Matrix Solvers
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Lecture - 34
Convergence Rate and Convergence Factor for Iterative Methods

Welcome. In the last session, we try to look into the matrix representation of Jacobi or Gauss-Seidel method or any basic iterative solver method; we classified Jacobi and Gauss-Seidel as basic iterative solvers. And from the matrix representation, we also found out the iteration matrix G. And observed that for certain property of G, there is matrix norm of G must be less than 1. If we start with any guess value, the iteration should converge to the exact solution of A x is equal to b, and that is because the error between the guess value and the actual solution, which is maybe at k-th iteration level, this is x k minus x x star, x star being the actual solution, this error will reduce in the next iteration level, as it is multiplied them it by the iteration matrix G.

So, if this multiplication reduces the length of the error vector, then after number of iterations the reductions will be multiplied on each other, and then it will go to an infinite small value,. And therefore, this should converge to the exact solution. Now, there is another issue, which we are discussing that, when the iterations converge to the exact solution, we can see that there is practically no difference between the guess an updated value, they also converge. So, we will also look in the convergence from the perspective that the guess an updated value mean practically same, they are also converging, and what are the requirements for them.

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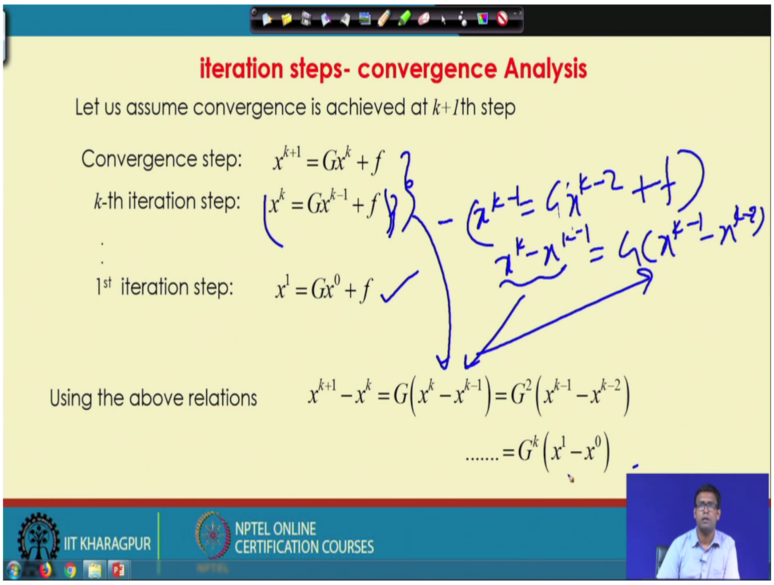

So, we start with assume that convergence is achieved at k plus 1th step. The convergence step then is x k plus 1 is equal to G x k plus f. Now, the step before that, which is k-th iteration step, here it is not converge. Convergence means the difference from the exact solution and the actual solution is of the of a very small number, and we make a criteria for that, say this number is10 to the power minus 8. When the difference is less than 10 to the power minus 8, it will tell that it has converged.

So, let us assume that this convergence has been done in k-th iterations k plus 1th iteration step. And in k-th iteration step, though it is not converged, but the relation of the guess and updated value is similar, x k is equal to G x k minus 1 plus f so on, we can come down to first iteration step, which is x 1 is equal to G x 0 minus f. Using the above relationships, we can write x k minus 1 is equal to G of x x k plus 1 minus k is equal to G of x k minus 1. By using these two, we can write this x k plus 1 is equal to G of this. Now, again using the next one x k, we can write that x k minus 1 is equal to G of x k minus 2 sorry G of x k minus 2 plus f.

So, if we subtract from these two, we will get x k minus x k minus 1. So, this minus this, this minus this is minus x k minus 1 is equal to G of x k minus 1 minus x k minus 2. So, the if now this we substitute by G f x k minus minus x k minus 2, we will get G square x k minus minus x k minus 2 and so on, we will get G to the power k x 1 minus x 0, till we come to this equation. So, the difference is also being multiplied by G to the power k, reference from the guess and updated value.

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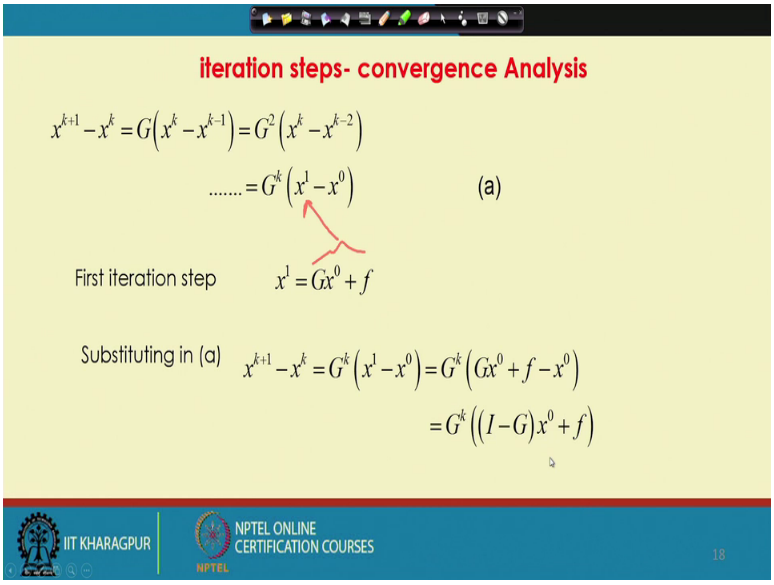

So, we write x k plus 1 is equal to minus x k's so on, it is G to the power k x 1 minus x 0. And the first iteration step is x 1 is equal to G x 0 minus f. So, substituting x 1, so what we will do, we will substitute x 1 into the relationship a. We can write that this combination G x k minus 1 x k etcetera, G to the power k I minus G x 0 plus f.

Now, I minus G is a the initial requirement was that I minus G must be a non-singular matrix, so if it is a non-singular matrix. For any x 0, which is multiplied by G to the power k this should will be a very small number, if G also again has a matrix norm less than 1.

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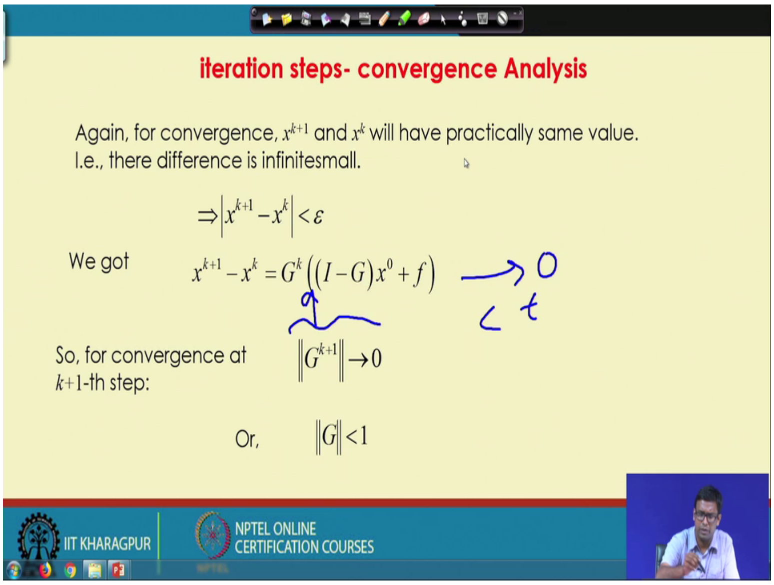

So, we can say that for convergence, x k plus 1 and x k we have practically same value. And therefore, that difference is infinite small, which will need that x k plus 1 minus x k is less than epsilon, we got x k plus 1 is minus x k is G to the power k I minus G x 0 plus f. x 0 can be assume that I minus G is a non-singular matrix, which is requirement for a Gauss-Seidel step. x 0 is a is an arbitrarily chosen vector, so x 0 can be anything. Whatever be the value of x 0, this value has to be less than a very small number epsilon, this value has to be less than a value has to be an infinite small value.

And that needs that the convergence of k plus 1-th step which is x k plus 1 minus x k will be same convergence in that term, will happen only if the norm matrix norm of G k plus 1 goes to 0. Then this value if G k plus 1 is if this happens, then this value should also go to 0 or should be less than epsilon. And this is again possible, if matrix norm of G is less than 1.

So, we can look into two steps things, one is that if modulus of if matrix norm of G is less than 1, then the solutions will converge in a sense that the difference between two guess an updated will be is essentially 0, very small number. The solution the iterations will converge to a value. And, the converge value will be the exact solution. So, both things are satisfied, if the matrix norm of G is less than 1.

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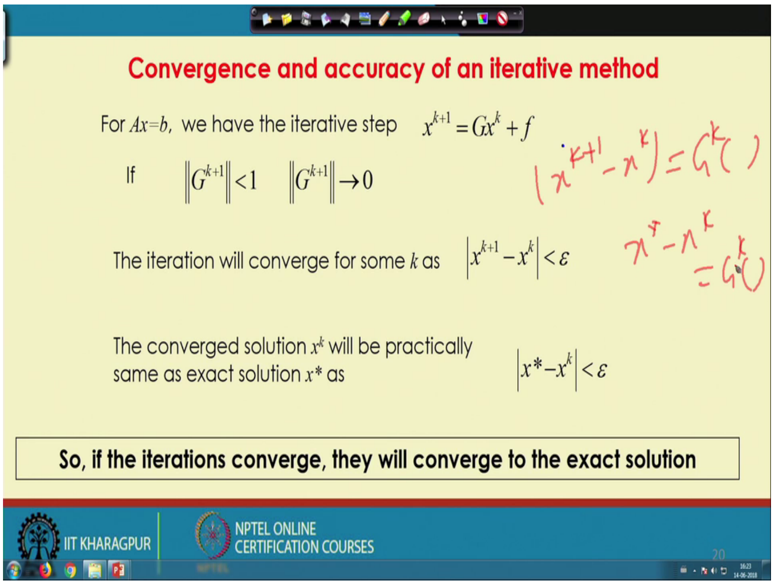

So, A x is equal to b, we will look into convergence and accuracy of the iterative method. A x is equal to b, we have the iterative step x k plus 1 is equal to G x k plus f. If matrix norm of G is less than 1, then if sorry if matrix norm of this is not G k plus 1, if matrix norm of the if G is less than 1, matrix norm of G to the power k plus 1 is basically multiplying the matrix several times, this goes to 0.

And once this happens, the equation iterations converge for some k for which the difference between guess an updated value is very small. Also the converge solution x k becomes practically same as the exact solution x star, because these two differences are small. So, therefore, if the first thing happens, if the first thing happens, this will happen for this particular criteria, and which for which this should also happen, or if the reverse happens like the solution practically converges to the exact solution, we will see that the iteration guess an updated value has also converge to the same value.

So, if the iterations at all converge, if we come into a stage that the x x value is not being updated guess an iterations are being same, then the iterations have converged to the right solution. And that is the Robustness of this particular technique, Gauss-Seidel or Jacobi or we will see successive over relaxation as a class of these methods. These class of technique that if we can have converging solution converging iteration, the final result must be same as the actual result. So, we can sum it up as, if the iterations converge, they will converge to the exact solution.

Now, the questions come that under which case iterations will converge, of course we know that by now that mod G matrix norm of G should be less than 1. And when matrix norm of G will be less than 1, we have to look into the A matrix, because we are doing very two steps to get G matrix from we are doing one type of splitting of A in Gauss Jacobian, we are doing another type of splitting of A in Gauss-Seidel, but this is basically splitting of the matrix A.

So, through splitting of the matrix A, we are getting the iteration matrix G. The iteration matrix G has a norm less than 1, this is our requirement for convergence of Gauss-Seidel Jacobi methods. And this requirement is satisfied based on what based on how is the A matrix. So, you have to look into the A matrix, and see that in under which cases splitting of A matrix gives us a G which has matrix norm less than 1.

Another thing we can also see that the value x k plus 1, if we go to the maybe we can go to the previous slide and or here we can see also, that this value x k plus 1 minus x k is something is this is like this value x k plus 1 minus x k is G to the power k into something. Similarly, x star minus x k is also G to the power k into something. So, all this is some initial value multiplied with G to the power k. So, what is G, if G is a very small number within very less number of steps, G to the power k will be a infinite small number.

If G has to be less than 1, but G G is close to 1, say for example G is 0.9, then it will take large number of steps. Consider to the case, G is 0.1 matrix norm of G is 0.1, it will take less number of steps. So, how first will be the convergence that will also depending on, how is the G matrix. And how is the G matrix, that depends on how is the A matrix, because G is coming through splitting of A. So, we will look into the properties of A matrix for which these things will heard.

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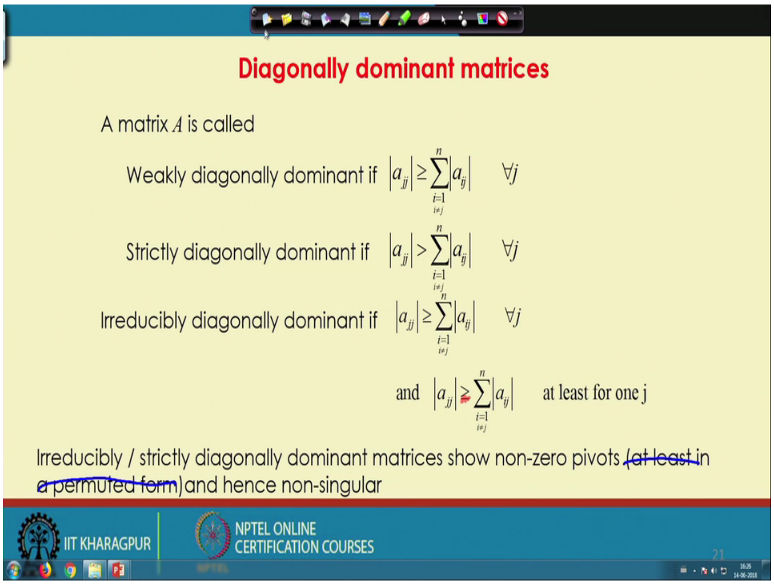

Matrix A is called weakly diagonally dominant matrix, we will look into few definition, and the definitions of diagonal domine diagonally dominant matrices. A weakly diagonally dominant matrix if the all the any diagonal term we considered any diagonal term that is greater than the all of diagonal terms of that particular sum of all of diagonal terms of that particular row in their absolute form greater than equal to, then we called weakly diagonal. They may be greater than the sum of off diagonals, or may be is equal to the sum of off diagonals.

We call it strictly diagonally dominant, if all the diagonal terms is greater than equal to sum of the of diagonal terms of that particular row, except the diagonal term of all diagonal terms. We call it to be irreducibly diagonally dominant, if for all j, it is weakly diagonally dominant, but there is at least one row or at least one j for which the diagonal dominance is there.

So, for all j, absolute value of the diagonal term is greater than equal to sum of the absolute value of the off diagonal terms. But, there is at least one j for which at least one row for which the diagonal term is absolute value of diagonal term is greater than this is not greater than equal to, this should be greater than greater than the sum of off diagonal terms. So, and the requirement is that another thing is that irreducible or strictly diagonally dominant matrices show non-zero pivots, at least in a permuted form, and hence non-singular; not in permuted from, also a non-permuted form also. Strictly diagonal matrix are reduced severally matrix as show non-zero pivots. So, the permuted form is not important here.

And therefore, these matrices strictly diagonally dominant or irreducibly diagonal dominant matrices are non-singular also, where we discussing this, because it has a relation. Remember at the beginning when is introducing Gauss-Seidel, I was introducing Jacobi, I said that this is only valid for diagonally dominant matrices. Now, we will see that only for diagonally dominance or irreducibility diagonally dominant matrix, the G will be such that that the matrix norm of G is less than 1, and that is why these methods will be valid.

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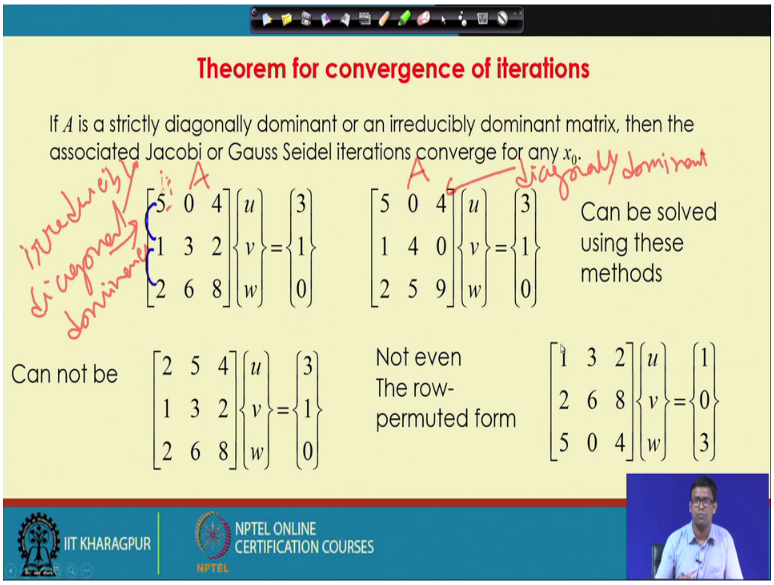

If A is strictly diagonally dominant or irreducibly diagonally dominant matrix, then the associated Jacobi or Gauss-Seidel iterations converge for any x 0. This is the theorem for convergence of iteration; when the Jacobi and Gauss-Seidel iterations converge, when G has a matrix norm less than 1. What is G? G is the G is a split comes from splitting of A.

And this condition if we look into A comes as if A strictly diagonally dominant or an irreducibly diagonally dominant matrix, then associated Jacobi or Gauss-Seidel converge that means, G will have matrix norm less than 1. So, for which type of matrices, this will converge, for example the first matrix 5 0 4, 1 3 2, 2 6 8 is A matrix. So, these are the A matrices.

Now, if I look here, this is a diagonally dominant line, 5 is greater than sum of this. This is weakly diagonally dominant 1 plus 2 is equal to 3, 2 plus 8 is equal to 6. However, this becomes an irreducibly diagonal dominance is there irreducibly diagonally dominant matrix. And this is the diagonally dominant matrix. So, for these two, this A x is equal to b can be solvable using Jacobi or Gauss-Seidel.

If in case, this would have been 4 in case instead of 5, this is 4, we could not have been able to solve this equation, because then it would have been a weakly diagonally dominant. So, for weakly diagonally dominant matrix, Gauss-Seidel or Jacobi cannot work. You can try, you will discuss about writing our own program, but you can try also this by hand, even paper, pencil, you can try few steps, and see what is happening to this. You will see that the values are not converging x k plus 1 minus x k, if it is weakly diagonally dominant x k plus 1 minus x k, it is not reducing, it is increasing or remaining constant at a I value.

If it is not a diagonally dominant matrix like if we look into the first row 2 plus 4 is equal to 6, which is 5 plus 4 is equal to 9, which is greater than 2, then it cannot be solved. Also if we think of doing a row permutation of this matrix, like I will permute these rows, I will again permute this and this, if we think of a row permuted form of this matrix, is it does not remain a diagonally dominant matrix.

However, the solutions will remain same, because row permuted form and the actual form should give us same solution. But, if we do row permutation, the matrix losses diagonal dominance and we cannot solve it using Jacobi or Gauss-Seidel. Though row permutations remain the solutions in solutions exist, and solutions are same with a diagonally dominant matrix or irreducibly diagonally dominant matrix, and the row permutated matrix form of that matrix solutions are same. However, the row promoted from, as it does not remain diagonally dominant, cannot be solved using Jacobi or Gauss-Seidel method. Why, because of the fact that G has been changed.

So, in case, we get a matrix which is solvable, but does it is not in a diagonal dominant form, we can try row permutations, and give it to a diagonal dominance form. And only in the diagonal dominant form, the matrix can be solved. This cannot be solved using a Jacobi or Gauss-Seidel method, because this is though it these two matrices are permutated form of each other. However, 2 plus 2 is equal to 5, which is greater than 1, so this is not a diagonally dominant form, so it cannot be something (Refer Time: 18:55).

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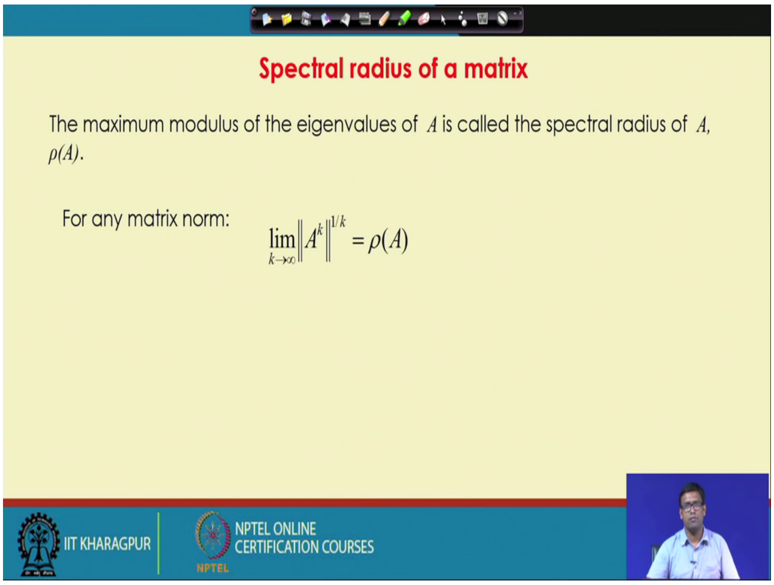

Here comes another important definition is that the maximum modulus of eigenvalues of A is called spectral radius of A or row A. And this is given as any for any matrix norm, we can take 1, 2, 3 up to p or infinite matrix norm. Matrix norm of matrix A to the power k to the power 1 by k and as limit, k goes to infinity, this is row of A. So, maximum and this is same as the maximum modulus of eigenvalues of A spectral radius.

So, any matrix raised to a high power, and then we take matrix norm, and then take say that root of that matrix 2 to the power, it was raised with the limit is the spectral radius or the largest eigenvalue of A. This is a definition.

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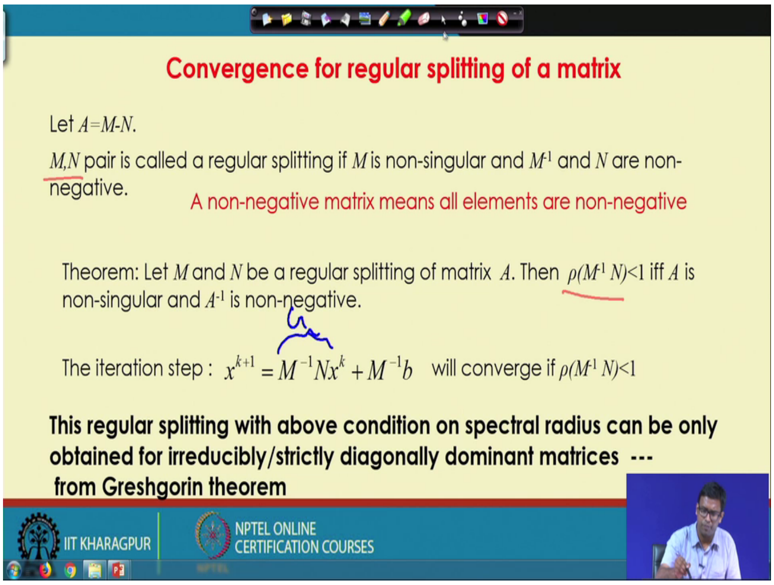

So, let A is equal to M minus N. M, N pair is called regular splitting, if M is non-singular and M inverse and N are non-negative. Then A is equal to M N is called a regular splitting of matrix A. A non-negative matrix means all elements of that matrix are non-negative. Let M and N be a regular splitting of matrix A. Then row M inverse N is less than 1, if A is non-singular and A inverse is non-negative. So, if M and N are regular splitting of A, then rho inverse N is rho rho of M inverse N less than a 1 is non-singular, if A is non-singular and A inverse is non-negative.

And the iteration step in that case, the iteration step x k plus 1 is equal to M inverse N x k plus M inverse b will converge to rho of M in converge, if this rho M inverse N (Refer Time: 21:08). This is basically the G matrix right this is basically the G matrix. This will converge, if the matrix norm of G G to the power k is very small or matrix norm of G is less than 1, which is spectral radius of G is less than 1. So, if spectral radius of G is less than 1, x k plus 1 G x k plus f will converge. And this will happen, if M N leaner regular splitting of matrix A that means, M is non-singular and M inverse and N are non-negative.

And now, there is there was a scientist named Greshgorin, who found out the theorem on the diagonal dominance of a matrix and its eigenvalues. And as a corollary of the theorem, we can say that this regular splitting with as above conditional spectral number row I mean M inverse N is less than 1, can only be obtained for irreducibly or strictly diagonally dominant matrices. This is comes from Greshgorin. So, if the matrix A is strictly diagonally dominant or irreducibly diagonally dominant, we can get a regular splitting of M and N following our Jacobi or Gauss-Seidel method. And then, we will also see rho of M inverse N, where M is M inverse is and N are non-negative, rho of M inverse N is less than 1. A inverse will also be non-negative, and A is non-singular in that case, if we have dominant or diagonally dominant matrix.

So, rho G is less than 1, and diagonally dominance, they are actually related. And this relation comes through Greshgorin theorem. I I am not going into detail of Greshgorin theorem here, but this discusses how should be the depending on the eigenvalues of the matrix, how should be the diagonally dominant diagonal term and off diagonal term and their arrangements.

So, now we what we got is that if the matrix is non-singular, A inverse is non-negative, if it can have a regular splitting, which is for a irreducibility diagonally dominant or strictly diagonally dominant matrix. Then G should have convergent G should have a spectral radius which is less than 1, therefore matrix norm of G must be less than 1, and matrix norm of G to the power k must go to 0.

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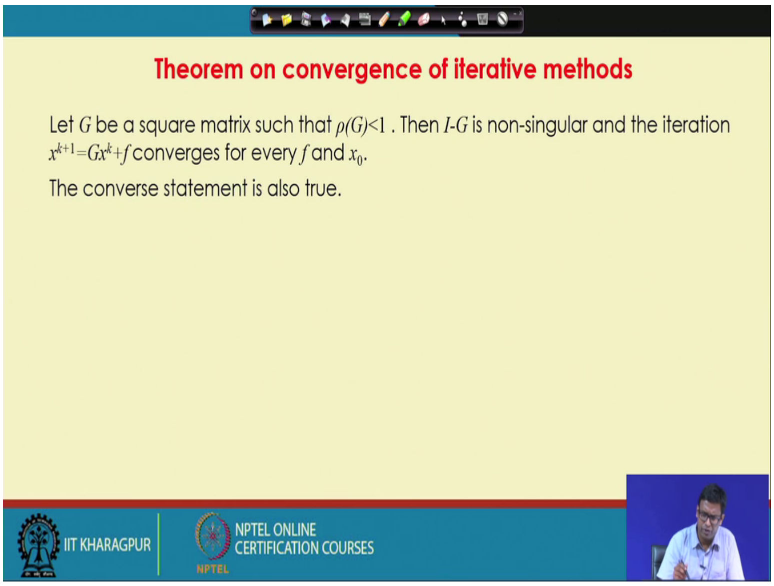

So, let G be a square matrix, such that rho of G is less than 1. Then I minus G is non-singular and iteration G x k plus 1 is equal to G x x k plus 1 is equal to G x k plus f, this iteration also converges for every f and x 0. And its converse is also true.

So, if G is a square matrix, so that rho of G is less than 1 obviously, I minus G will be non-singular, and the iterations will converge, and that converge statement is also true. And this happens, if and only if A is a diagonally dominant or irreducibility diagonally dominant matrix. So, these say these are the cases in which we will have convergence of the iterative methods. And this is very important Gauss-Seidel and Jacobi are very robust methods, but only for the matrices, which is dominant diagonally dominant or irreducibly diagonally dominant.

Now, you can see that the convergence depends on G rho of G has to be less than 1, spectral radius we made things much simpler and much easier to quantify at this stage. Instead of looking into the matrix norm of G, we can just consider the spectral radius of G or the largest eigenvalue of G. If that is less than 1, then the equation system will converge, and that should be done for any diagonally dominant or irreducibly diagonally dominant matrix.

Now, the question comes is that, how first will they converge, what will be the number of steps, when we Rana Gauss-Seidel Jacobi program, how many iterations will it need. If it needs a very large number of iterations, there is a much not much need of running the iterative methods, because probably the direct solvers can give us faster solution, but if it less takes less number of iterations, we can do it.

So, how to see what is the rate of convergence or how fast do they converge, how many steps needed for convergence of an iterative method. It is also important to see, which will do in the later section also. That if we can improve the rate of convergence, if we can do certain things, so that the convergence rate improves or if we can if we play with the splitting of the matrix, so that we can get faster convergence. So, we before going into this, we will try to define two more parameters regarding convergence.

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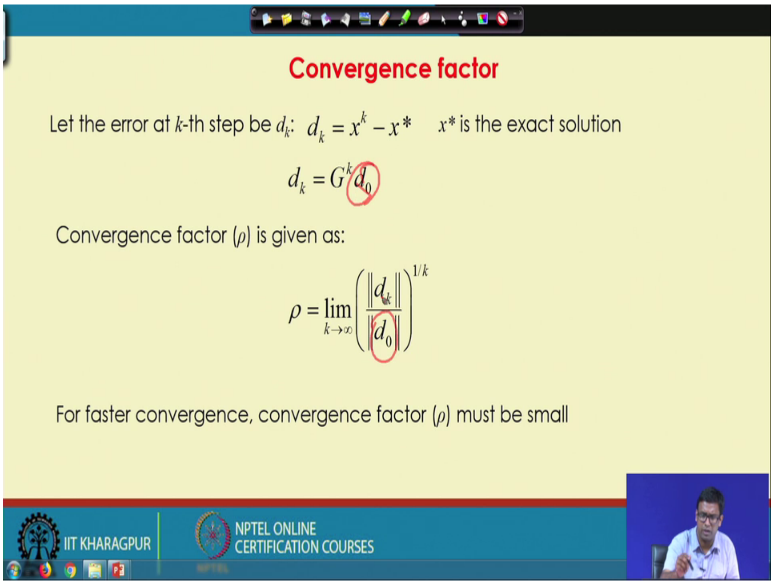

One is the convergence factor. Let the error at k-th step be d k, which is x k minus x star; x star is x exact solution. The difference between the Gauss Vector and the Solution Vector is the error, we define it as d k. d k is G to the power k into d 0 initial error into G to the power k, you have seen that earlier.

The convergence factor which is defined by rho is given as rho is equal to limit k goes to infinity. So, this rho is not spectral radius, rho of a matrix is the spectral radius, simple rho is the convergence factor. There are two rows must not be convergence. Row limit k goes to 0 d to d k d by d 0 1 1 by k to the power 1 by k. For faster convergence, the convergence factor must be solved must be small. So, how should we determine convergence factor, we after k-th iteration after a large k is a large number iteration, we see what is the solution, what is the error, and the ratio between these two errors to the power 1 by k is the convergence factor.

If the factor is small that means, its the error is reducing in a first way, so convergence, so we need less number of iteration steps or convergence factor. However, if we look into the previous slide, this definition depends on the term d 0 right. We have to start with a d 0, and see what is happening.

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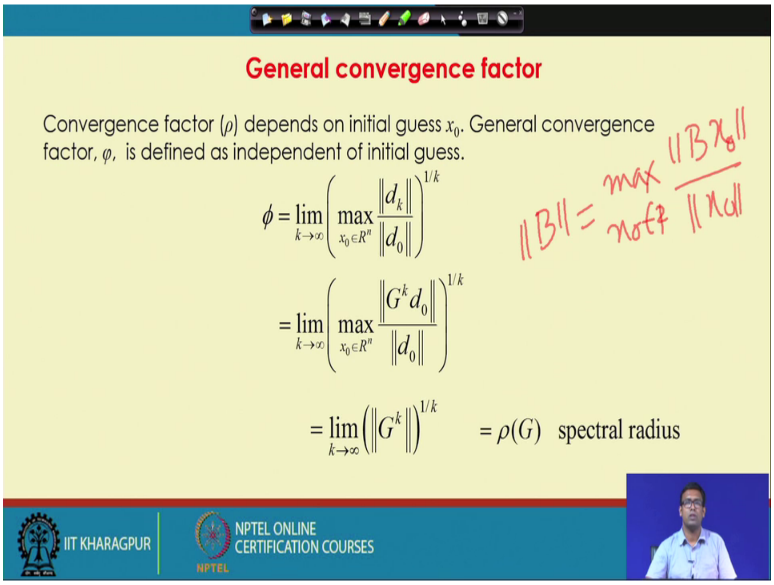

So, a better definition is given which in which, we do not need to a have we do not have to need we do not need to depend on x 0, we call the general convergence factor. And it is defined as independent of initial guess phi general convergence factor is limit k goes to 0, maximum of x 0, which belongs to r to the power n d k by d 0 to the power 1 by k. And this is maximum d k is G to the power k d 0 maximum of matrix norm of G to the power k sorry vector norm, but these are ratio of vector norm G to the power k d 0 by d 0 1 by k.

Now, by definition, this is the definition of say vector norm of any vector B is equal to maximum of x 0 belongs to R matrix norm of any matrix B is maximum of x 0 belongs to R B x 0 by x 0. So, this is the definition of matrix norm of G to the power k. So, we get limit k goes to infinity, matrix norm of G to the power k to the power 1 by k, which is nothing but spectral radius.

So, if we try to see maximum of the convergence factor maximum value of the convergence factor for any guess x any gauss x 0, we will see that is the general convergence factor, which is spectral radius. So, spectral radius now the smaller the convergence factor, faster is the convergence. Therefore, smaller the spectral radius, faster should be the convergence. We need to have a matrix G, whose largest eigenvalue is a small number; therefore we should get first convergence.

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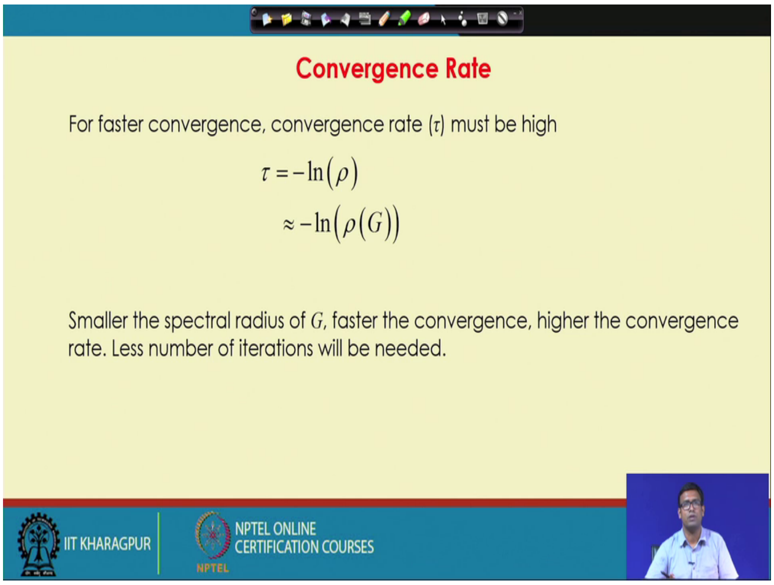

And there is another term called convergence rate. For faster convergence, convergence rate must be high, so it is kind of has an inverse relation with this vector convergence factor. If the convergence factor is small, then convergence is faster. If the convergence rate it rate is high, convergence factor is faster. And convergence rate is defined as minus log of rho tau is equal to minus log of the logarithmic of convergence factor, which is minus logarithm of equivalent to general convergence factor and minus log of row G.

Smaller the spectral radius of G, higher is the convergence rate, and therefore faster the convergence had the convergence rate. So, less number of iterations will be needed. So, we need to see that and that will give an idea, that how we can increase the improve the convergence criteria through splitting, which one will give us faster convergence Jacobi Gauss-Seidel or some gradient of it, for which, we get smaller spectral radius of G.

If that spectral radius of G is equal to 1, there is no convergence it is skilled right, I minus G is singular, it cannot converge. If spectral radius of G is less than 1, that will this will converge and as small as it will be in spectral radius this is the modulus of the eigenvalues, so this cannot be negative. So, as small as it be as it is a in between 0 to 1; as small as it be faster will be the convergence.

So, by in next class, we will look into some matrices and look into the spectral radius of this of the G associated G matrix. And see for Jacobian Gauss-Seidel, how is the convergence rate. And we will also see if we can do something with the matrix or we can design a better matrix solver, which we will have less condition smaller spectral radius, so that the matrix the solver convergence in a faster way than Jacobi and Gauss-Seidel. We will look into this in the next class.

Thank you.