Matrix Solvers
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Lecture – 03
System of Linear Equations

Hello welcome to the second topic of this class on Matrix Solvers, this topic is System of Linear Equations. We will see how state of linear equations can be expressed in terms of matrix equation and you see different operations on the matrices and are the effects of them on the actual solution vector. Rather how the matrices can go through sets of operations without changing the solution much.

And as example of this will considered very simple equation systems, but those are very simply simple equation systems will not spend much time on solving the equation systems, rather will solve see the nature of the equation systems.

So, we take a 2 equation 2 unknown systems, 3 x plus 4 y is equal to 5 2 x plus 7 y is equal to 0 as I said we are not going to solve this equation, but these equation can be written in a matrix form where x y is the unknown for which I am seeking the solution we call this as the solution vector x.

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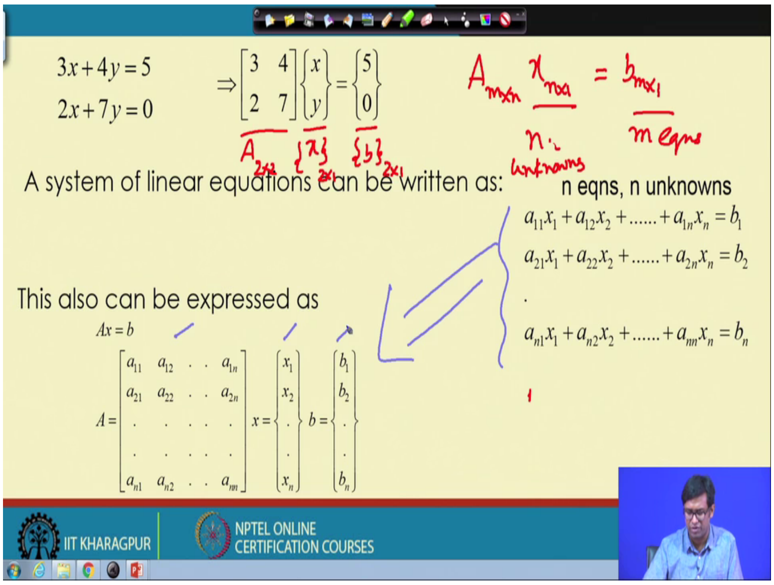

The right hand side is another vector b and the coefficients of the equation is written as a coefficient matrix A, A has a order 2 into 2 x has an ordered 2 into 1 b has an order 2 into 1. In a general sense I can write these as A m into m into n into x n into 1 is equal to b m into 1 and what does that mean. It means that there is a coefficient matrix which has m rows and n column x has n columns and the right hand side has m rows. So, right hand side defines what are the number of equation. Therefore, it has m equations and n unknowns.

So, we can see here bigger equation system or much larger equation system can be expressed in terms of matrix equation. For example, I go to a n equation n unknown system or will have completely all the equation complete set of solution n equation n unknown is there and these can be represented as a matrix multiplication where the. So, this is same as the same system of equation, where they are coefficients are given as a coefficient matrix A, the unknowns are as the column vector x and the right hand side is an another column vector b.

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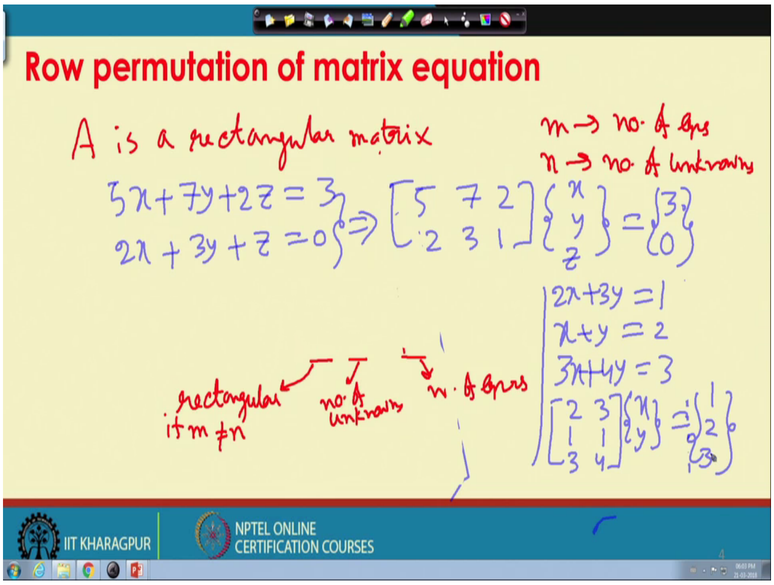

So, equation system can be expressed as a metric metrics system of unknowns and coefficient matrix using matrix multiplication operations. Now, there can be like the earlier example what I wrote in the board is the cases where the number of equation is different than the number of unknowns. And where an example can be A, in this case a will be a A is a rectangular matrix because the number of unknowns is not equal to the number of equations.

So, m is number of equations n is number of unknowns, so when we write m into n, x n into b m into 1 x gives number of unknowns and b gives us number of equations. If they do not match then A will be rectangular in form, if m not equal to n right. We can get some of the examples like that for example, if I have an equation 5 x plus 7 y plus 2 z is equal to 3 and 2 x plus 3 y plus z is equal to 0. I will have I can express these as a metrics equation as this can be expressed as 5, 7, 2, 2, 3, 1 and the number of unknown is 3.

So, x y z, but the number of right hand side column vector number of elements in right hand side column vector is 2, which is 3 and 0. Similarly there can be case when number of equations are more than number of unknowns for example, we can see another case where we have 2 x plus 3 y is equal to 1 x plus y is equal 2 and if we add them we get another equation 3 x 3 x plus 4 y is equal 3 and this can be expressed as 2, 3, 1, 1, 3, 4 x y is equal to 1, 2, 3.

So, all the cases number of equation is equal to number of unknown, number of equations is greater than number of unknown, number of equation is less than number of unknowns, all these cases can be presented in a matrix equation. If we have the ideal case where number of equation is equal to number of unknowns so, that we get a nice solution without spending much equations extra equations without solving extra equations. So, will have a square coefficient metrics A otherwise will have a rectangular coefficient matrix A, if number of equations and number of unknown do not match with each other.

So, the next point is what happens if we have a row permutation of the matrix equation and again for simplicity we will consider the cases with 2 equations 2 unknown case, this is 3 x plus 4 y is equal to 5 2 x plus 7 y is equal to 0. Now, if we do a row permutation that means, if I write equation change the order of the equation say this is equation 1 and this is equation 2, if I re-number the equations as it these becomes my new equation 1 and this becomes my new equation 2.

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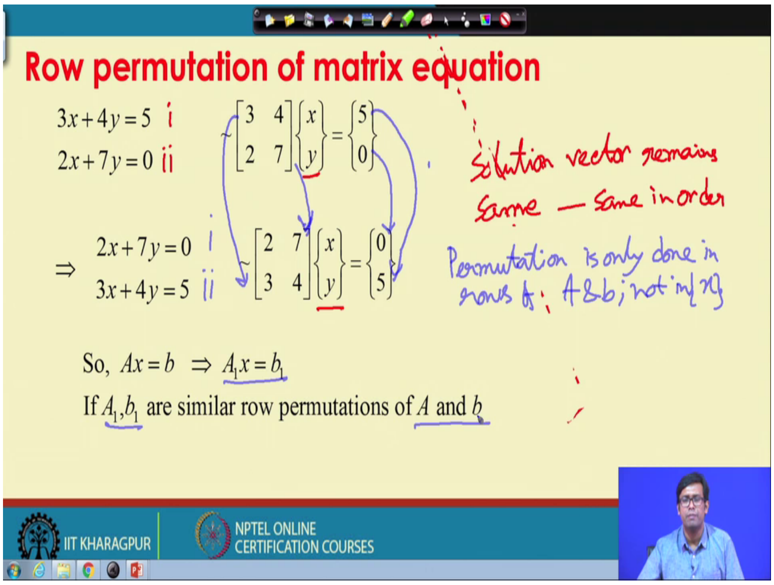

If I renumber the equations with see that the matrix coefficient matrix is do permuted that is first row comes into the first row comes into the second place and the second row goes to the sorry second row goes to the first place. And same thing happens with the v b b vector the first floor in the b vector which is a column with that goes to the second place and the second row element comes to the first place. So, however the matrices or the column vector x y which are the solution vector solution vector remains same.

And not only same in value because we solve the equation will get the same result, but also same in solution vector remain same not only values same in order also. So, the permutation is only done in the rows of or will right permutation is only done in rows of A and b not in x, x vector the vector x remains the solution vector that remains the same one.

So, solving x is equal to b is same as solving the equation A 1 x is equal to b 1, if A 1 b 1 and A and b are row permuted form and the similar low permutation operation is taken clear place in both A and b matrices.

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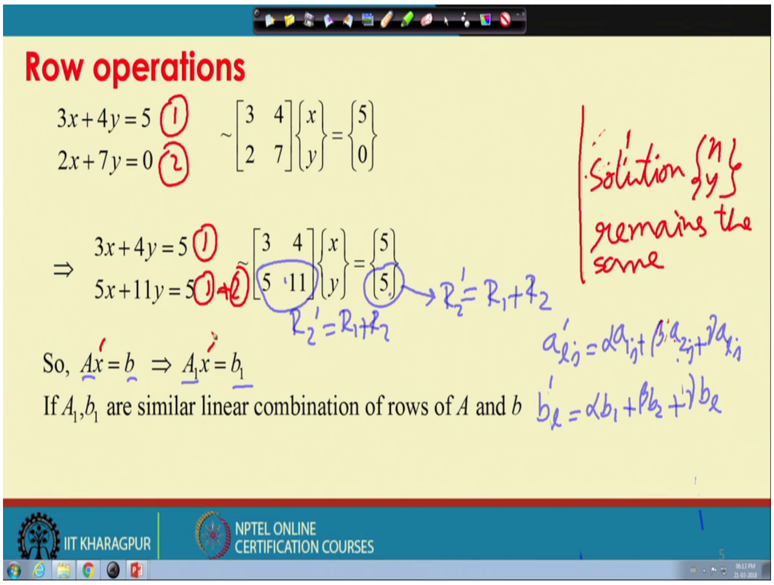

Now, will see some other row operation that is we have 3 x plus 4 y is equal to 0 and we have 2 x plus 7 y is equal to 0 this is equation 1 and this is equation 2. Now, if I change my equation to, so that equation 1 is added with equation 2 the solution will be remaining the same. So, if I replace this set of equation as 3 x plus 4 y is equal to 5 which is the original equation 1 and equation 2 is now equation 1 plus equation 2.

The equation system basically gives us similar solution. So, what do we write is that that the new row or this is the new row R 2 prime is equal to R 1 plus R 2. And same operation has been taken here new this new row which is R 2 prime of b is equal to R 1 plus R 2. So, we add rows of A to get a new row and we add rows of b to get another new row and we see the solution remains same.

So, solution which is x and y remains the same, so row operation terms of when 1 row is expressed as linear combination of 2 or more rows and this is done in both A and b similarly the equations remain similar in sense the solution vector x remains same both in terms of it is magnitude and in order. We can we can consider a much bigger matrix also and will see that the similar methods are existing there that is if we have a large matrix with number of rows.

And will see that say elements of row l l j which is a elements of row l is expressed as in the new matrix expressed as alpha into elements of row 1, beta into elements of row; it should not choose the wrong they there should not be any mismatch in terms of the operations in the rows of coefficient matrix and then matrix and b. Then you will end up in this case you are doing one operation in columns in b another operation in columns in one operation in rows of A and another type of operation in rows of b will not remain the get the same solution vector. So, matching of the operations is important.

So, alpha a i j and then beta a 2 j plus say gamma alj and similar operation we do in all coefficients of the all terms of the right hand side matrix b that is b prime l is equal to alpha b 1 plus beta b 2 plus gamma b l. So, we considered one particular row and do the operations and we consider another similar row of b and do the operations and what is the operation; operation is basically the one elements, all the elements of this row is expresses linear combination of the elements of the other rows.

The linear combination means you multiply the elements of one particular row with some number and add with which sub scalar and add with elements of the other row multiplied with another scalar. So, multiplied by a scalar and then addition this kind what is said as a linear combination. So, if we can do linear combinations of different rows both in A as well as the right hand side vector b will keep the same solution vector. So, both the cases in the matrices A and A 1 b and b 1 are linear combinations of rows of A and b it can be solved and the solution vector x will remain same.

So, whatever we do with these matrices if we have A x is equal to b in and hypothetical case I find that solving A x is equal to b is not trivial. So, what I do I for that trivial is not simple, we have to do lot of computing lot of tricks are to be followed in solving x is equal to b and A b and A is not a small matrix a is a very large matrix.

Now, we think of changing this matrix into A 1 x is equal to b 1, the a row operation either row permutation or linear combinations of row. Now, the new equation system A 1 x is equal to b 1 is probably easier to solve using out metrics solution algorithm. And will see when the solution is obtained the solution of A 1 x is equal to b 1 is exactly same as A x is equal to b, because these row operations are not doing anything messy with the solution vector x; the solution vector x is reaming the same.

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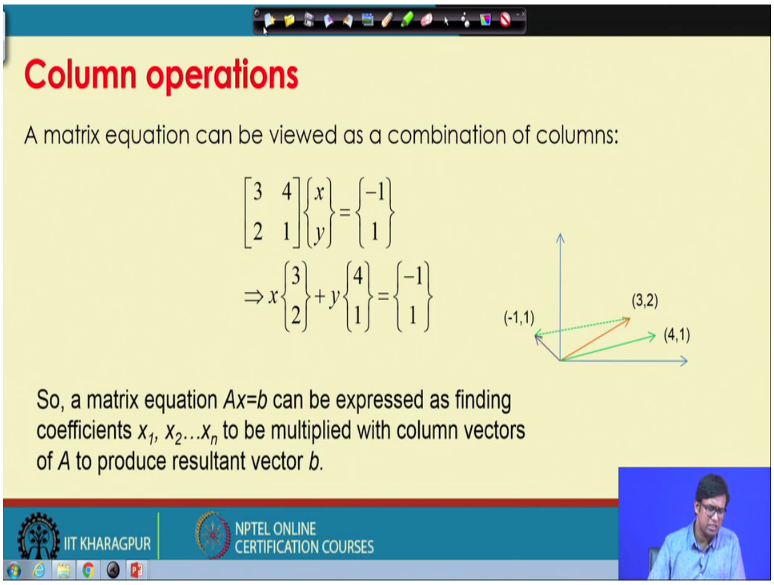

So, we go to the next slide which is column operation. A matrix equation we have expressed that a metrics equation as a row combination of rows that is the variables multiplied with the coefficients gives us a matrix forms. And each of the coefficient in a particular row is multiplied with all the elements of the solution vector to give one particular element of the right hand side that is b.

Now, we can also this these as a column operation or a matrix equation can be seen instead of just writing the writing the equations and looking into from the matrix multiplication point of view. We can see a metrics equation as a combination column that is how if we have this a similar equation 3 x plus 4 y is equal to minus 1, 2 x minus 2 x plus y is equal to what which is the metrics equation is A x is equal to b; A is 2 into 2 x by x, I have now denoting the x which is written in red ink is a column vector x is now and it has member x and y.

So, this multiplication can also be looked as a factor that the column the column 3 to the first column, the first column 3 2 is being multiplied by a scalar x and the second column 4 1 is being multiplied this is the second column is being multiplied by another scalar x and when we add this we get. So, this is the vector multiplied by a scalar this is another vector multiplied by a scalar when we add them we get a resultant in vector.

So, this matrix equation now can be viewed as a vector equation we are doing linear combination of vectors and getting resultant vector. So, if in 2 D space we can see these two other vectors 3 2 and 4 1 they are the vector, basically 3 2 is 3 i plus 2 j 4 1 is 4 i plus j and the resultant vector is 1 1. Now, this if I solve this equation on solving I will get probably write it down on solving and we can do it by mental maths will get x is equal to y is equal to minus 1; that means, I will add a minus of this vector with this particular vector.

So, we do this operation we add this 1 and we get the resultant vector. So, these equation becomes a vector addition on b and a matrix equation A x equal to b can be expressed as finding coefficients x 1 x 2 up to x n which can be multiplied with the column vectors to produce a resultant vector b we can sorry. Now, we can look into a bigger matrix equation and we see that similar expression is possible.

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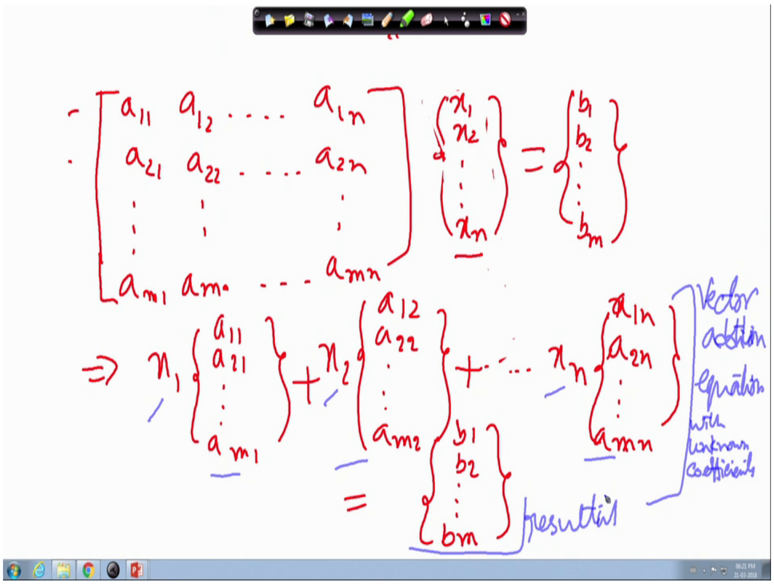

Let us consider a matrix equation which has number of entries a 1 1, a 1 2, it goes up to a 1 and we take a rectangular metrics; that means, number of equation is not equal to number of unknown a 21, a 22 and it goes up to a 2 n. And there are many entries in all this case and the last entry a a m1 a m2 and it goes up to a mn and this is multiplied with a column vector. Now, I know that there are n columns in a therefore, x will have n rows from the matrix multiplication the rule matrix multiplication.

So, this is x 1 x 2 and it goes up to x n and the solution how many elements will the solution vector have or the right hand side vector or not the solution vector. The right hand side vector will give us exactly same number of rows that the metrics coefficient metrics is having because this is the number of equation. So, this will have b 1, b 2 and it will come up to b n. So, this is in the sense interesting that the number of columns in coefficient matrices is equal to number of elements in the solution vector.

And the number of rows in the coefficient matrix is equal to number of elements in the right hand side vector. So, and will let us see the right hand side vector is called a member of the column space of coefficient matrix or it can be expressed or what will do in the next step is expressing the right hand side vector has linear combination of columns of the coefficient metrics and the solution vector is this solution vector this is this a similar entries as the rows of the coefficient matrix.

So, solution vector is called a member of the or it can be shown as the member of the rows space of the coefficient matrix that is we can have linear combination of the rows of the coefficient matrix we can get the solution vector nevertheless. So, now our job is to express this in terms of a vector equation, and what we can write here is that x 1 into a 11 a 21 and it goes up to a m1 plus x 2 into a 12 a 22 and it goes up to a m2 and it goes it will be go on up to x n which is now a 1 n a 2 n and it will come up to a m n and this is the resultant will be b 1 b 2 up to b n.

So, the matrix equation is now expressed as a vector addition equation that we are adding number of vectors with unknown coefficients. So, we are adding number of vectors with unknown coefficients with this these are the unknown coefficients to get the resultant and we have to see what are these coefficients. So, matrix equation can be expressed as find a coefficients x 1, x 2, x n which will be multiplied with column vectors of a where produce the resultant vector which is b.

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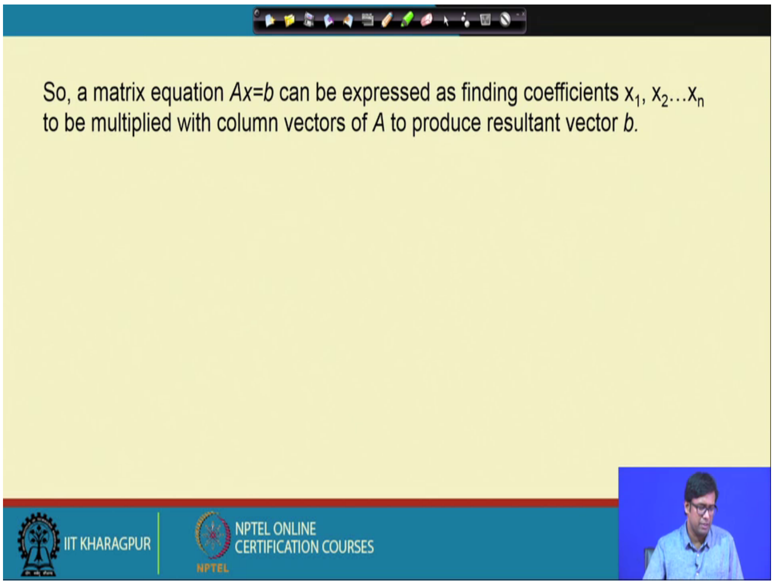

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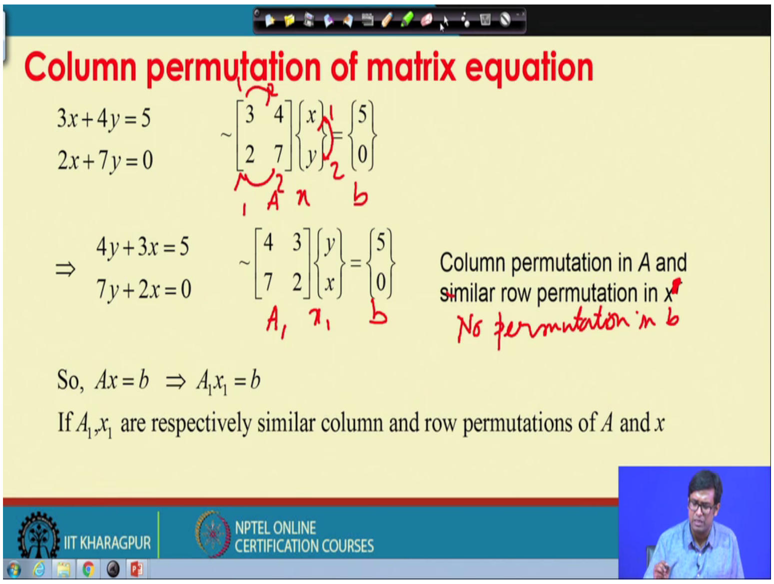

Now, if we do a column permutation for example, we have these equation and if I do a column permutation; that means, the 4's the columns 4 7 column 4 7 x is move to column 3 2 so we send this. So, we send these column here in the column permutation we send column 2 to column 1 and column 1 comes to column 2.

So, what happens if we write the expressions in a column permuted way will see y comes first and x goes later and the matrix equation becomes 4, 7, 3, 2 which is the column permutation. So, this is A x is equal to b and these becomes A say A 1 x 1 because the members of the column vector is also permuted, but the b vector remains same.

So, column permutation of A has similar so this is this should not be the statement rather this should be yeah this not x transpose in x. So, the column permutation it is not x transpose correctly it will be x. The column permutation in A is from column 1 and column 2 is permuted and in x will see the rows are permuted 1 will go to 2, 2 will come to 1. The column permutation is in A is similar to column permutation in b, but no permutation in b. Column permutation in A is similar to permutation in x, but row permutation in x, but there is no permutation in b.

So, what we get is A x is equal to b is same as A 1 x 1 is equal to b A 1 x 1, A 1 and x 1 and x 1 are similar column and row permutation A and x the way the columns of A are permutated in the same way the rows of x n is permutated.

And essentially the solution remains same only the order of the solution matrices order of the solution vectors changes. Column operation is like linear combinations that can be column operations like linear combinations that can be column operations like linear combinations also can be perform.

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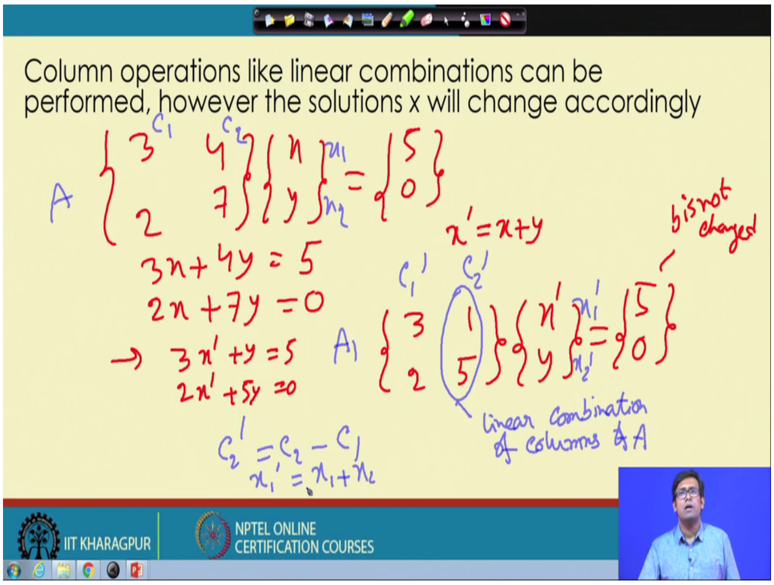

However, the solution x will change accordingly. What is that if we see accordingly linear combining of the column if we see that example we have a matrix 3, 4, 2, 7 which is multiplied with x and y is giving me solution 5 and 0. So, we can get the equations which is 3 x plus 4 y is equal to 5 2 x plus 7 y is equal to 0, now I can assume a new.

So, this is little more complex that is the solution the solution vector will be much changed. We can assume a new solution vector a element on solution vector x prime which is x plus y and we can write this equations as 3 x prime plus y is equal to 5 and 2 x prime plus 5 y is equal to 0, which will give me the equation 3 1 2 5 into x prime y is equal to 5 0. Note b is not changed and the second column so, this is say A and this is A 1.

Second column of A 1 is linear combination of columns of A what is 1 5 we can find it this is basically this is basically this column minus this column, 1 5. So, column or the second column which is 1 5 of a 1 is subtracting the second column first column of A, from second column of A, which is we can write C 2 prime or this is say this is C 2 prime this is C 1 prime this is C 1 and this is C 2 is equal to C 2 minus C 1 and x 1 prime is a new x is equal to x 1 plus x 2 or the. So, this is this is x 1 and this is x 2 and this is x 1 prime and x 2 prime.

So, the way one column is changed in from A to A 1 there will be a relationship following which the solution vectors will be changed. However, the solution vectors will be linear combination of the solution of the members of the solution vectors of the previous solution matrix.

So, result of row or column operation or combine column operation thus new solution vector is either same as original solution vector which is for the row operation the solution vector does not change.

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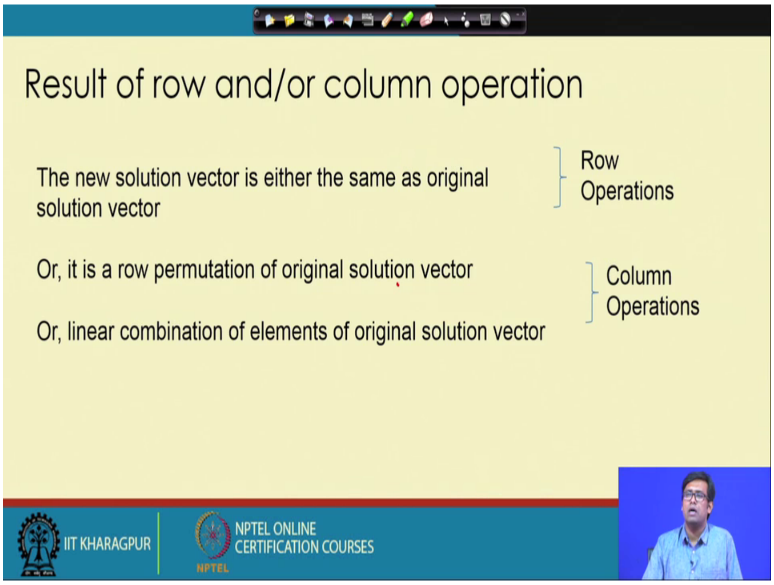

Or the solution vector is a row permutation of a original solution vector which is for column operation for column permutation the solution vector goes to a row permutation or linear combination of elements of original solution vector. So, this is row operation and this is column operation.

Therefore if we do row column operations like interchange the row columns to permutation or add row with another row add a column with a another column and after multiplying with a scalar; do this type of operations in rows and columns solution essentially remain similar. Either it is a same solution or it is a permutated solution or it is a linear combination of the original solution vector.

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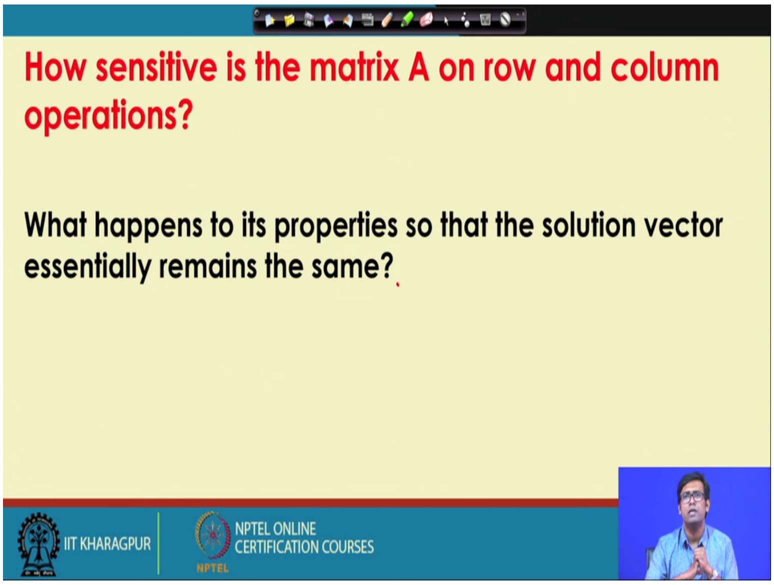

So, one very important thing is that how sensitive if the matrix A on row and column operations, that is we are this course is focused on matrix solvers. We will discuss about solving the matrix equation and will see in certain stages that the matrix solvers perform well for certain properties of the matrices.

And if the matrix is diagonally dominant one of the matrix solver will perform well; that means, diagonal entries are larger or equal to the of some of the diagonal entries. If the matrix is symmetric one of the matrix solver will perform in this case and once we get a matrix which is non-diagonal dominant not symmetric we can do row and column operation and transforming to the form we are seeking from.

So, we have to see what is the sensitivity of matrix A to this operations, what will happen to the solutions will the solution remains same in row operation that is the best advantage. In column operation also solution is either permuted or a linear combination of the original solution vector. So, this is very important thing which is to be checked and what happens to the property of the matrix why the solution vector remains essentially the same.

So, if you look into the matrix what happens to it is property. So, the solution of the essentially vector remains same and this is one very important part of discussion in this course. As the class will progress will have more ideas about the properties of A and how they are sensitive to and column operation, but in the very next class will see certain property which is determinant of the matrix which does not alter much with the row and column operation.

There is probably some sign changes in certain cases of row and column operation and that is a key factor where the solution remains very similar once row and column operation is done. So, today will we have discussed up to these up to the expressing matrix equation as the system of linear equations and the effect of row and column operations.

Next class we start discussion on determinants

Thank you all.