Matrix Solvers

Prof. Somnath Roy
Department of Mechanical Engineering
Indian Institute of Technology, Kharagpur

Lecture – 33
Basic Iterative Methods: Matrix Representation

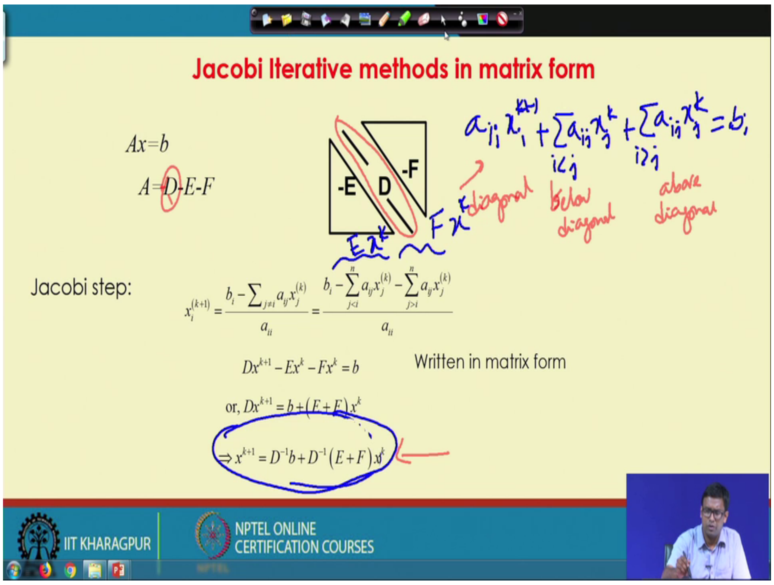
Welcome. We have been discussing about basic iterative methods namely Jacobi iterations and Gauss Seidel iterations. In last class we try to discuss about the this algorithms for this basic iterative methods.

And also try to give an idea about what is convergence of the iterative methods or when should we say that the iterations have converged to the exact solution or near exact solution with the difference being very small number which is kind of the machine precision of the particular computer is of that order. And we will say that the solution have converged to that solution. So, we will start with a guess value and follow the iteration steps and at the end we will get a converge solution which is the practically the exact solution of the equations.

So, these methods we have discussed in the last class. And we look into more detail of these 2 methods Jacobi and Gauss Seidel iterative methods in this class in here and also in the subsequent class. And we will try to have better insight on the convergence of these methods what are the requirements for these methods and as well as what are the methods to improve convergence. So, that we can get the converge solution in less number of iterations in less time and also doing less computational work.

Since, start looking into the both this method Jacobi and Gauss Seidel methods. So, first let us look how should this method like look like when we represent them in the matrix form. So, we are discussing about Jacobi iterative methods first and then we look into it the matrix form, because last class we have only looked into the equations individually and look looked it as a collection of equations not as a matrix equation. So now we will we will start looking into it as a matrix equation.

(Refer Slide Time: 02:11)



So, it solving the matrix equation Ax is equal to b. Now if you see in Jacobi in each step what were we doing is basically we are starting with a guess value and we are updating the guess value based on the iterations. So, the updated value maybe the updated value maybe x k plus 1. So, this is the fall out of first iteration.

We start with first iteration with the guess value x 0 and the updated value is x 1. So, this is for the ith term and for the term for ith equation will only update the ith component of the x vector or ith x, which is a a ii which is the diagonal term xi k per at k plus 1th level. Then this is the ith equation in the k th iteration or rather k plus 1th iteration ith will looking into the ith equation.

So, this will be sum of a ij x j at k th level and i less than j. So, except i is equal to j all the terms will be considered as the at the at the previous value or at the guess value, ij x j k and here i is greater than j is equal to bi means right. So, this is what are the form of Jacob iteration have given in the last class.

So, this a ii now if we look into these terms, this is the diagonal term this a ij x j k, but i is less than j; i is less than j means they are the sub diagonal terms or the lower triangular part terms. So, this is below diagonal and this is above the diagonal of the matrix.

So, a matrix can be composed into 3 components. One is the diagonal another part is a lower triangular form, which is the below diagonal part another part is a about triangular part which is the above diagonal part. And with x vector and the updated x vector are multiplied with these 3 matrices for Jacobi the x vector is multiplied the guess x vector is multiplied with only the below and above diagonal term and updated vector will be from the diagonal term. And in Gauss Seidel the updated vector is multiplied with diagonal and below diagonal term and the guess will be updated with the above diagonal term.

So, you can look into the we can say that the matrix Ax is equal to b for matrix equation a can be split into 3 components. One is a diagonal component D another is the lower triangular component minus E another is the upper triangular component minus F.

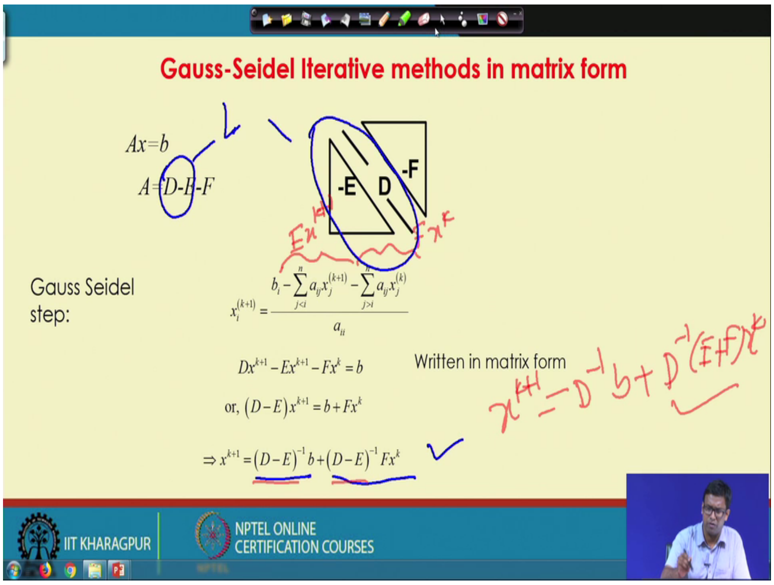
And now we can write the Jacobi step as x i k plus 1 which is the updated x vector from here I can directly write it is b i minus j is not equal to i a ij x j at kth level by a ii, which will mean that bi minus the terms of x vector which will be multiplied with the lower diagonal part. So, this is basically this part is basically my E, instead of minus am writing it E because I have taken it into the right hand side and this part is F.

So, this is E x and this is F x and this x is at kth level. So, we can write as the matrix equation D x at k plus 1 minus E x which is at the guess level kth level minus F xk is equal to B. This is the matrix form of this equation and we can club them and write that D x k plus 1 is equal to b plus E plus F x k or x k plus 1 can be obtained as D inverse B plus D inverse cf E plus f x k.

So, though this is a matrix equation and we are getting inverse of the matrix because this D is a diagonal matrix, D inverse is getting D inverse is nothing but dividing it by the term 1 by a ii. So, D inverse is basically a matrix where all the inverse of all the diagonals are present in that matrix. And this is trivial to find out the inverse. So, this step is not basically a matrix inversion stable rather simple algebraic step one operation algebraic step to find out x k plus 1.

So, nevertheless this is the matrix representation of a Jacobi step, is a matrix representation of a Jacobi step. This is in same light we can also represent a Gauss Seidel step. So, let us look into a Gauss Seidel step.

(Refer Slide Time: 07:58)



The Gauss Seidel equation will be that updated vector will be multiplied with a lower diagonal term also. So, this is E x k plus 1. And with the upper diagonal only with the upper diagonal the guess vector is multiplied. So, this is F x k.

So, we can write the Gauss Seidel matrix form as D x k plus 1 minus E x k plus 1 minus F x k is equal to B which is D minus E x k plus 1 plus is equal to b plus F x k. And x k plus 1 is D minus E inverse b minus D minus E inverse F x k again if we look into matrix form this is inverting a lower triangular matrix because this D plus E minus E or this part gives me a lower triangular matrix l.

So, inverting a lower triangular matrix is also not very difficult and then that is how we can get this step that easily for. So, both Gauss Seidel and Jacobi can be formed as a matrix equation in which x k plus 1 or the updated x vector comes out as some inverse multiplied with another matrix into x k plus some inverse multiplied with the v vector. So, will try to look into the general form and this is important because the properties of these iterative steps which will determine how first will they converge or what are the restrictions on their convergence will come through looking into these matrices. This is related with the properties of this matrix because.

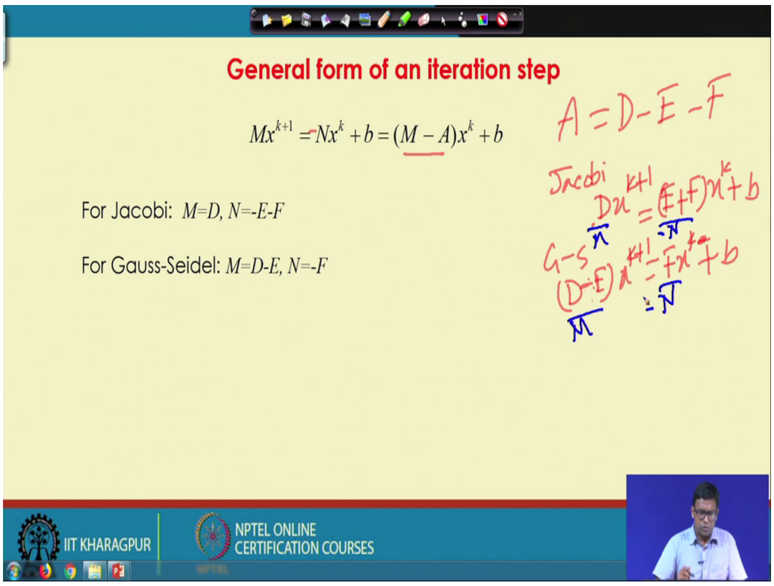
Now, we can see that this is right now is a matrix inversion problem for example, D earlier we got D inverse E plus F x k. So, if D in the in the Jacobi method we got the step to be x the Jacobi method we got the step to be x k plus 1 is equal to D inverse b plus D inverse E plus F xk and the requirement for this is that D inverse cannot be a singular matrix. So, D must have all nonzero vectors here.

Similarly, requirement for Jacobi so, this is this is an apparent requirement we look into the requirements in much more detail in next few through after next few slides. Here the requirement is that D minus E inverse has to exist. So, D minus E cannot be a singular matrix. So, diagonal minus the lower triangular cannot also be a singular matrix hm. So, all these require requirements for the case matrices for which we can get solution through these iterative methods will come out as this analysis.

So, it is important that you should look into this mate iterative steps in their matrix form. So, nevertheless both these forms like the Jacobi form and the Gauss Seidel form both this form has something similar.

And similarity is that in appearance they look similar the updated vector is some vector which contains some in inverse multiplied inverse of a matrix multiplied with the solution with right hand side vector b plus, inverse of some matrix multiplied with another matrix may be in 2 which is multiplied with the guess value x. So, this is a general form of this matrices it is like x k plus 1 is equal to say F plus M inverse N x k or something like that. So, let us look into the general form.

(Refer Slide Time: 12:32)



The general form of an iteration step both Jacobi and Gauss Seidel is M x k plus 1 is equal to N x k plus b where N is M minus A. So, a component of one component of A is subtracted from A and this comes as M or whatever is the M the remaining is M minus A if we sum at the converge state when if it converges to one particular solution. So, m x is equal to M minus x b which will A x is equal to b.

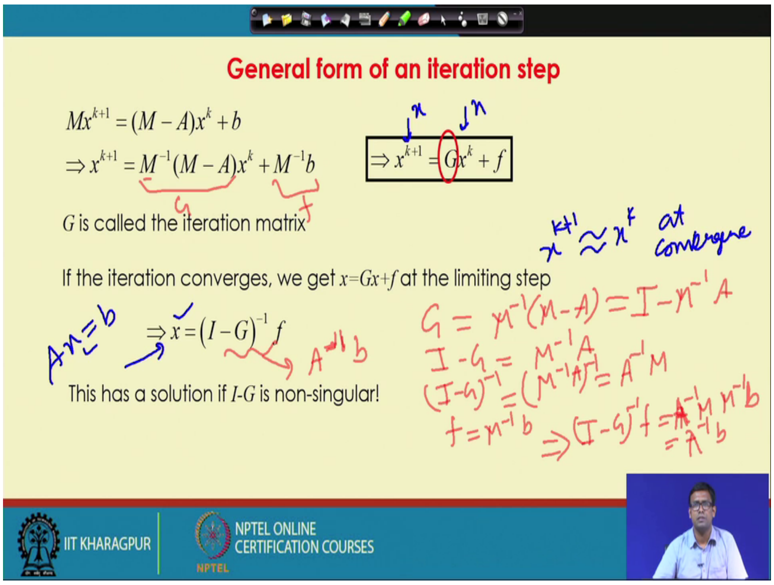
And if we think if we recall A is equal to D minus E minus F diagonal minus lower triangular right diagonal minus the lower triangular by minus of lower triangular. So, diagonal lower triangular an upper triangular part. And then Jacobi it is D x k plus 1 is equal to E plus F x k plus b. And in Gauss Seidel this is D plus E xk plus one is equal to F x k plus B.

So, you can see that if we add if this is M and this is n. Similarly, here this is M and this is N if we add M and n this should give me A. So, we can write that for Jacobi M is equal to D, N is equal to minus E minus F. For Gauss Seidel M is equal to D minus E it is not D plus E rather it is D minus c because this is D. This is D minus E (Refer Time: 14:36) and this should be minus n rather this should be minus N.

So, N minus N is equal to E minus E minus F and minus N is equal to E plus F and minus n is equal to my F here. So, this is minus N. So, nevertheless so, both this can be expressed as a generate general form which is M x k plus 1 is equal to minus N x k plus B.

So now, if we further look into this steps.

(Refer Slide Time: 15:25)



That M x k plus 1 is equal to M minus A x k plus b is a general form of the of any iteration step in Gauss Seidel Jacobi. So, you can write M x k plus 1 is M minus A x k plus b x k plus 1 therefore, will be M inverse M minus a x k plus M inverse B and this is written as a general form x k plus 1 is equal to G x k plus f. So, this becomes my hm. So, this will become G and this is F now and this G is called the iteration matrix.

So, write this is equal to G and this is equal to F. This G is called the iteration matrix with this G we multiplied our guess value matrix with this G and add v that F and get the updated value. And finally, this G is something which is being multiplied with the guess value and then we are shifting the shifting it also the subsequent multiplication with G and shifting by F will should finally, take me to the converse solution, that is the id of any iteration step.

So, when the iteration will converge will get. So, when the iteration will converge it will practically converge to one particular value. And we will see later that it asymptotically converges to this value. At the converge step x k plus 1 and x k does not have much difference. They are essentially same. So, and we can we can put it into the equation also in the converge step as it is step as it will satisfy B minus x k is equal to A x k plus 1 is equal to 0. We will see x k plus 1 and x k are being essentially similar.

So, we get as converse solution where x cannot be further updated excess converse to a particular solution and the equation then x k plus 1 when the iteration will converge will get x k plus 1 is nearly equal to x k at convergence. So, if we will look into x k plus 1 is equal to G x k plus k both will be some well both will have same value x and will get a equation x is equal to G x plus f at the limiting step. Therefore, x is equal to I minus G inverse of f.

Now, if we look into this particular equation x is equal to i minus v inverse of F this is the solution this x is the solution of the equation. So, this x should also satisfy Ax is equal to b then when the equation satisfy x is equal to b x x is converse to the right solution. So, x is we can get the convert step x is equal to G x plus say for x is equal to I minus G inverse of f.

So, x is equal to I minus G inverse of f must be this x must be a solution of x is equal to I minus G x is equal to b this a 2 x s should be same. So, they must be coming from the same solution. So, let us see what is i minus G inverse. G is equal to we know that G is equal to M inverse M minus A which is nothing but I minus M inverse A. So, I minus G is equal to M inverse a therefore, I minus G inverse is equal to M inverse a inverse which is the order will change A into A inverse multiplied by M.

Similarly, F is equal to M inverse B. So, that should imply that I minus G inverse f which is x is equal to is equal to a inverse M into M inverse b, which is a inverse f which is a inverse b. So, this is same as x is equal to A inverse. So, essentially through the iterations we are finding out x is equal to A inverse b. We are solving the right equation on D.

However, we are doing iteration because we are not doing a direct solving solution like Gauss Seidel or lu D composition which essentially inverses the a matrix. Why are not we doing this we are not doing this because your same Gauss Seidel is extremely expensive it takes if there are there is a 3 by 3 matrix it should take 3 2 cube or order of 27 operations and if this is a very big matrix a number of operations will be large then therefore, computing time will be very high the perturbation due to round off errors will be very high etcetera.

So, you are not doing direct inversion we are doing an iterative method which is same as inverting another matrix I minus G inverse. So, I minus G must be invertible here. Instead of solving a inverse a finding I minus G inverse, I minus G must be having some properties for which the I minus G inversion should be simpler or iteratively what the iteration steps will do will basically find out an I minus G inverse.

So, you remember that this G is independent of x k, G only has certain components of A the decomposition of A and multiplication among these decomposed components or their inverses.

Now, this what we are doing we are not solving a inverse we are trying to find I minus G inverse. So, it is important that along with a I minus G or the G matrix should have some properties for which it is simple to or it is easy to invert I minus G inverse also the issues like condition number etcetera along with a now we have to look into I minus G also. So, that it is less perturbed by the numerical errors.

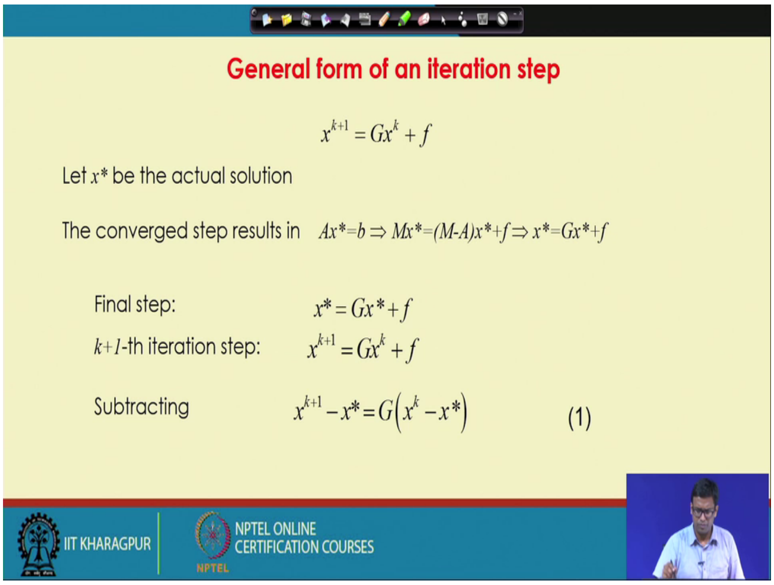
E inverting different matrices needs different efforts. For example, if we have a full matrix of n into n order. Inverting is something like doing a Gauss Jordan step over the matrix which will take in cube operations, but if you have a diagonal matrix it will take less operations, if we have a triangular matrix will take little more operations, but not as many as the for a fully densed back matrix.

So, inversion also depends on the nature of the matrix. And we have to see that how easy is to invert I minus G inverse, and I also see that if whether I if I minus G is a singular matrix we cannot invert it.

So, also we will also need to see in which cases we can invert I minus G and what exactly happened to A or to the equation system in case G becomes a singular matrix what happens to the iterations. Because we are not directly inverting it to it going through certain steps since G become a singular matrix or G is very close to a, sorry I minus G becomes a singular matrix or G becomes very close to an identity matrix what happens there we will look into these cases also.

So, we now have to concentrate on G matrix in in next few slides we will do that and see how this I minus G inverse happens and what is it is implication, when we think of inverting this through an iterative method. So, one direct fall out of this formulation is that thus there this has a solution if I minus G is nonsingular. So, G cannot be an identity matrix I minus G must be a nonsingular matrix.

(Refer Slide Time: 24:07)



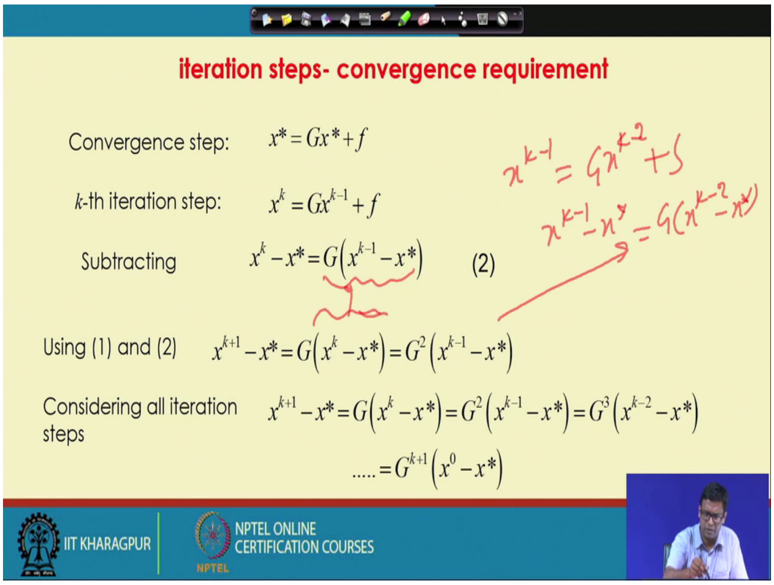
So, we further look into these things say that a general form of an iteration step what we got x k plus 1 is equal to G x k plus f let x star be the actual solution. So, Ax star is equal to b and we have seen that this will give us M x star is equal to M minus A x star plus f which is x star is equal to G x star plus f what we are doing last class. Last slide that if x when we have exact solution this will this is equivalent to writing x is equal to b in a decomposed way we will get x is equal to G x plus a for x star is equal to G x star plus f.

So, the final step will be x star is equal to G x star plus f. Final iteration step because it has converse to the actual solution. The iteration step before that say some steps before that we have a k plus 1 at iteration step, which will be x k plus 1 is equal to G x k plus f which is the general form of the iteration step. What is happening in final step x k it has a k plus 1 and k both converge to the final solution. So, you can write x star we can write x turning state of that.

So, if we subtract this 2 we can see what is the difference between updated solution and the actual solution and you call that error during any iteration is x k plus 1 minus x star, which is G of x k minus x star. So, the error or the difference from the actual solution and the guess solution is multiplied by G times when we are coming to the updated solution.

So, this multiplication must reduce the error and that then only we can approach to the convergence of the solution. If we write the series x k minus x 1 x 0 minus x star x 1 minus x star x 2 minus x star and go on up to x k minus x star, this must be a monotonically decreasing series so, that we can get a converse solution. Let us go to the slides. So, let us call this our relation 1.

(Refer Slide Time: 26:23)



In the convergence step is x star is equal to G x star plus f. The k th inter the iteration step will be x k is G x k minus 1 plus f subtracting we get x k minus x star is equal to G x k minus 1 minus x star, which is the relation 2.

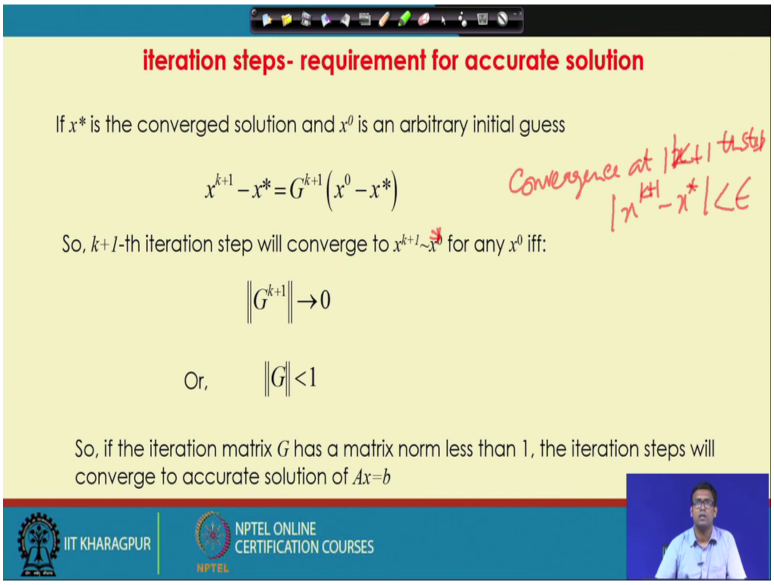
And now using 1 and 2 we can write x k minus x star is equal to G of x k minus x star what we got earlier and we have also seen that if we carry on this relation sorry we have seen that x k minus x star x k minus x star is G of x k minus x k minus 1 minus x star. So, this will be G square x k minus 1 minus x star.

Similarly, we can if we further write like x k minus 1 k minus 1 iterative step is j x k minus 2 plus f. So, if we subtract them will get x k minus 1 minus x star is equal to G into x k minus 2 minus x star. So, here we can further substitute this and we will get G cube, G square x k minus 1 will get this as G cube x k minus 2 minus x star. And we can go on and still get finally, get G to the power k plus 1 x 0 minus x star.

So, what is the initial gate x 0 and the after kth iteration the guess k plus 1 th iteration guess k plus 1 the related as the difference from the actual solution x k plus 1 minus x star is G to the power k plus 1 x 0 minus x star. So, this the initial error is being multiplied by k plus 1 th time after k plus 1 th iteration, is being multiplied by k plus 1 th power of G.

So, every day you may multiplying the error by G. And finally, the initial error will be multiplied by G to the power k plus 1 to get x well k plus 1 F x k plus 1 minus x star. So, if G is a small value or if G is less than 1, then we will do multi several multiplication and finally, get a convert solution.

(Refer Slide Time: 28:57)



So, x star is the converge solution and x 0 is an initial arbitrary initial guess. If it happens we can write G x k plus 1 minus x star is G if k plus 1 x 0 minus x star.

So, at k plus 1 either iteration step the solution converges to any to the final value x star for any x 0 any guess x 0. If this difference is very small, we will say that this has converged, if convergence at x k plus one th step will tell me that the difference x k plus 1 minus x star is less than epsilon; epsilon is the very small number.

So, this will converge to this will start for any x 0 this will start with a finite value and go to a very small value if the norm matrix norm of G to the power k plus 1 goes to 0, then this difference if this norm goes to 0, then this norm will also be these different norm of the vector will also go to 0. And this is possible if G has a matrix norm less than 1. Because any number less than 1 we multiply it several time it reduces to further smaller number.

So, if the iteration matrix G has a matrix norm less than 1 the iteration steps will converge to the accurate solution of Ax is equal to b. We can get the right solution of Ax is equal to b through iterative method if the iteration matrix G has a matrix norm less than 1.

Now will in next session will further check that what happens to xk plus 1 minus xk because we are telling that xk plus 1 and xk will be practically same at the convergent step how which under which condition this happens. And what are is the implication of this condition that modulus of matrix norm of G is less than 1.

How does it important well we look into the property of the matrix a or what should be the properties of the matrix A for which we should get modulus of G less than 1. We will look into the it next classes and I also discuss how does it being implemented by Jacobi and Gauss Seidel what is what is the exact value form of G and what are the exact more matrix norm of G matrix in for different matrices if we think of Jacobi and Gauss Seidel we will look into next few classes.

Thank you.