that. So, we can see that up to suddenly will J will reduce, because J is initially reducing here. And there will be local minima on this; we will probably go up to there. And then see, how we can again move.

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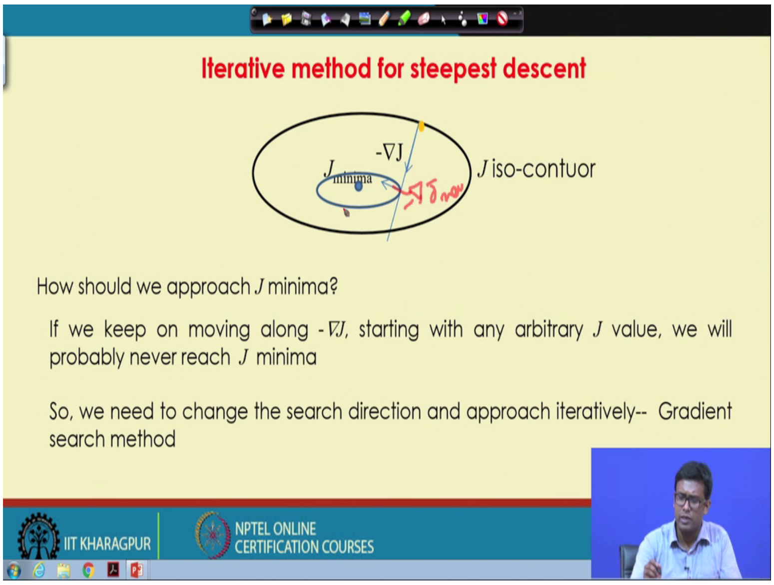

So, we need to change the search direction and approach iteratively. We will see we will go along minus grad J, and reach to one particular level. And then, again I will calculate the nu minus grad J here. And move along this direction, I will probably enrich somewhere from there, I have to again calculate minus grad J and move along that direction that is the iteration technique necessarily here. So, we need to change the search direction, and approach iteratively. And this method is called the gradient search method. In the next class, we will look into more detail of the gradient search method.

Thank you.