Matrix Solvers
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Lecture-23
Best Estimate

Welcome. So, in last few classes we looked into fundamental subspaces associated with the matrix solution Ax is equal to b, and we also looked into orthogonality between row space and null space column space and left null space. So, what we learnt from there that left null space is orthogonal to column space and row space is orthogonal to null space. The idea that left null space is orthogonal to column space is important to realize the fact that if the right hand side vector b has some component in left null space this vector is not a member of column space. And therefore, we cannot solve Ax is equal to b, Ax is equal to b is only solvable when b is a column space vector.

So, b has some component in column space we cannot solve Ax is equal to b. Then many cases we end up in situations where we have seen earlier where Ax is equal to b equation is given, but b is not in column space. Especially when we have independent columns, but the columns are not spanning the entire the entire r n, and b is a member of r n which cannot be formed by combining given columns column vectors of the matrix a. So, we say that there is no solution, but in special in optimization we in the area of optimization we have to deal with situations, when we get a number of equations the equations do not have particular solution. Why does not have any particular solution because, the columns of this equation is not spanning the entire column space r n or r n.

And, the right hand side solution vector b is away from the column space. However, when we are optimizing it is for example, say we got certain observations from experimental methods and found the equation. Now, we are seen that the right hand side matrix b is not in the column space so, you cannot solve this equation. But in practical cases especially for engineering applications, or when we do something like data science we end up in situations when we need a solution; however, the solution does not exist.

So, for example, I am trying to design a particular surface. And for designing the surface I got equation for normal’s at few edges of the surface. And now I cannot solve this equations once we can solve this equation I can get the coefficients, which will be multiplied with different splines and I can design the surface. However, we cannot solve this equation I mean in a situation why I cannot solve this equation, because I do not have enough columns to span the entire vector space r m and b lies in space which is not in column space of r m. So, what we do there especially in engineering applications.

We say that we try to find out an approximate solution; that means, that the solution will not be correct. I will not find out any x which will satisfy Ax is equal to b. Of course, I will not find any x because b does not lie in column space of a. So, I cannot represent b as combination of columns of a I cannot write Ax is equal to b. So, I will be in a equation where I cannot solve Ax is equal to b. However, will try to propose some estimate of x, which will not satisfy Ax is equal to b. So, in we will write Ax that is not equal to b or b minus x has some error.

X is not exactly equal to a not exactly the solution of Ax is equal to b, x n estimate of the solution of Ax is equal to b. And therefore, it has some error and we will try to minimize the error. And, what we call will find out what is the best estimate for x in case of the equations Ax is equal to b when the equations does not have any particular solution. And today's class we will look into these type of equations and the main idea is that if b does not lie in column space of a we cannot solve Ax is equal to b, but there can be some component of b which is on the column space of a.

So, we will take the b vector will project it into the column space of a. And for that projected part of b vector which is b c lying in the column space of a we will solve Ax is equal to b, and we will get a solution x will say that this is the best estimate of x. Why is it best estimate because we will also show that this estimate reduces or minimizes the error between b minus x. So, we will discuss this in this particular class. So, this is on best estimate and normal equation. Normal equation is the equation which will solve not, we will we cannot solve Ax is equal to b in leave of Ax is equal to b we will solve a normal equation form of Ax is equal to b. Let us see what is that.

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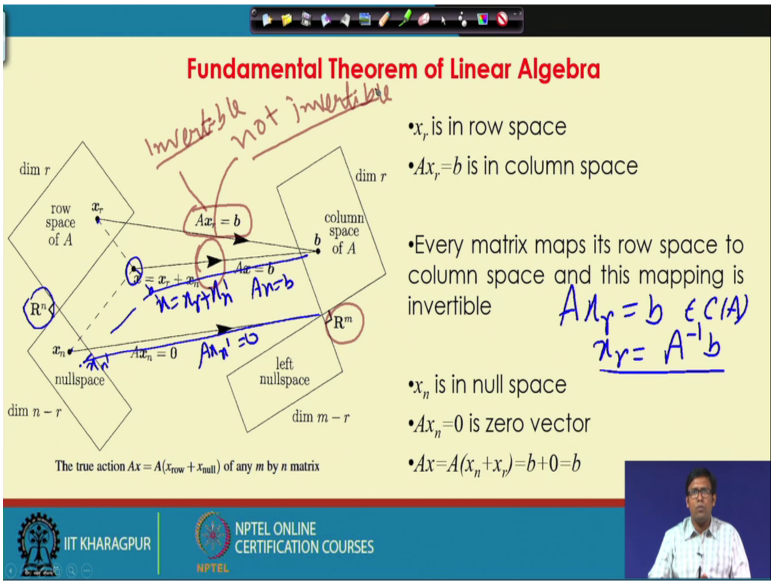

So, we will go back to the fundamental theorem of linear algebra which says, that in r n the row space exist and in r m the column space exist; that means, there are n number of columns. So, each row has n elements there n number of rows each column has m elements. And row space is has a dimension r which are the number of independent row and it is same as the dimension of column space and number of independent columns and n minus r is the dimension of null space.

So, if all the rows are not independent, then n minus r is the I am sorry if number of independent rows is less than the order of the vector space r n, then n minus r is the dimension of the null space. So, if all the rows and if all the rows do not represent the basis of r n, there are some dependent rows and n minus r will be the based on the dependency of rows, will be the dimension of null space. Any solution x n in the null space when multiplied with the vector with the matrix a goes to origin 0; so, x n is a null space and x r which is the solution of the equation Ax is equal to b must lie in the row space.

Because whichever does not row space is orthogonal complement of null space whichever does not lie in null space must lie in row space. So, I take x r which is a row space vector x r is combination of linear combination of rows of a multiply it with the matrix a. I get a column space vector b and I take another vector x n which is not a row space vector which is a null space vector, but in r n I multiply it with a I get a 0 vector.

So, while add them and I get the final result Ax is equal to b where b is a column space vector b is the column space vector means, b is a vector in the space subspace of dimension r which is formed by r independent columns wherever columns are belongs to the vector space r m.

So, if there are r independent columns r is not equal to n n minus r will be the dimension of left null space. And there is if m minus r is non0 there is a possibility that we can get some vectors we can think of some vectors, we can propose some vectors which do not lie in the column space. In case m is equal to r left null space is only the 0 vector. So, there is nothing on 0 left null space, but otherwise there can be something on left null space, which does not lie in the column space. And therefore, if we get a b which is lying in the left null space we cannot solve Ax is equal to b.

So, one important observation is that every matrix maps it is row space to column space and this mapping is invertible. In a sense if I have Ax r is equal to b, if b is in the column space and b belongs to the column space of a is always invertible and I can write x r is equal to a inverse b. And we have seen that this if a is a rectangular matrix we will think of a left inverse, but this is this has to be there the row space, and column space they are one into one map and this mapping is invertible if b lies in the column space.

Similarly, x n is in the null space. So, x n is equal to 0 is a 0 vector, and Ax is equal to x n plus x r is equal to b plus 0 is b. So, x n lies in the null space x r lies in the column space and the final solution x is addition of row space, and null space it is anywhere in the in r n with by might not be in the row space if null space is non-zero. This also will map to Ax is equal to b; however, this mapping is not invertible if I try because there are infinite if null space is non 0 there can be another null space solutions. So, this can be also another x n one which is similarly giving us Ax n 1, Ax n 1 is equal to 0.

And we can get another solution x is equal to x r plus x n one. And this Ax is also b. So, there are infinite number of x's if null space is non 0 which is giving me Ax is equal to b. So, this mapping is not invertible. This mapping is not invertible, simply because this is not one to one this is a many to one mapping. However, Ax r is equal to b this is always an invertible mapping. Anyway we now try not to look into the left hand left part of this diagram that what happens to row space or null space rather we will try to see if b does not lie in column space, if b is away from the column space what will happen to the system of equations.

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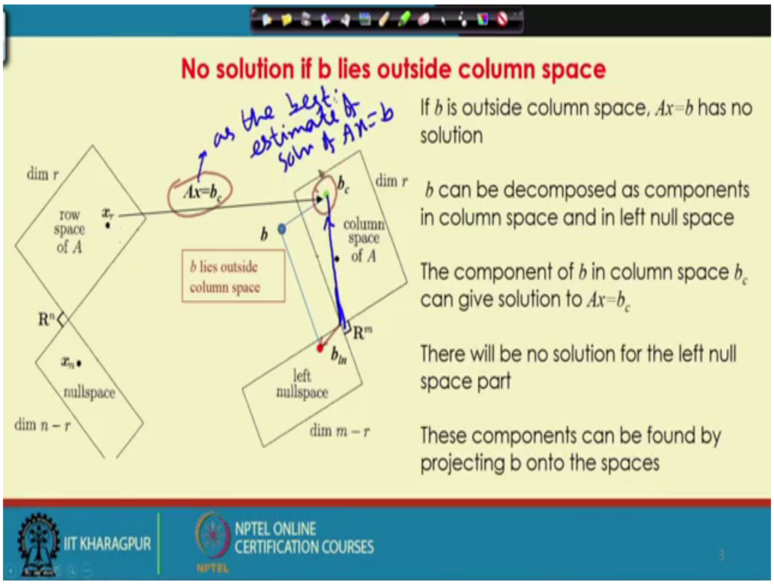

So, we have n similar situation. We have similar situation; however, b lies outside the column space. So, we cannot have any vector in a row space x r, which multiplied with a is may x r is equal to b because b is outside the column space. So, Ax is equal to b does not have any solution. B can be decomposed in 2 components. One is along the column space we project b along the column space and we get a vector b c this becomes b c this is the vector comp a sorry like this.

So, this is projection of b on the column space and we can also make a projection in the left null space and this will there will be another vector b, b ln which is in the left null space. So, b can be decomposed into 2 vectors one is it is projection in the column space another is a perpendicular part, which is it is projection in the left null space. And this is possible because column space and left null space are mutually orthogonal spaces in r n and their orthogonal compliments of each other. If something does not belong to column space that must belong to left null space.

So, nevertheless b can be decomposed as components in column space as well as in left null space. The component of b in column space is b c and this can give a solution Ax is equal to b c right. Because b c is a column space vector and now I can have solution of Ax is equal to b c. And what will do here that we will we will call that this solution this x as the best estimate best, sorry estimate of solution of Ax is equal to b. Ax is equal to b c the solution of that can best estimate the solution of Ax is equal to b. That will be the proposition here. There will be no solution for the left null space of course, Ax is if b is in left null space there is no solution. These components b c and b ln can be found by projecting b into the onto the spaces.

So, if b lies outside the column space then there can be 2 components of b, a b can be decomposed into 2 components. One is it is projection in the column space another is projection on the left null space which is orthogonal to column space. For left null space b projection of b or b ln x is equal to b ln really have does not have any solution, because b ln is completely away from the column space is orthogonal to column space. The projected part of b on to the column space Ax equal to b c will have a solution. For any b for any right hand side vector which lies in the column space Ax is equal to b will have a solution. So, Ax is equal to b c will have a solution and this solution we will call this as the best estimate of the.

Solution of Ax is equal to b which is not solvable, Ax is equal to b is not solvable because b is not on the column space we project b in to the column space and get b c. Ax is equal to b c becomes solvable. And this solution is the best estimate of the solution of Ax is equal to b. And we go we will go ahead with this. That now so now, our target will be to find the solution Ax is equal to b c and to find the projection of b on column space b c.

And we also have to justify why are we telling it to be best estimate, whether this is the least error minimized minimal error solution of Ax is equal to b or if I calculate b minus Ax that will not be 0; obviously, because Ax is not equal to b. We are solving Ax is equal to b c we are not solving Ax is equal to b. So, b minus Ax will be some value, we call it error is it minimum if we do it in this way let us look into this.

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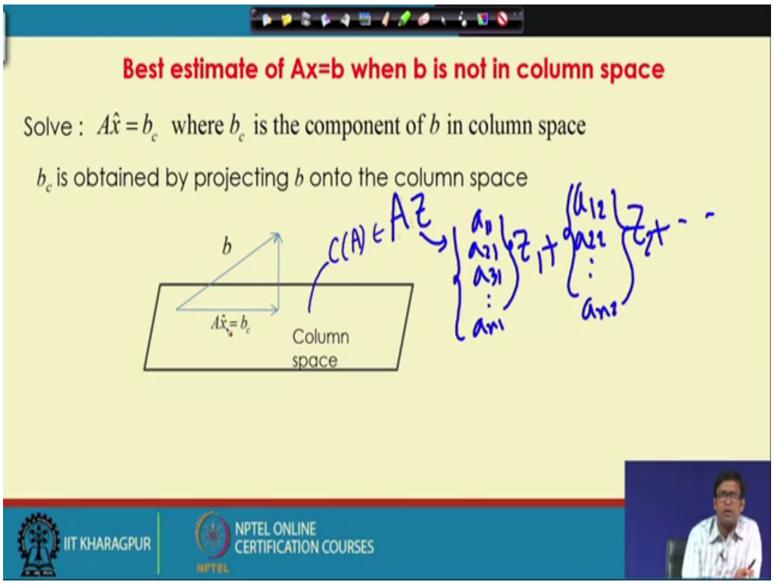

So, how to find this best estimate of Ax is equal to b, when b is not in the column space. We will try to solve Ax tilde is equal to b c, where b c is in the is the component of b in column space or b c is the projection of b in the column space b c is obtained by projecting b onto the column space. So, we have a matrix b we project it into the column space. And we get Ax tilde is equal to b c why because there should be some b c is in the column space. So, there should be some x tilde which is some value which is multiply coefficient which is multiplied with the columns of a and the linear combination will give us b c.

Or we can write any column space vector. So, any column space vector any vector on column space is of the form of z, or other value right, a z is of the form of; so, column of a multiplied by z 1 plus column of first column of a multiplied by z 1. So, what is a z this is like a 1 1 a 2 1 a 3 1 a n one first column multiplied by z 1 plus a 1 2 a 2 2 a n 2 with z 2 plus 1. So, any column space vector will be of this form. So, when b is projected to the column space we should get some x tilde for which we can write Ax tilde is equal to b c. Because this x tilde vector each component is multiplied with each component of a which each column of a and linear addition will give me a column space vector b c.

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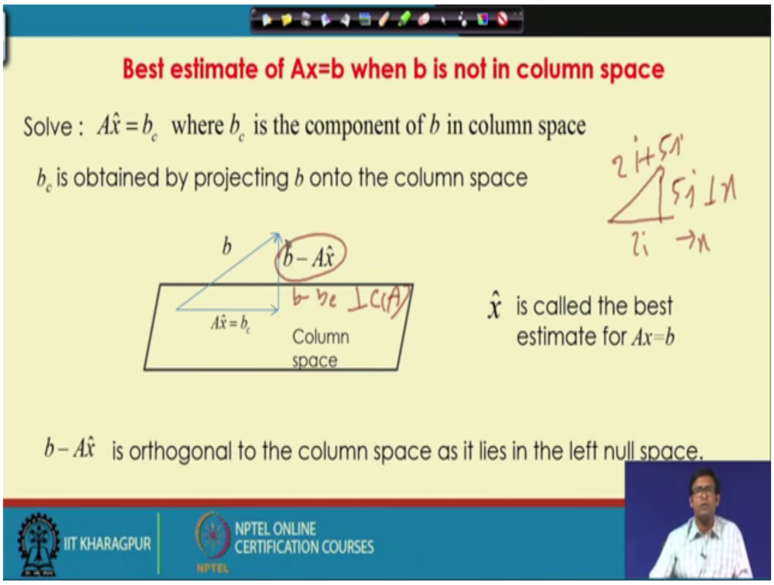

So now if I try to solve this equation, x tilde is called the best estimate of for Ax is equal to b and b minus Ax becomes orthogonal to column space Ax tilde is on the column space what is or b c is rather b is the vector which is inclined to column space. So, b c is it is projection to column paces.

Therefore, b minus b c is a part which is perpendicular to column space. We think of taking a vector 2 i plus 5 j. So, 2 i is the component along x direction. So, this is 5 j which is perpendicular to x direction therefore, this b minus b c if b c is along column space b minus b c is perpendicular to column space. Or b minus Ax is the component as b c is equal to Ax tilde b minus Ax tilde is the component which is orthogonal to columns space as it lies in the null left null space; that means, any vector which is perpendicular to which lies in the left null space must be orthogonal to column space.

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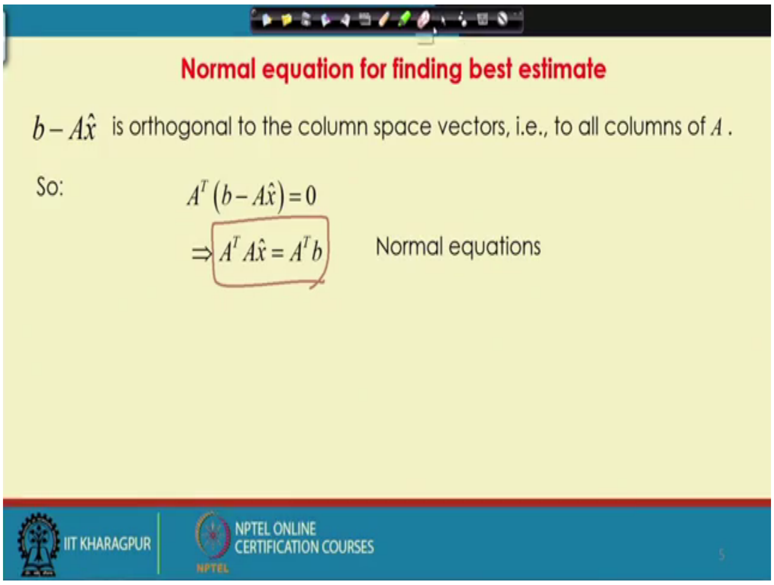

So, b minus Ax is orthogonal to column space vectors or it is orthogonal to all columns of A. So, we can write A transpose B minus Ax dot product of b minus Ax with all the columns of a must give me a 0 vector, each column is dot product of A with each b minus a x. So, with each column a must be 0 because b minus Ax is orthogonal to the column space of a. So, it is orthogonal to all vectors of the all column vectors of the matrix A. So, we multiply A transpose with b minus Ax we will get 0s as a row of the solution vector. So, it is a 0 vector.

And with some modification, we get like with rearranging it we get the equation A transpose Ax is equal to A transpose B and this we call as the normal equations. So, earlier we had an equation Ax is equal to b which is not solvable, with propose that we will not solve for Ax is equal to b, rather we will solve for the component of b which is which lies in the column space, which is the projection of the column space and the equation for that is of, of course, not Ax is equal to b rather it is it is something else. And, this equation we obtained as A transpose Ax tilde is equal to A transpose B, where x tilde is the best estimate of x and this equation A transpose Ax tilde is equal to A transpose B is called the normal equation.

Now, we have to see that whether this equation is solvable; that means, A transpose B it should lie in the column space of A transpose A b does not lie in the column space of A transpose A and we obtain that through just projecting B in the column space of A. So, it is ensured and we also have to see that if this equation is solvable then some process like gauss elimination should hold and we should not get a 0 pivot when doing gauss elimination or lu decomposition type of things. So, A transpose A must be a nonsingular matrix or A transpose A must be invertible.

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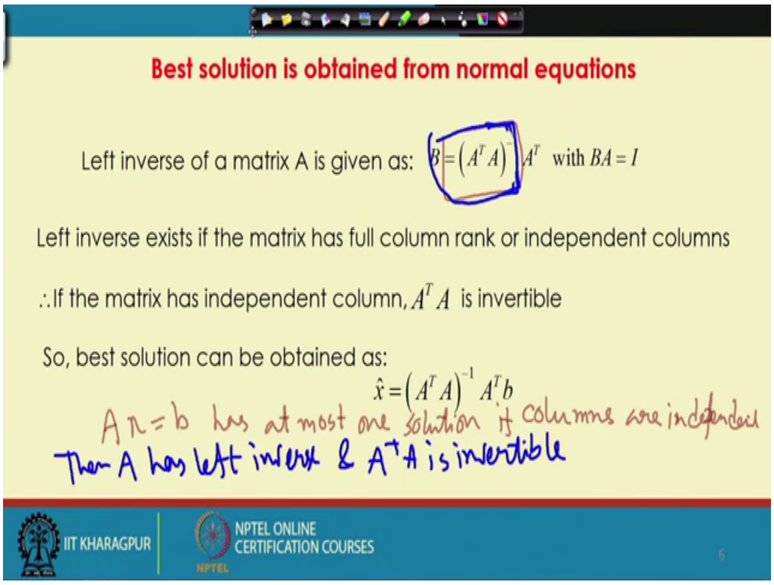

And we look into it that left inverse of matrix a is given as B is A transpose A inverse A transpose where what is left inverse the inverse the where matrix which is multiplied in the left of matrix a gives an identity matrix. Left inverse exists if the matrix has full column rank that is or the matrix have independent columns. So, if the matrix of all independent columns then the left inverse will exist. And if left inverse exists; that means, A transpose A is invertible left otherwise left inverse will not exist.

So, if we get a matrix which has A which has all independent columns then A transpose A is invertible. Or we should be able to get a solution of A transpose Ax is equal to A transpose Ax tilde is equal to A transpose B or x tilde can be obtained as A transpose A inverse is equal to A transpose B. So, what we can write is that that Ax is equal to B we have seen when the left inverse exist Ax is equal to B has at most one solution if columns are if columns are independent as well as spanning then Ax is equal to b has just one solution, but Ax is equal to B has at most one solution if columns are independent it may not have any solution. But, if it has any solution it will have at most one solution it cannot have infinite solutions if Ax the columns are independent.

And in that case, then A transpose A then if this has at least one solution then a has left inverse and A transpose A invertible. So, we have a matrix which has independent columns. Now, Ax is equal to b is not solvable because this columns are not spanning enter r m and b lies outside the column space. Still we can have an estimate of Ax is equal to b which is A transpose Ax tilde is equal to A transpose B. This equation is solvable because matrix with independent columns will have A transpose A A transpose A to be sorry will have A transpose A which is invertible. Now, what we will check is that, that this equation system gives will check that this equation system gives the least error solution of Ax is equal to B.

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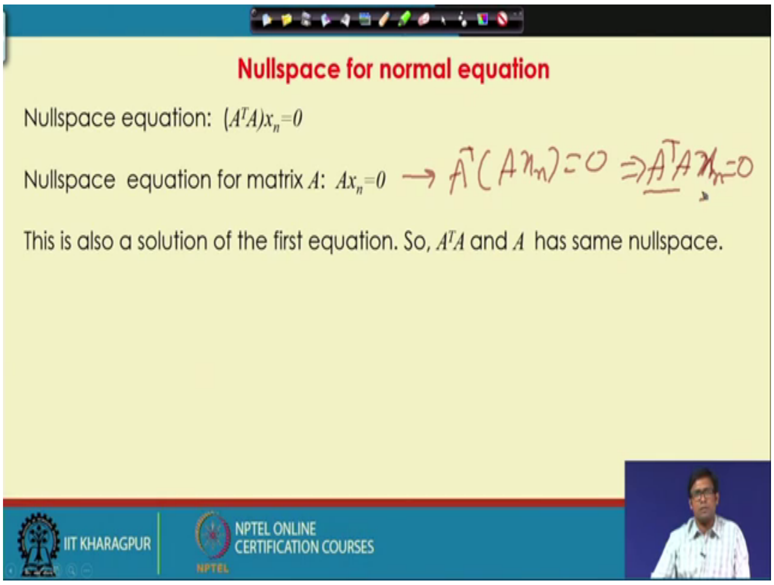

But before that we wants also check this that, what will be null space of A transpose a. So, if A transpose A is invertible what we understand that null space of A transpose A must be 0, but let us see how what will be null space of A transpose A. Null space equation gives us A transpose Ax n is equal to 0. And null space equation of the matrix x n A A is Ax n is equal to 0.

So, if I substitute if I multiply this with sorry if I multiply this with A transpose A transpose Ax n is equal to 0, and we get A transpose Ax n is equal to 0. So, this is also the solution of first equation. So, A transpose A and A has some same null space. So, this should (Refer Time: 26:39) at a later stage that if A has independent columns; so, but A does not have all independent rows. Therefore, there should be some null space of A.

And how this is related with null space of A transpose A, but A transpose A and A will essentially have same null space which is evident here. Now what I was telling earlier that we have to check that the best estimate is actually a best estimate; that means, the error obtained while doing finding best estimate is minimum and we can check that.

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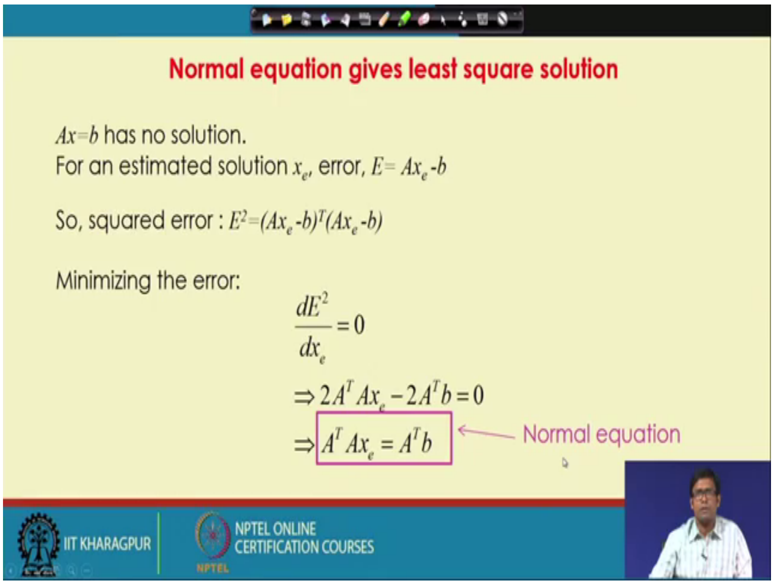

The normal equation gives least square solution, instead of normal equation if we do a least square method do least square find least square of error and try to find out the solution which gives least square error solution, it will be the same one Ax is equal to b has no solution. So, we let us estimate x as x e and the error e is Ax e minus b. So, the squared error is e square which is and we try to find out the squared error in many cases, because it is an absolute value which is Ax e b transpose Ax e. So, this is a vector transpose of vector. So, it is a scalar this is the sum of the error in each of the rows of A.

And, if I try to minimize it then we will get the equation d e square d x e is equal to 0. So, of course, we have to see that the second derivative is positive for minimal value, but then we find out A transpose Ax is equal to A transpose B, which is nothing but the normal equation. Now, we can and that shows that normal equation is also the least squared equation and the error is minimized when you are solving normal equation and that is why this is a term used by statisticians that A transpose Ax tilde is equal to A transpose B is the best estimate of the solution of Ax is equal to b when Ax is equal to b does not have any solution, or B lies outside the column space.

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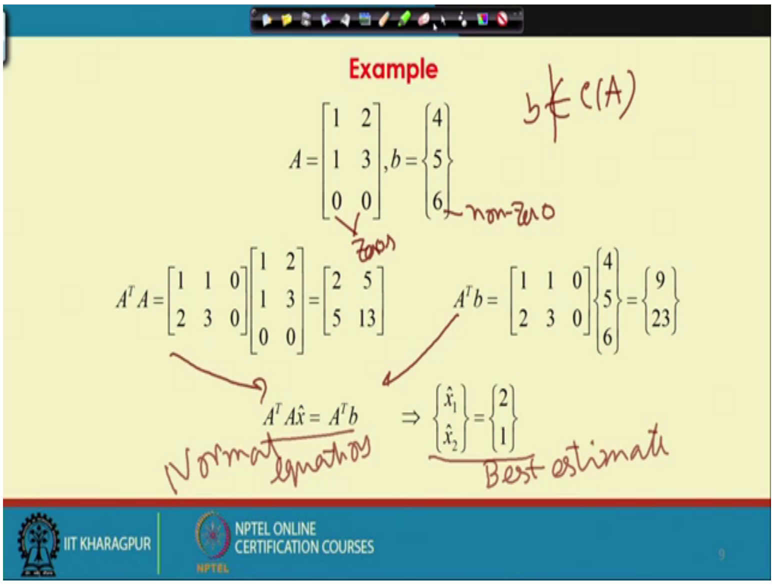

So, look in to a example quickly before we finish take the matrix a is 1 1 0 b is 2 3 0 b is 4 5 6. So, we can quickly check that these are 0s and this is non 0. So, the column of a does not have A 0 and the third component, B has all non 0 third component. So, b really does not lie in column space of A B is an independent vector of column space ok.

. So, we find out A transpose A which is 2 5 5 13, and A transpose B is 9 20 3 and A transpose Ax tilde is equal to A transpose B which gives us x 1 tilde x x 2 tilde is equal to 2 1. So, this is the what will say that this is the best estimate of Ax is equal to B. This is the best estimate and this comes from solving the normal equations. So, you just substitute A transpose B here substitute A transpose A here and we can solve this because it is a 2 by 2 system we can quickly check it also.

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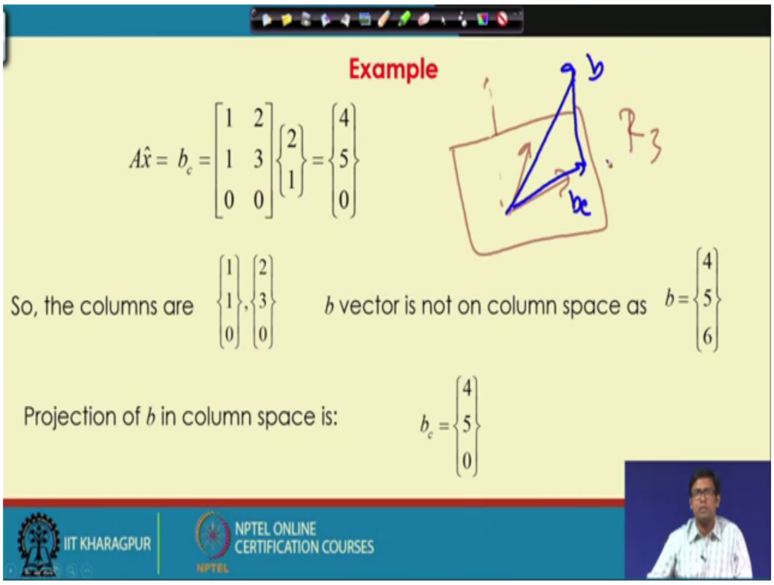

Now, what we will do we will substitute this x into this x tilde in to the main equation and see what is the error. So, Ax tilde and Ax tilde is the projected value of b in the column space is b c is a into x tilde is 4 5 0 the columns are 1 1 0 2 3 0. So, this is basically projected value b vector was not in column space and b c is projection of the b vector in the column space. So, column space here was 1 1 1 0 2 3 0 means 2 vectors like this right, which is forming the a particular plane in r 3 and this is the this is r 3 this should not use here and the vector b was away from the r 3 this is b.

And now we project b in this column space and this becomes b c which has a 0 component in the z direction. So, this is the projection of b in column space it is verified. So, in next class we will start looking in detail on the projection operators, and what are the general forms of operator which can project a vector b onto the column space, and we explore detail about the projection process.

Thank you.