Matrix Solvers

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Lecture – 24
Projection Operation and Linear Transformation

Welcome. So, last class we discussed about solution of Ax is equal to b, when b is not in the column space, or when there is no solution is existing of Ax is equal to b. However, we tried to solve it and we obtained normal equation, and found out best estimate of the solution of Ax is equal to b. So, in that process what we did we projected the vector b into the column space of A so that Ax tilde is equal to b c has a solution b c is the projection of b in the column space of A, and x tilde is the best estimate.

So, we will this class we will look into more detail about the projection operation, and we will explore the concept of linear transformation how it is important in context of projection operation and other operations on b vector. So, this operation is a geometric operation on b vector by which we projected the b vector onto A space for which we have the solution of Ax is equal to b, or the bx is equal to projection of b. Ax is equal to b has no solution. So, we are doing an approximate solution using the projection. So, let us look into detail on that.

Best estimate of Ax is equal to b, when b is not in column space. This is recapitulation from last class.

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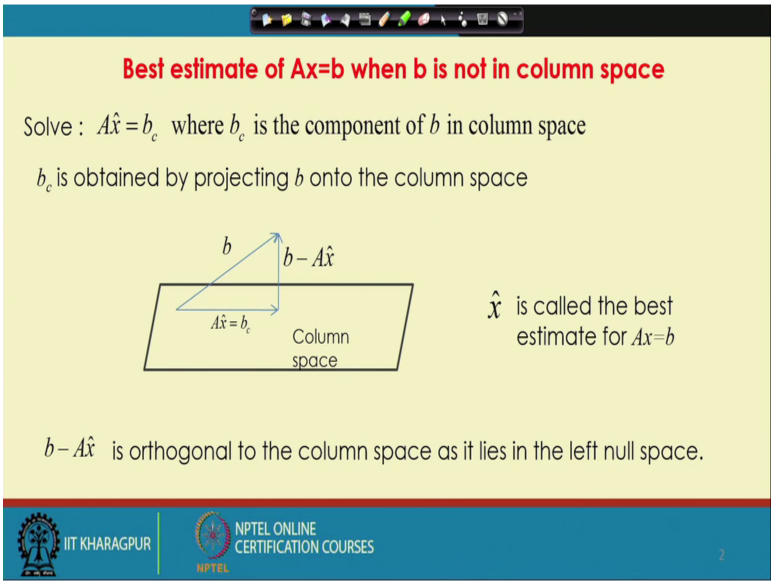

Solve Ax is equal to b c Ax tilde is equal to b c, when b c is the component of b in the column space. Or b c is obtained by projecting vector b's column space, and you have the vector b you project it into onto the column space. And get the vector b c which is Ax tilde which you can solve and the error is b minus Ax. And x tilde is called the best estimate for Ax is equal to b, and b minus Ax tilde is orthogonal to column space as it lies in the left null space.

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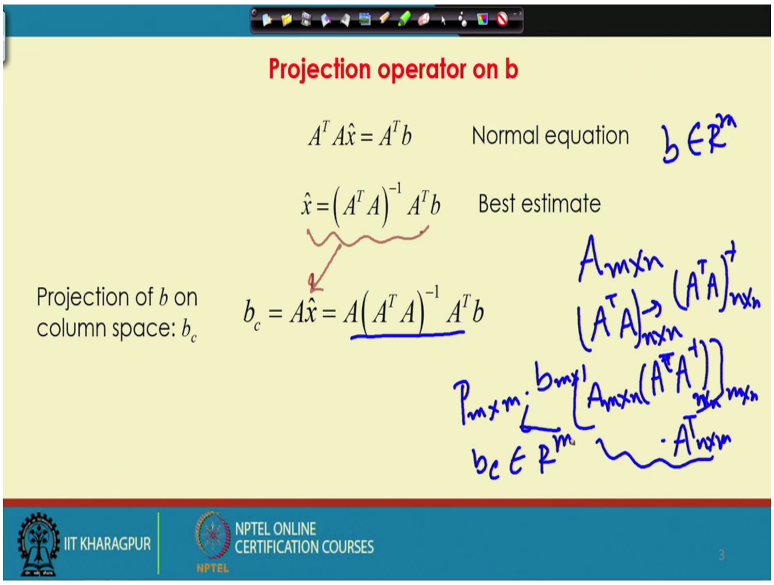

Now, this projection operator on b projects the entire equation and we have seen took dot product of b minus Ax with Ax tilde b minus Ax tilde with Ax tilde as they are mutually orthogonal and made it 0 and obtained, the normal equation which is A transpose a x tilde is equal to A transpose b.

And A transpose Ax tilde is called the normal Ax transpose Ax tilde is equal to A transpose b is called the normal equation, which can be solved when we have seen that inverse exists for A transpose a in case of a having independent columns. And this can be solved to get the best estimate. So, x tilde is A transpose A inverse A transpose b is the best estimate.

Projection of b on the column space of b c and we looked through an example that take find out x tilde multiplied with A and we should get b c which is projection of A, b on the column space b c is equal to Ax tilde. And this is if I substitute x tilde if we substitute x tilde x tilde and multiplied in the equation b c is equal to Ax tilde, we get b c equal to A A transpose A inverse A transpose b. So, this is projection of b into b c. That means now what is it? A transpose a say for example, I take a of the order m into n.

So, A transpose A is a matrix of the order n into m into n into n. So, n into n, and A m into n into A transpose and A transpose A inverse we will have same order. And so, we can see A transpose A becomes square matrix. A transpose A inverse this product, this has n into n has again m into n. And then we multiply these with A transpose which is n into m.

So, finally, the entire matrix, this entire matrix becomes a projection matrix of order m into n into n into m so, m into m. And this m into m, now m is the number of n is the number of rows or number of elements in the column; so b lies of R m b also has m components this multiplied with b which has m rows and one columns will give us another b c which also lies in R m. So, this b and b c will be vector same R m, only they will be b will be projected from one particular inclination to the column space of a.

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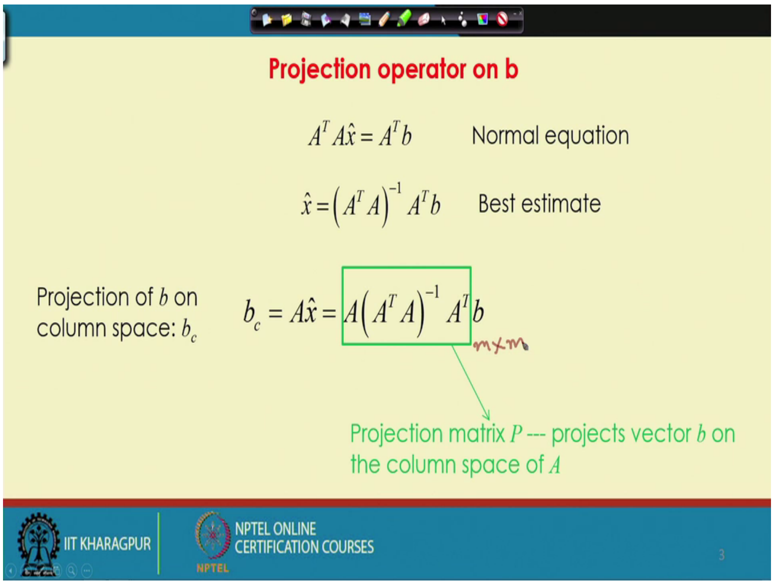

And this matrix so, we got p as a matrix what is the dimension of this matrix? This matrix we got as m into m. This is called a projection matrix, it projects vector b onto the column space of A. Remember, the last example we discussed at the end of the previous class, found out x tilde and multiplied with A and solve. Discuss this and we verified that this is projection of b onto the column space of A. And this projection is obtained by multiplying b with a vector. So, projection is a geometric operation; however, this is obtained by multiplying the vector with a matrix.

So, a geometric operation can be substituted by a multiplication with a matrix. And this is something what we will discuss as linear transformation in the next few slides.

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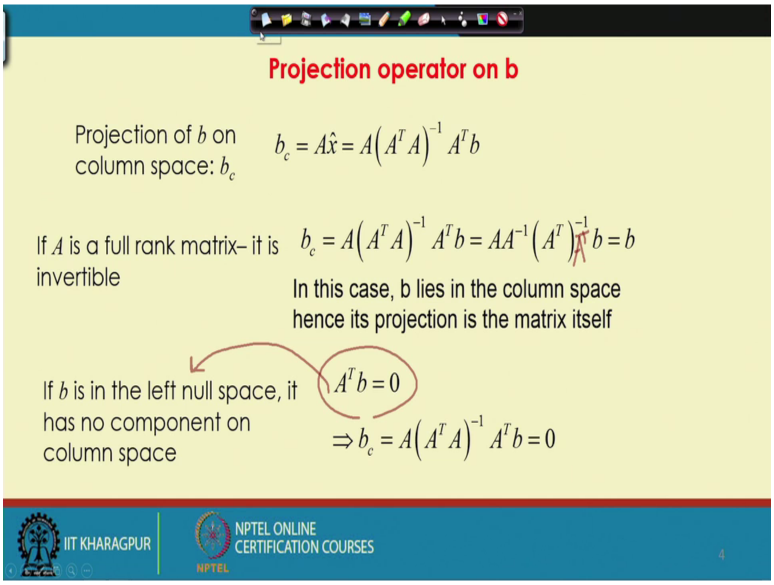

Projection operator on b is b c is equal to Ax tilde A transpose A inverse A transpose b. We must verify that if b already lies in column space of A, what will be there? So, a vector which is lying in this space; if I project this vector; so, this is a general projection; like, I have a space like this my hand is a space a subspace, and there is a vector which is away from the subspace. If I project it this will be the projection.

Now, I have the vector lying in the space already. B is already a member of column space. What will be the projection? Projection, will the vector itself. So, the projection matrix should be an identity matrix which when multiplied with the vector gives back the same vector. And we let us check that when b is on the column space of A, b is itself on the column space. If b is on the column space A must be a full rank matrix. Then any b it should be in the column space. So, A transpose A inverse or A is invertible. So, we can write A transpose A into A transpose A inverse. A transpose is A inverse A transpose inverse and sorry; A inverse A A into A inverse into A transpose inverse there will be another A transpose into A transpose.

So, this will give me the b itself. And if b lies in this case b lies in the column space in the projection is itself is the matrix is the vector itself in the matrix is a vector itself. If b is in the left null space, then b is in the left null space means, any vector in the left null space when multiplied with A transpose will give me 0. So, A transpose b is equal to 0. And we can find out b is in the left null space means, b is already perpendicular to this plane. So, what is the projection of a line perpendicular to a plane is the point or is the origin itself, because everything is connected at the origin is 0 vector.

So, when A transpose b is equal to 0; that means, b is in the left null space of A we can see b c gives us 0, ok. So, these are 2 quick checks on the projection operator which is consistent with the case b is in the column space and b is in the left null space.

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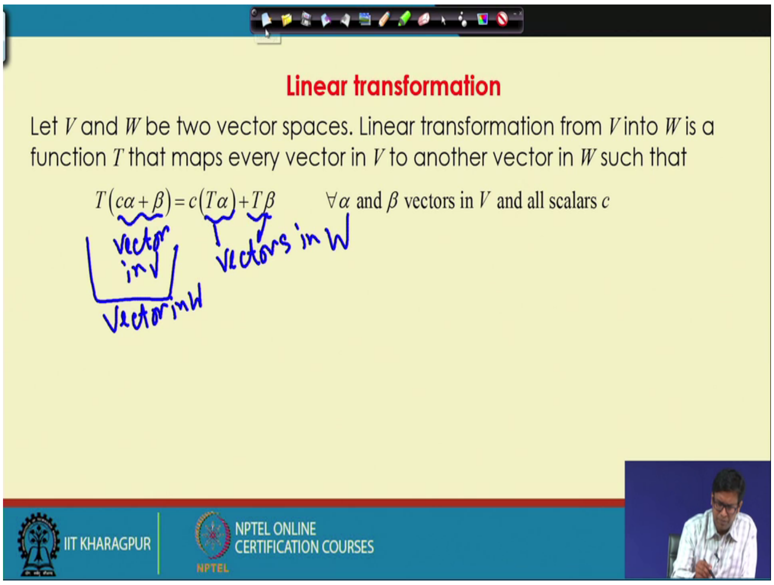

Now, let us see what is linear transformation, and we will say that this is a linear transformation. If V and W be 2 vector spaces, linear transformation from V to W is a function T that maps every vector in V onto another vector in w, such that T applied over c alpha plus beta alpha and beta vectors are in b and c is a scalar is c into T alpha plus T beta.

So, if I take a combination of 2 vectors in a vector space V and there is a linear transformation like projection which is taking vector from one space and projecting it to another sub space. And if and we will call it linear, if linear combination of transformation of 2 vectors is same as transformation of if linear combination of transformation is same as transformation of the linear combinations. Then we say that to be a linear transformation, and here T c and T alpha, like they are vectors in W. So, c alpha plus beta is vector in V, and T c this is vector in W. T is a matrix which when multiplied with a vector, it takes the vector onto another space, say exactly what we saw for the projection operator.

We took a vector, which is not in the column space, projected it and it is now in the column space so, it is a linear transformation.

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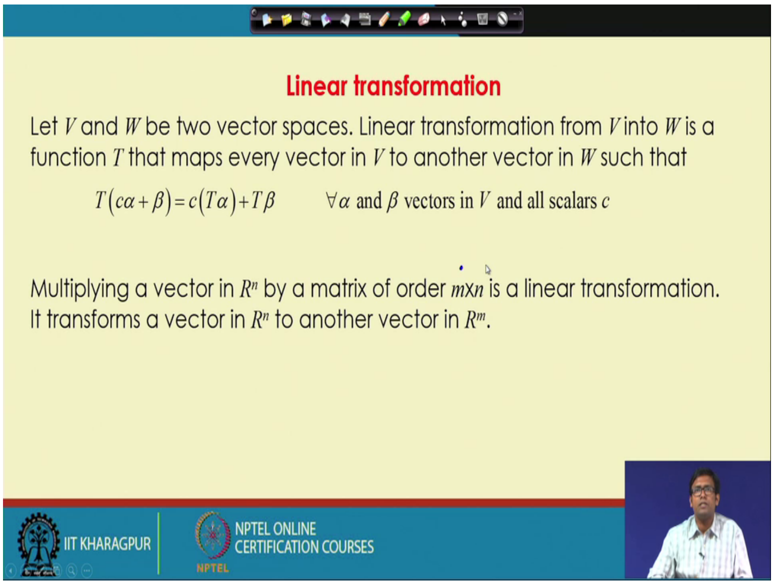

Multiplying a vector in R n by a matrix of order m into n is a linear transformation. So, we will see multiplication by any matrix is a linear transformation. And that is consistent with the definition we have given here. Now if I multiply a vector in R n with a matrix of order m into n, the resultant will be a vector R m. So, a vector in R n can be transformed to a vector in R m. The dimension of the vector spaces can change during direction of the real coordinate spaces can change during linear transformation for also. A vector can be transformed, but victor will always be a vector, but the number of components in the vector can be changed due to linear transformation.

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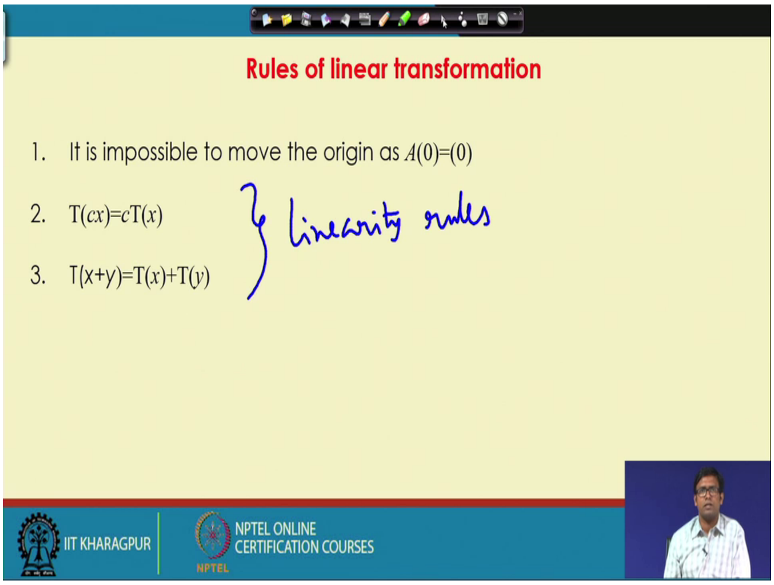

And of course, there are few rules of linear transformation. It is impossible to move the origin, linear transformation of origin is always the origin. Linear transformation of a vector multiplied with a scalar is linear transformation of the vector. And then multiply scalar with a linear transformation. Similarly, linear transformation of addition of 2 vectors is addition of linear transformation of these 2 vectors. And these are the general rules of linearity which holds in case of linear transformation.

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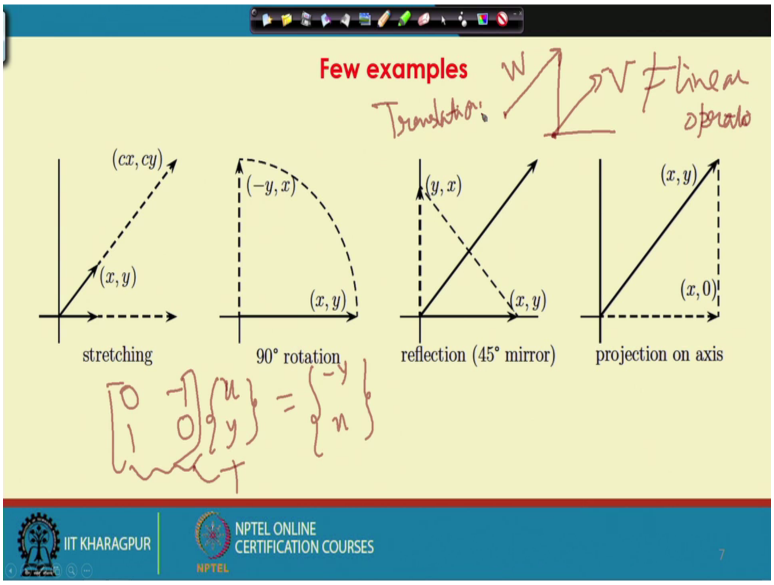

Few examples of linear transformation can be stretching of a vector 90-degree rotation of a vector reflection of a vector for doing 45-degree mirror projection of on an Axis. So, like here we can say that if I have the vector x y, and the 90 degree rotated vector is minus y x. So, this is obtained by multiplying this with 0 minus 1 and 1 0. So, this is a matrix which is you can say this is the T matrix, which when operated over x y will give a minus y x vector. And this is linear because we can have 2 vectors and check this that the rules of linear operation is staying in all these cases.

. So, all geometric operations can be thought of linear operation except translation, why not translation? For example, I have a vector here. I translate it to a vector W here. This is not a linear transformation, why simply because the origin is shifted. And any 0 vector will not remain a 0 vector in the transformed space. So, except translation this is not linear operation except translation all the geometric operations can be thought as linear transformations.

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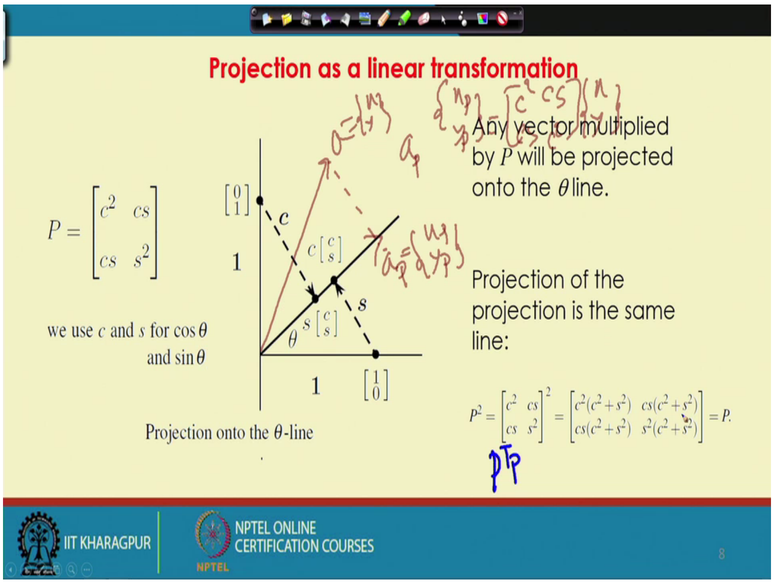

We are very much interested in the projection operation. So, we see what is in a projection operation. So, projection operation is basically multiply if I think of projecting a vector on onto another line which makes a makes theta with x axis. So, it is basically multiplying it by the projection operator c square c's c's c square where c is cos theta and s is sin theta. So, for example, I take a vector say A here, and I will like to project this vector along this. So, we will get something a p for example. And we can write a p. So, this is say x y this is x p this is x p y p. So, we can write that x p y p which is a p is equal to this operation the projection operation c square cs cs c square x y.

So, any vector multiplied by p will be projected onto the theta line projection of projection is the same line. So, if I again try to project a p on theta line, a p is already theta line. So, it is the same line. So, we can say that p square is equal to projection of projection once it is projected and then it is again projected. So, we multiply p and p is a square matrix. So, p square is basically we give it as, sorry, p transpose p as it square matrix it is same it is nothing but p matrix. So, when we project a line once more it is already projected in the phase it becomes the same line. So, projection operation doing n number of time is same as doing projection operation once.

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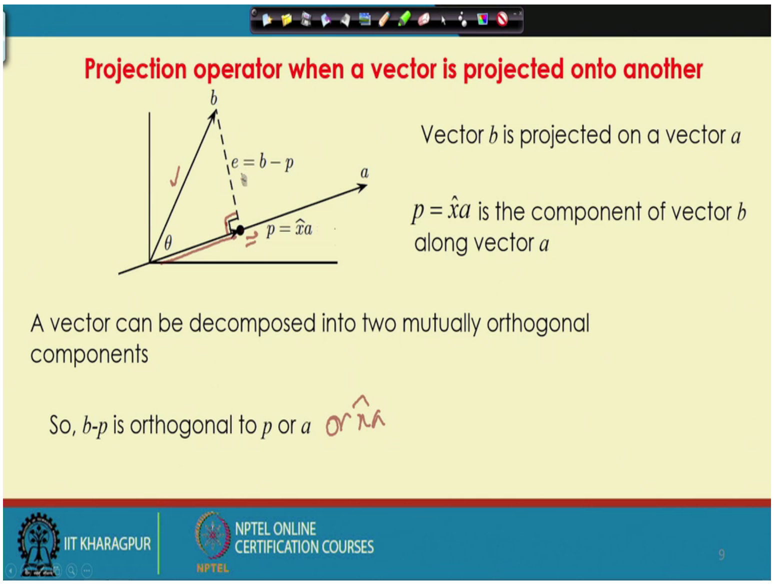

So, we will look detail onto the projection matrix, vector b is projected on a vector A, and this is projection operation. P is equal to x tilde a is the component of b along a. So, b is b is projected along a the new vector is along has the same direction of a. So, it is only a multiplied by certain constant it is either stretched version of vector a or a shrinked version of vector a. So, vector is either magnified or shrinked. So, we multiply a constant a scalar with vector a, and we should get the projection.

A vector can be decomposed into 2 mutual components. So, if there is one part which is along vector a, the other part should be mutually orthogonal part to this. So, we b minus p if p is the projected vector b minus p must be orthogonal to the projection, or must be orthogonal to p or Ax a p is equal to x tilde a or a. So, b minus p is orthogonal to. So, what we got it? We had a vector v, we projected it b, we projected into a and this is p is equal to x a. And they are perpendicular to the projection and the component which is left out. What is left out? We take the original vector subtract the projection out of it that is a component which is left out. This must be orthogonal to the vector.

And we can see that that if we have this particular plane, and we project a line here, this will be the projection. And this is the orthogonal part to the projection plane. So, b minus p the left out part must be orthogonal to the projection b minus p should be orthogonal to p or p is along p is x a p is along a. So, b minus p is orthogonal to a. So, the projection operator when a vector is multiplied to another b minus p is orthogonal to a, or we can write a transpose b minus p is equal to 0 which gives us a transpose b is equal to a transpose x a, or b is equal to a transpose b by a transpose a.

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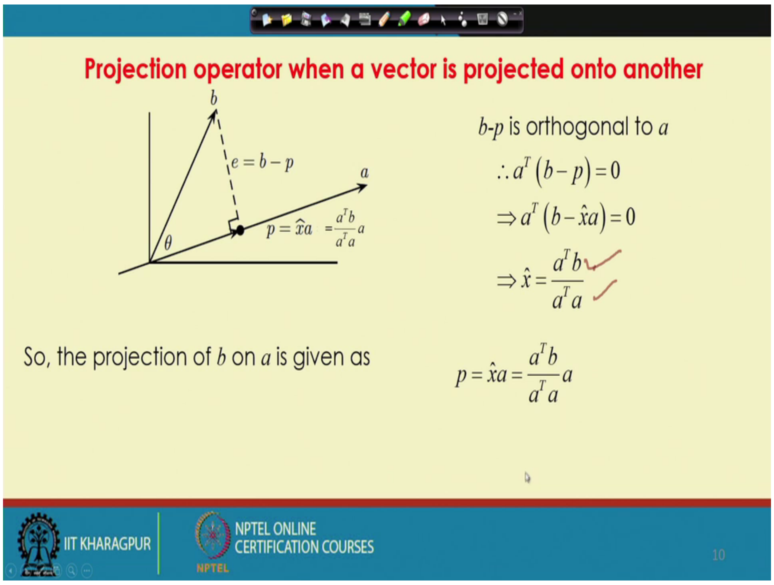

Now, sorry x transpose x tilde is equal to a transpose b by a transpose a. So, what is x tilde? X tilde is a scalar, and we can see it is a transpose b is a scalar a transpose b is a scalar, a transpose a is a scalar. So, it is a scalar by a scalar. So, the projection of b on a will be given as x tilde multiplied with a, it is magnification or shrinking of a is a tilde b by a into a tide a. So, we need to only get the constant which gives us dot. So, what is it a transpose b is dot product between a and b. And a transpose a is the square of the magnitude of a. So, we take and this is basically the angle between x tilde is basically cosine of the angle between we can check a and b.

And we multiply it with a the vector, the cosine tilde is the magnitude of b into cosine of that angle. And we take this is the magnitude of the transformation with respect to a. Projection with respect to and we multiply it with a and get the projected value.

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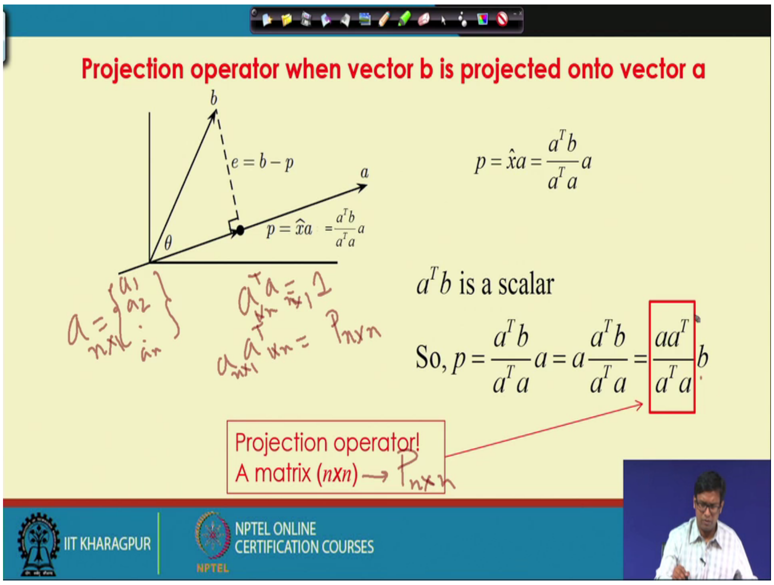

Now we will do so, this projection, we will do little more (Refer Time: 21:10). So, we got the projected part of b on a p is equal to x tilde a is a transpose b a by a transpose a. a transpose b is a scalar. So, we can write p is equal to a transpose b is a scalar. So, we can write p is equal to a transpose b a transpose a by a, or you can take a in the left hand side. Because it is a scalar multiplication with scalar has not it is not a matrix multiplication.

And this will be a transpose b by a transpose a. And we end up with a transformation matrix, what is a? A transpose by a transpose a. This is the projection operator. a transpose a is a scalar; aa transpose is not a scalar. So, if a is a vector which is a 1 a 2 a n. So, a is n into 1; a transpose a is 1, because this is 1 into n into n into 1. But aa transpose is n into 1 into 1 into n is a matrix p of the order n into n. So, it is a square matrix. Transformation in terms of projection is through a projection operation which is a matrix p which is a square matrix. a matrix, and this matrix we can denote by capital P. It is a square matrix.

If we multiply the original vector b by the square matrix n into n we should get the projected value, projected vector of b on the on a. And this projection operator is only depends on the vector a. For any vector b, I can get it is projection on a and this matrix is same for all the projection operators. And this is a projection operation matrix as this is a matrix multiplication the projection process should also be a linear transformation.

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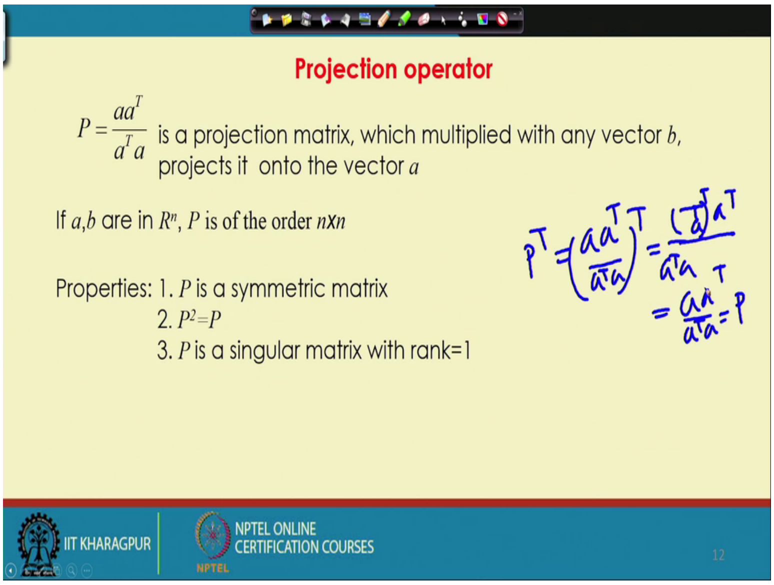

So, we will look into further detail. P is a a transpose a transpose a is a projection matrix. Which multiplied with any vector b will project the vector b onto vector a. If A b are R n in R n in real coordinates space of dimension in P should be of the order of n into n. P is a symmetric matrix and we can see that, very quickly that P transpose is equal to aa transpose A transpose a whole transpose. a transpose a is a scalar. So, it should come out of it. And aa transpose is again a transpose transpose into a transpose divided by this. So, this is again aa transpose by a transpose a is equal to p.

So, P transpose is equal to P. That means P is a symmetric symmetric matrix. P square is equal to P; we can check that if we multiply p if we do p transpose p that will again give us P. And P is singular matrix with rank one and this is I think little interesting which you should look into here.

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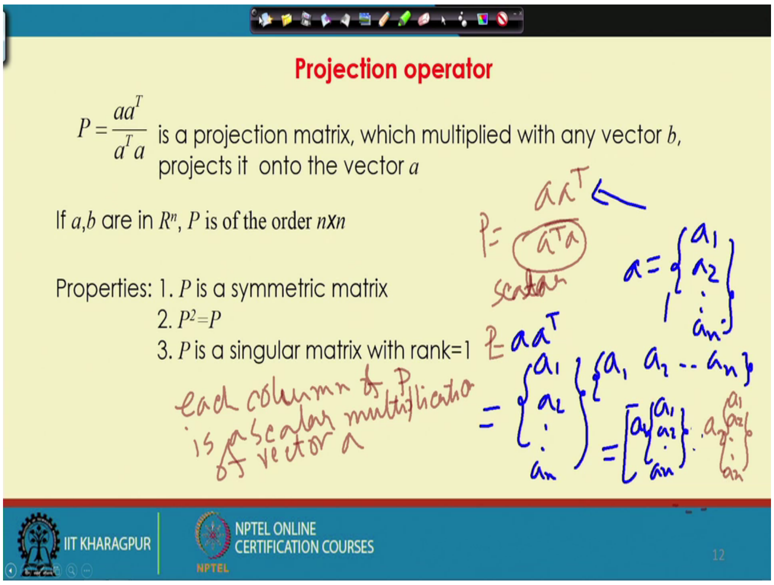

So, P is equal to aa transpose by a transpose a. The matrix part of p this is this is a scalar. This part is a scalar. The matrix comes for from aa transpose. What is aa transpose? A is a 1, a 2, a n, aa transpose is a 1, a 2, a n into a 1 a 2. What is it? This is the larger matrix which has a 1 into a 1 a 2 a n plus not plus this as the first row.

The second row is a 2 multiplied with all these things. So, a 2 into a 1 a 2 a n and so on. The last row will be a n into this. So, last column. First column is a 1 with the in multiplied with the first column of a with a column with a vector, a 2 multiplied with a vector. So, each column of this is a multiplication of a vector. So, we can write each column of P this is P equal to of p is a scalar multiplication of vector a. So, there is only one independent column, and it is should have rank 1. And is this a square matrix, it is a it should be a singular matrix, because the determinant should be 0, because all the columns are dependent on only one column.

So, this is some interesting aspect about the projection operator. So, in discussing this in little detain, because this has extremely significant importance in matrix solver both as direct solvers and in iterative solvers which will start very soon after, maybe a couple of weeks we will start working on iterative solver. Not only that projection operation is very important, if we try to understand several physical phenomena, several important phenomena like vector calculus vector algebra complex numbers. This is important a good understanding in projection operation can help us in certain things. That is why I am going into little detail on the projection operators.

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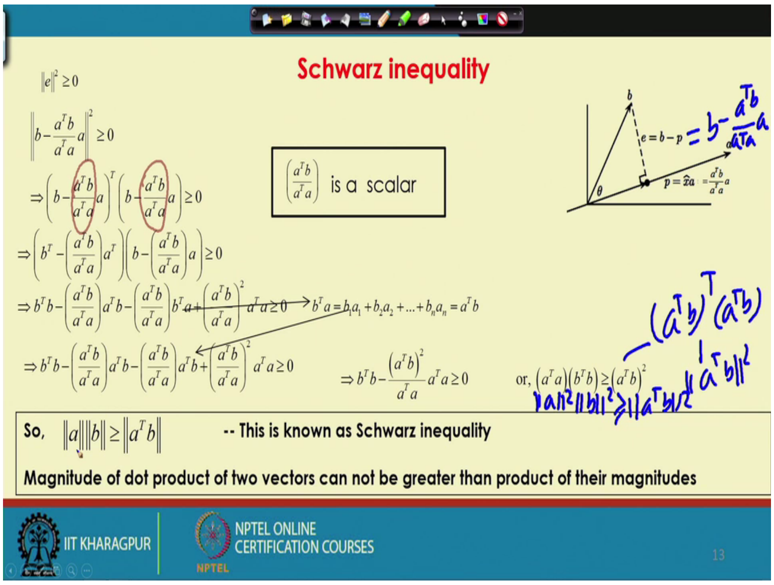

And what we end up here this, this session will end is called Schwarz inequality. So, we will start with the projection of a vector onto another vector. And see that the here or the left out part when we project b onto a the left part is b p is the projection b minus p. This left out part is either if p b is on a this is 0 or it is greater if something greater than 0 non 0. So, magnitude of e square greater than equal to 0. And b and this is we have obtained that p is a transpose b by a transpose aa. So, we substitute it that e is equal to the left out part is b minus A transpose b by a transpose a into a.

So, this should have a length greater than equal to 0. And we to find out this length square we get square of the length is greater than equal to 0. So, b minus a transpose b a transpose a transpose into b minus a transpose b a transpose a is greater than equal to 0. A transpose b a by a transpose a both a transpose b and a transpose a are basically both dot products so, both are scalar. So, this 2 terms and this is a scalar, this terms term is a scalar. So, we can do the multiplication giving this a scalar multiplication here.

So, we write bb transpose minus a transpose b a transpose a transpose is b minus a transpose b a transpose a greater than 0. So, as this is a scalar the transpose is not important here. It will remain same. So, and we multiply it we get b transpose b by a transpose b by a transpose a transpose b minus. So, b transpose b minus a transpose b by a transpose a into a transpose b. And then b transpose a into a transpose b by a transpose a minus here. Plus, a transpose b into a transpose by, sorry, a transpose b by a transpose a whole square into a transpose a is greater than equal to 0.

And what do we observed that, this is b transpose a is basically b 1, a 1, b 2, a 2, plus b n an which is same as A transpose which is dot product between 2 vectors. So, we substitute b transpose a into a transpose b into the equation. And finally, we got b transpose minus a transpose b whole square by a transpose a into a transpose a is 0 which gives us a transpose a into a transpose b transpose b is greater than a transpose b whole square. Or if we write it in terms of the mode of the vectors, that is given as mod of a a transpose a is mod of a b transpose b is mod of b, or the length of b and this is basically a transpose b transpose into a transpose b; which is mod of a transpose b square.

So, we get a transpose a a square into b square length second norm of a square into second norm of b square or length of a square into length of b square is length of a transpose b whole square. Or we can write a transpose mod a into mod b is greater than equal to mod of A transpose b. This is known as Schwarz inequality. So, if we have 2 vectors, we take their dot product, the magnitude of the dot product cannot be greater than the product of their magnitudes; it is an extremely important postulate in linear algebra as well as geometry, that magnitude of the dot product is less than equal to product of the magnitude of the vectors. And that gives us that cos theta should always be less than equal to 1.

.So, we have spend some time on the projection operators, and un in next class what we will start looking into is that, that if we can utilize the idea of projection operator like we will project one vector to another vector and get a component along the vector. And, get another component which is e or the left out component which is perpendicular to vector. So, from one vector if we can create 2 orthogonal vector. And orthogonal vectors are always of mutually independent. So, in that sense if we can create a set of mutually independent orthogonal vectors, and that can use that for certain purposes including solution of matrix equations.

So, we will start that discussion from the next class.

Thank you.