Matrix Solvers
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Lecture - 21
Left and Right Inverse of a Matrix

Welcome. We have been discussing few elementary concepts of linear algebra in last few classes. We have discussed about linear independence, spanning, basis, dimension this definitions and then we also discussed about fundamental subspaces related with the matrix A, in the context of solving Ax is equal to b equation. And these spaces are row space, column space, null space and left null space.

We will continue the same discussion and we will try to look more from the perspective of solution of Ax is equal to b, and using the definitions of these spaces if you can build our understanding more sound on solvability of x is equal to b whether, solutions exists and when the solutions are when there is a unique solution of x is equal to b. So, we look into the perspectives in more detail, using the basic understandings of linear algebra, which we have developed over last few classes. So, today's discussion is on left inverse, right inverse and the fundamental subspaces.

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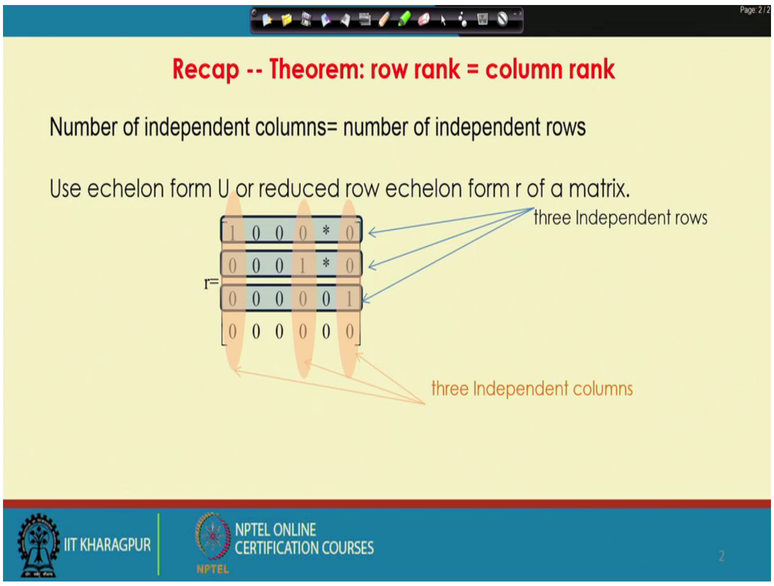

The important part which we will recapitulate in this discussion before studying this discussion is the theorem with says row length is equal to column length. And that implies that number of independent columns in a matrix is equal to number of independent rows. And this can if you can remember, this has been shown by considering the echelon form or reduce to echelon form of a matrix, the r form from r of a matrix a, where we got the pivoted rows and number of pivoted rows is same as number of pivoted columns. And these are the independent rows and independent columns of matrix a when transferred into reduce to echelon form. Therefore, the number of independent rows and number of independent columns is exactly same as the number of pivots in the reduced to echelon form. So, these numbers are same.

Therefore, number of independent columns and number of independent rows same implies the dimension of the row space and dimension of the column space is same or row rank of the matrix or column rank of the matrix is same.

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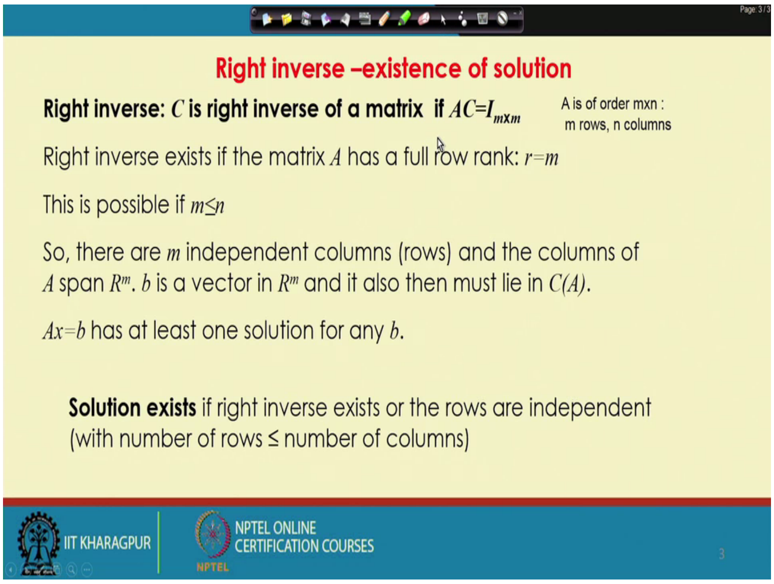

Now, we come to an important definition which is right inverse. So, this definitions right inverse and left inverse are typically for rectangular matrices; in case of rectangular matrices inverse of matrix really does not exist. Instead what you say that there is something called right inverse of a matrix which is multiplied on the right side of the matrix and gives an identity matrix. C is right inverse of matrix A, if AC is identity matrix I when A is the rectangular matrix with order m in to n; that means, A has m rows and n columns.

Now, it is not guaranteed that right inverse will also exists. Like if we take a square matrix if the matrices singular, then inverse of the matrix will not exists. Similarly if we take a rectangular matrix, the right inverse may not exist and similarly if we follow something like a Gauss elimination step, we will see 0 pivots and we will see that right inverse is not existing here. So, when the right inverse will exists and it has a relation as evident from the title of the slide, it has a relation from with existence of a solutions.

When should a right inverse exist? The right inverse exist if the matrix A has full row rank r is equal to m. The matrix A is of order of m in is into n. So, it has m rows and n columns and all these m rows are independent. So, it has a full row rank or the rank of the matrix or dimension of the row space is equal to m, then only we will say right inverse will exists. That means, if we consider the matrices has collection of linear equations, we have all independent linear equations and this is possible if number of column is greater than or equal to number of row.

Because, if number of column is less than m, then we cannot have m number of independent rows; maximum number of column is actually maximum number of independent columns. So, maximum number of independent columns will be the number of rows, number of independent rows. So, the rank will be n which is less than m. So, n has to be greater than equal to m in order to have a full row rank matrix. And therefore, there are m before coming into this point we should write there are if the matrix is a full row rank r. So, it has r independent rows that will imply what number of independent rows is equal to number of independent columns sorry not r this is m it has m independent rows.

So, it has m independent columns. So, out of n columns the matrix has n columns out of n columns m are linearly independent and column space is a subspace of R m, m is the number of rows. So, there are total n components in a column in a column vector. So, column space is a subspace of R m. So, we have R m vectors as column vectors and there are m independent vector.

Therefore, the entire R m can be span by few of the column. So, we can say that there are m independent columns which is the number of independent rows, they are same and the columns of A span R m because there an any m independent columns each column vector belongs to R ms. So, there are total m vectors independent vectors which can form a basis of R m or can span the entire R ms.

So, few of the independent columns are m of this independence column will span the entire column R m space. B the right hand side vector is the vector in R m therefore, b column space span R m, b lies in R m b must also lie in R m column space. So, if right inverse exists if all the rows are independent interestingly this is coming from the row perspective; if all the roes are independent the column space spanning entire R m and therefore, b must lie on column space. So, what we will can tell is that A x is equal to b we will have at least one solution for any b.

Because b is on the column space, it may have more than one solution because there are more column space the number of equations are less than the number of variables, there are more column vectors than the number of vectors in a basis, there are total n vectors m are sufficient to span the space. So, there can be more than one solution, but there is at least one solution, because this ensures that b lie in the column space.

Therefore, existence of right inverse ensures that the solution exists if right inverse exists or the rows are independent and this is with the case that number of rows is less than equal to number of the columns. And we can say that this is due to the fact that columns span entire R m. So, this is from the comes due to the property of spanning of R m. That columns will span entire R m, b is a member of R m therefore, b must lie in the column space. So, solution will exist with light inverse exists now let us see what is left inverse.

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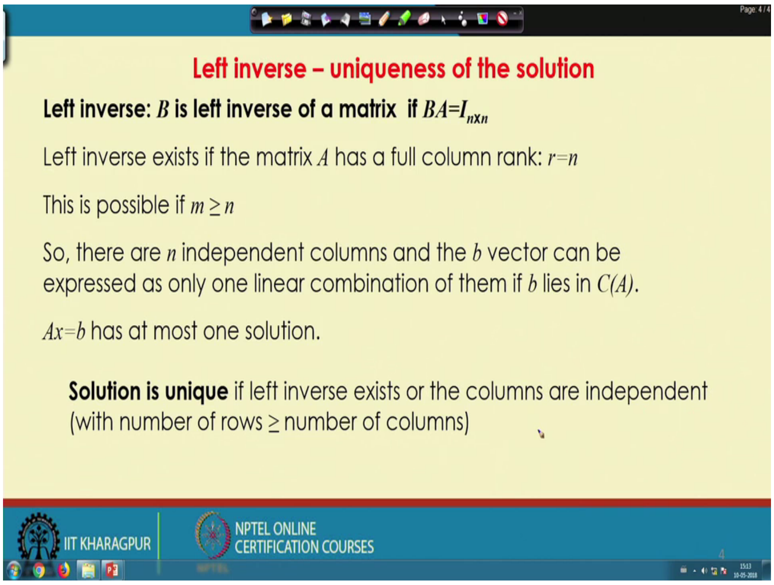

Left inverse gives uniqueness of the solution, left inverse is it is B is sorry it is not right inverse the term will be left inverse, B is left inverse of a matrix if B multiplied in the left of the matrix and gives an identity matrix. And if the matrix is of A is of the order m into n, then B multiplied before a gives me gives as an identity matrix n into n. So, B will be n into n. So, both are rectangular matrices left inverse and right inverse both are rectangular matrices in the cases when A is the rectangular matrix.

And left inverse exists if the matrix A has full column rank or r is equal to n which is possible similarly when number of rows is greater than equal to number of columns, then we can have all columns independent and number of independent column is equal to number of independent rows. So, we will also have n independent rows. So, what interestingly we have there are n independent columns and therefore, b vector can be expressed as only 1 linear combination of the columns of b. If b at all lies in column space b may not lie in column space, because it does not ensure that the column space spanning entire R m.

However, the vectors in the column matrix in the matrix the column vectors of the matrix they are linearly independent therefore, there is at most there is only one linear combination of the column vectors, which confirm the victor b this comes from the property of linear independence. So, A x is equal to b we will have at most one solution and solution is unique if left inverse exists or the columns are independent with number of rows is greater than equal to number of columns. So, this comes from the uniqueness only and as well as left inverse comes from the property of independence.

I will write linear independence of columns. So, if the rows are linearly independent, all the rows are linearly independent then we have right inverse, which ensures existence of the solution if all the columns are linearly independent then we have left inverse, which ensures uniqueness of the solution.

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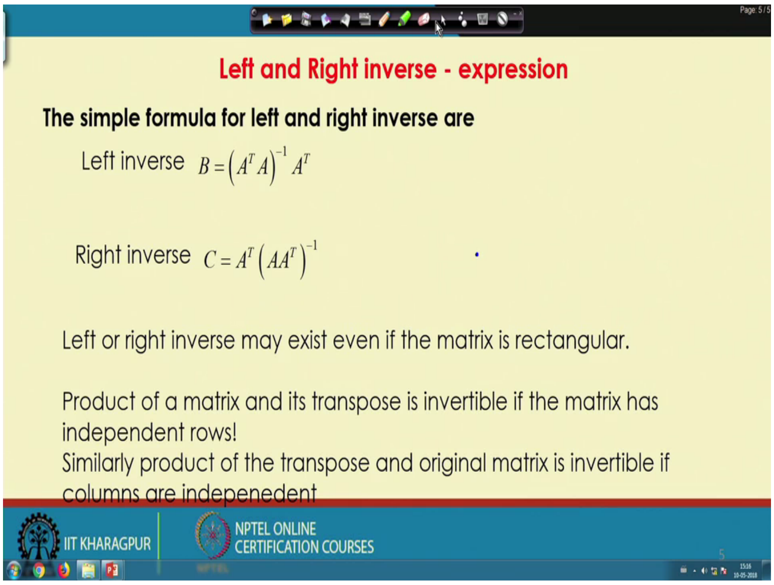

So, let us move ahead and we can check very easily that there are very simple formula for right inverse and left inverse left inverse is given as B is equal to A transpose A inverse into A transpose and therefore, we can write we can calculate it all right B A is B A is equal to A transpose A inverse A transpose into A which is identity matrix. Similarly, right inverse is given by the formula that C is equal to A transpose A A transpose inverse. Left inverse this also tells us one thing before coming into the next points that A transpose A or A transpose is invertible A maybe a square matrix, A may have independent dependent rows or dependent columns.

However, maybe not a square matrix, maybe a rectangular matrix may have dependent rows and dependent columns. However, A transpose A or A A transpose can be invertible in the few cases. A A transpose this is invertible, A transpose is invertible when we have independent columns, then because it has to be invertible in order of existence of B are left inverse.

So, if columns are independent then A transpose A must be invertible. Similarly A A transpose is independent when we have independent rows. Go to the next point left or right inverse may exists even if the matrices are rectangular. Product of a matrix and of course, for a square matrix they also exists will come in a while. Product of a matrix and its transpose, which is A and A transpose is invertible if the matrix has independent rows. And similarly the vice versa is product of transpose and the original matrix is invertible when the columns are independent. This is a small piece of the information, which we should note and it will be important that later stage when we look into more detail of independent rows and dependent columns some square matrix of a like that some rectangular matrix like that.

Now, we also should see what will happen for a square matrix for a rectangular matrix we can find out independent number of rows less than equal to number of column, number of column less than equal to number of row in both cases independent all independent rows and all independent columns gives us respectively right inverse and left inverse, which is related with existence and uniqueness of the solution. Now what will happen for a square matrix? This formula m less than equal to n m greater than equal to n, still holds because equal to m is equal to n as such that the matrix square.

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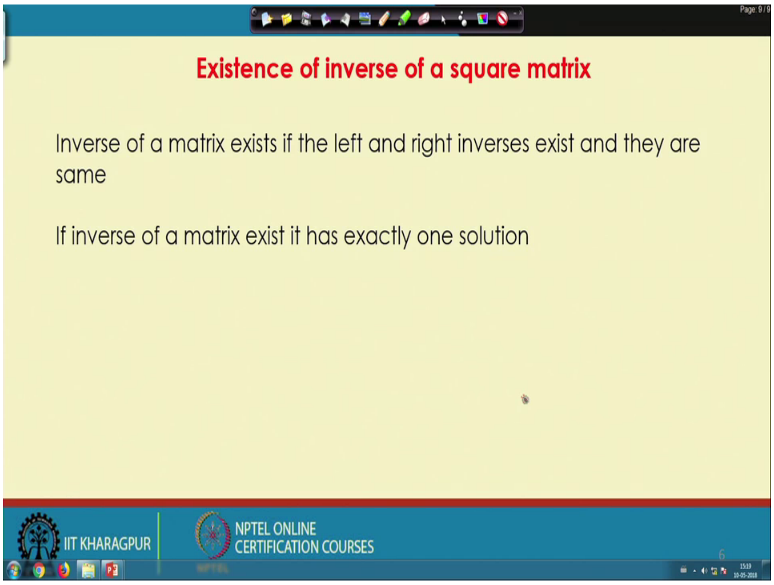

So, inverse of a matrix for a square matrix inverse exist, inverse of a square always is for a square matrix. Inverse of a matrix exist if right in and left inverse exist and they are the same. So, if both of them exists and both are same both of them can only exists for square matrix right because m greater than equal to n is existence of left inverse m less than equal to n is existence of right inverse, only when m is equal to n we will have both them inverses exists right and left and then they were same. And, then we will say that the matrix is a invertible square matrix inverse of the matrix. If inverse of the matrix exist it has exactly one solution why? Inverse exist means that left inverse exists and left inverse exist means that there is solution exist there is at least one solution.

Similarly, right inverse exists for inverse both left and right exist right exist means there is at most one solution. So, when both are both the solutions exist they has to be exactly one solution it displace at least 2 1 and at most 1. So, if inverse of the matrix exits and we have seen it in several cases, there is exactly one solution. However, the cases with rectangular matrices with left inverse existing we can get at least we will should get at least one solution, we can we may get many solutions if the matrices are rectangular one. Similarly, we can get at most one solution if right inverse exist; that means, b vector will lie in the column space.

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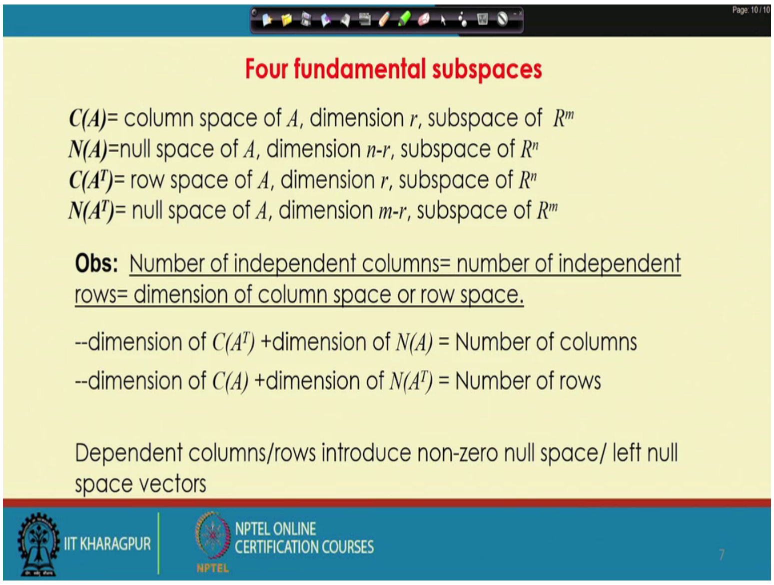

So, we again further explode the idea of existence and uniqueness especially existence of the solution and we now look into the definitions we obtained in last few classes on the fundamental subspaces, there are 4 fundamentals sub spaces column space which is denoted by C A which is a dimension r, which is the number of independent columns as well as number of independent rows. And it is a subspace of R m, m is the number of rows or m is the number of elements in one column vector. Null space of N A has a dimension n minus r and it is a subspace of R n where n is the number of rows and therefore, these are n is the number of column sorry and therefore, this is the number of elements in one particular row vector.

Row space of a is denoted by C of A transpose or column space of A transpose has same dimension as column space because number of independent columns is same as number of independent rows and it is sub space of R n, n is the number of components in row vector which is equal to number of columns. And, we have left null space or null space of A transpose sorry this is not this is not null space we should write this is left null space of A or null space of a transpose it has dimension in minus 1 and it is sub space of R m. So, column space and left null space are both subspace of R m and null space and row space are both sub space of R m.

And number of independent columns is equal to number of independent rows, which is dimension of column space or row space dimension of row space plus dimension of null space is equal to number of columns and dimension of column space and dimension of left null space is equal to number of rows, these are the few basic observations which we have made in next few places. So, we will look into an example and see how null space row space column space and left null space look like for one particular matrix, and we purposefully chose one rectangular matrix because m and n will be different and we can different dimensions of null space and left null space also.

Another interesting observation here was that dependent columns and rows introduce non-zero null space or left null space. So, if there is no dependent column then the null space is a 0 vector. So, r is equal to m and we will say that the because dimension of dimension of row space and dimension of null space is equal to number of columns, if all the columns are independent then dimension of row space sorry then dimension of row space if all the columns are independent, then dimension of row space and dimension of the number of columns are same. So, null space is a 0 dimensional vector which is basically a 0 vector and same for left null space.

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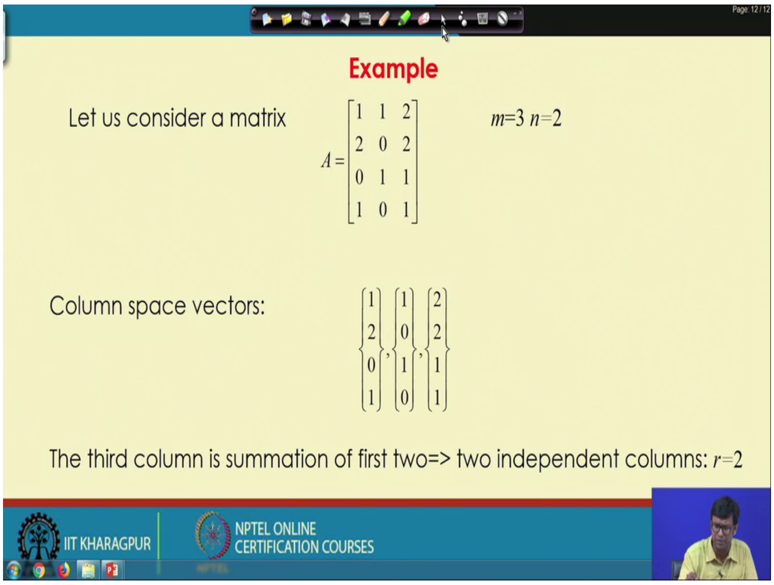

So, dependent columns and rows introduce non zero null space and left null space vectors. Now, we will as an example we will take an in the will take a rectangular matrix, which has dependent columns and rows. So, that we get non zero null space left null space and let us see how do they look like. So, let us consider the matrix A which is a sorry m is equal to 4 n n is equal to 3 everything it wrong I should correct it which is a rectangular matrix with m is equal to number of row is equal to 4 and number of column is equal to 3 4 into 3 matrix.

Now, what are the columns row vector now basically the columns and which is 1 2 0 1 1 0 1 0 2 2 1 1; now if we look into the column space vectors, the third column space vector the third column space vector is summation of first 2 column space vector. As the way I said purposefully I have chose a matrix which is dependent columns and rows. So, we what we get? We get that the third column is summation of first 2 columns therefore, there are 2 independent columns r is equal 2. So, rank of the column space rank of the column rank of the matrix is 2 rank of the matrix is 2 dimension of the column space is 2 because there are 2 independent columns therefore, we should be dimension of the row space also, but let us look into the row space.

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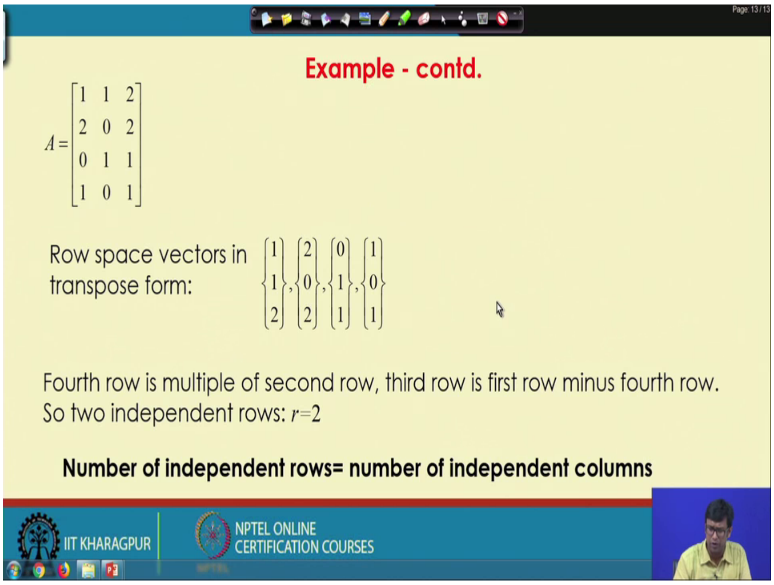

And the row space vectors are 1 1 2 2 0 2 0 1 1 1 0 1 they are written in this transpose form as following and then what we can see is that, that 4th row, this is the 4th row, this is multiple of 4th row is multiple of second row 4th row is basically half of second row and third row is first row minus 4th row. So, 4th row and third row are dependent on first and second rows. So, number of. So, the there are two independent rows and number of independent rows is equal to number of independent columns, that is also varies right here. Now, we will look into the null space.

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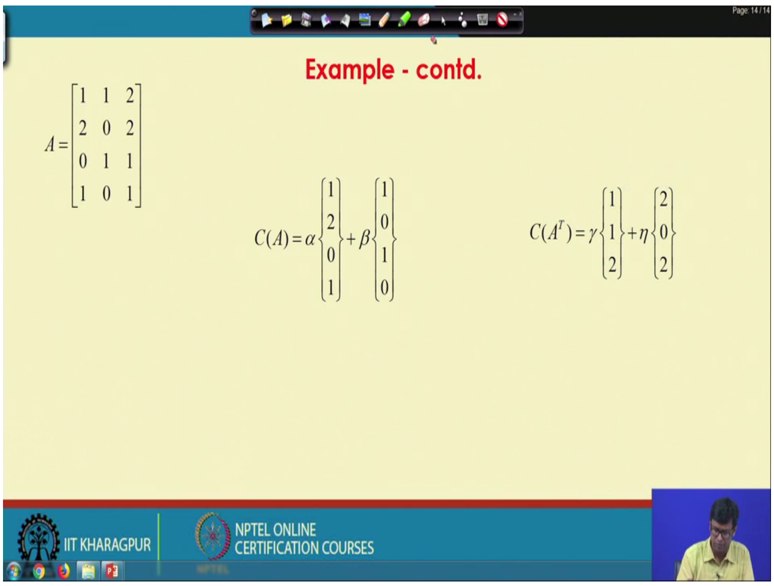

So, before that column space is given as linear combination of the two independent columns. So, any column space vector will have any column space vector any vector, which lies on column space will have this form: what is the form? Alpha is an arbitrary constant multiplied with one column vector plus beta another arbitrary constant, multiplied with other column vector and they can constitute any vector which is lying in the column space by choosing different values of alpha and beta.

So, there can be infinite vectors in column space is 2 vectors. So, in R 4 because they have 4 components in an R 4 they will construct a plane, we cannot visualize what is the plane in R 4, but there is, but they will construct a plane in R 4 what is further important is that if we chose alpha is equal to 0 and beta is equal to 0 its a 0 vector 0 vector is always a member of any vector space. So, it lie in the column space. And the row space will similarly look like transpose of the row space column space of a transverse is, some constant gamma into 1 1 2 plus some constant eta into 2 0 2, and we can really form any vector in row space will have this form. So, we can really construct the plane in 3 D by say 2 0 2 right. So, one vector will be this is xyz.

So, 2 0 2 1 vector will be something like this, and another vector will be x 1 y 1 and z 2 there will be another vector like this and the plane contain by this vector anything any vector in this plane. So, this has to be extended like this, any vector in this plane will be a row space. So, if I draw anything like this will belong to row space. So, this will be the row space, which is a plane in a three dimensional unit however.

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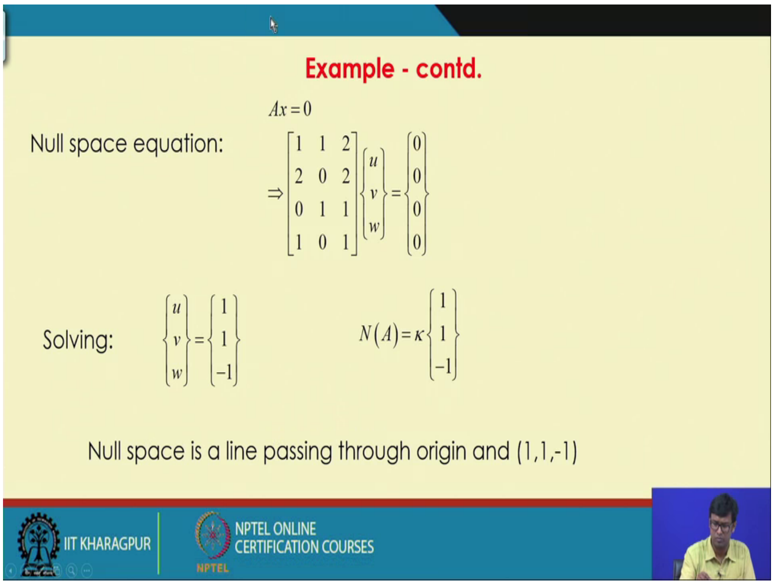

So, we now look into the null space and they solve the null space equation which is A x is equal to 0 and get null space as 1 1 minus 1 and 1 null space vector that. So, if any null space vector will be constant multiplied with 1 1 minus 1. So, this will be basically if I have sorry if I have the axis if I have the axis xyz, this is basically 1 1 minus 1. So, this is basically some line like this which passes to 0. And any vector along this line, will be the null space vector. So, null space is a line passing through origin and 1 1 minus 1.

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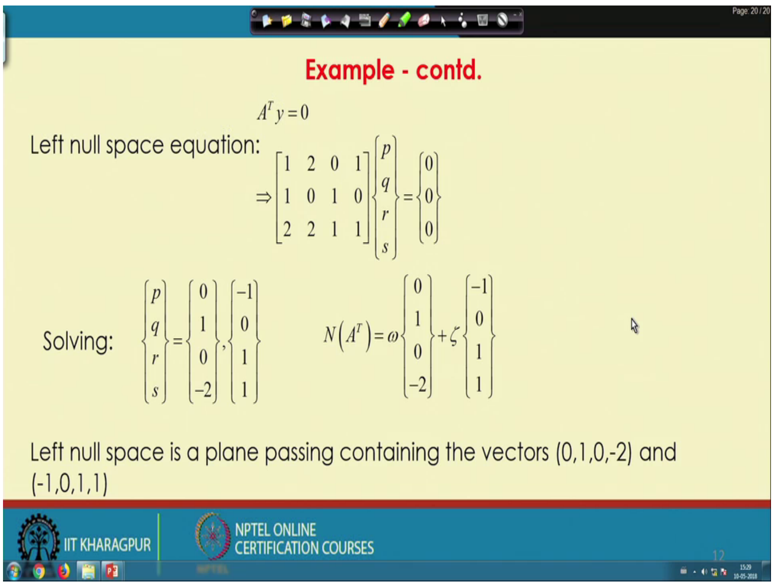

Similarly, the left null space equation A transpose y is equal to 0 and solving we get 2 vectors for left null space and any vector in left null space is a linear combination of these 2 vectors. So, this is also because there are 2 vectors, this is also a plane in R 2 R 4this is a plane in R 4. So, left null space is a plane passing which contains the vectors 1 0 0 2 and minus 1 0 0 minus 2 and minus 1 0 1 1 its again a plane.

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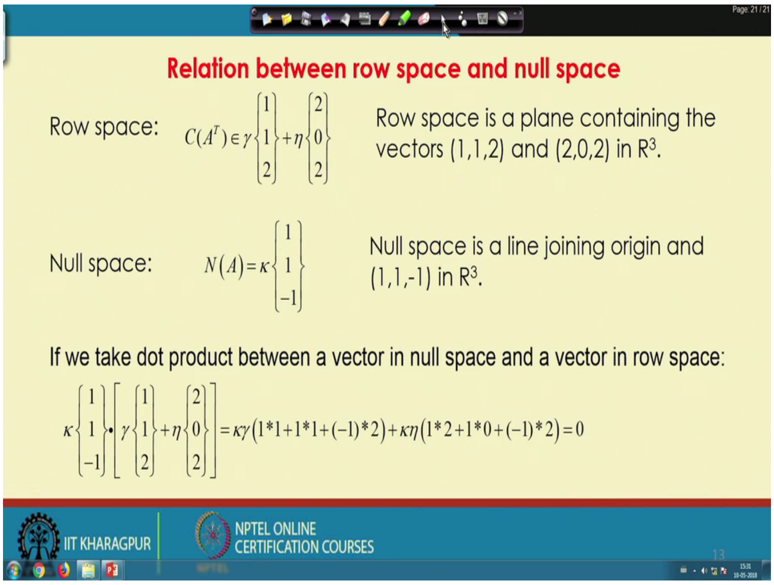

So, that is how we can find out all the spaces null row space column space left null space null space and check the dimension should come accordingly if we only have to find out the number of independent columns and rows, and that will give dimension of row space column space subtract that from number of columns and number of rows, get respectively the dimension of lift null space null pace and left null space. So, we will get row space which is this which has dimension 2 null space which has dimension 1 all their member in r 1 all they are three because they have three components in each of the vector. So, row space is the plane containing the vectors 1 1 2 and 2 0 2 in R 3 null space is a line joining origin 1 1 and minus 1 in R 3.

And if we take dot product between the vector any vector in null space, which is which will have this particular form with any of the row space vector which will have this particular form. And this dot product will be we can do it this will give a 0 and 2 vectors. So, dot product between 2 vector is 0 that ensure the if a dot b is equal to 0 this ensures that a and b are perpendicular. So, what do you are getting here at least for this particular example, a row space general form of a row space vector is perpendicular to another general form of null space vector. So, every row space vector is perpendicular to any of the null space vectors. Same observations this is an interesting observation property of the vector sub space will follow later.

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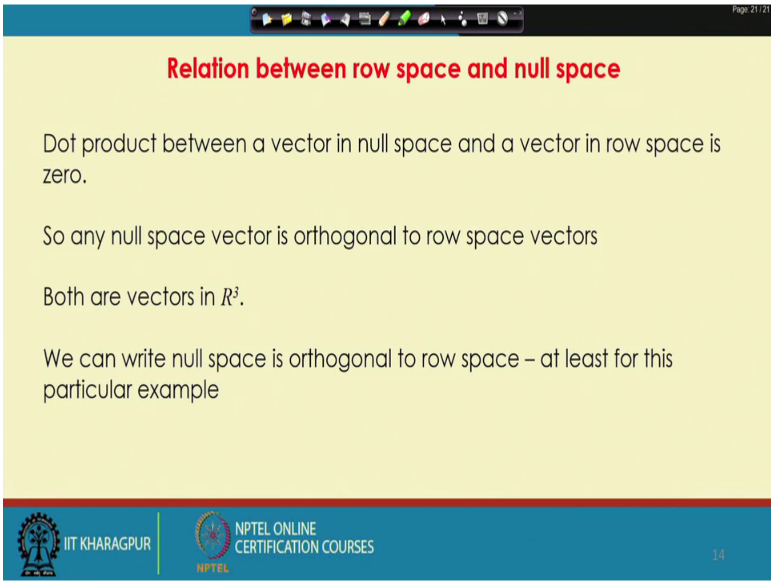

This same so, this what we can say relation between row space and null space dot product between a vector in null space and a vector in row space is 0. So, any null space vector is orthogonal to row space vector both are vectors in R 3. So, we can write null space is orthogonal to row space; that means, the null space is a line, null space is a line and row space is a plane. So, this is row space and this is null space null space is orthogonal to row space and that is why we can have any vector in row space and there contain joined at origin, any vector in row space perpendicular to any vector in null space. At least this is true for this particular example that null space is orthogonal to row space.

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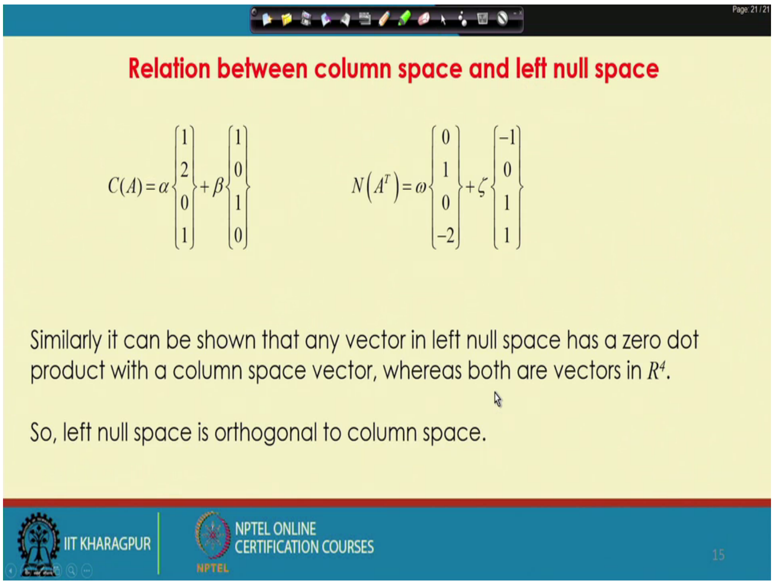

Similarly, we can look into column space and null space and left null space and say that any vector in left null space is a 0 dot product with a column space vector whereas, both are vectors in R 4. So, in R 4 the left null space is orthogonal to column space and one interesting thing comes out of the second observation that left null space is orthogonal to column space.

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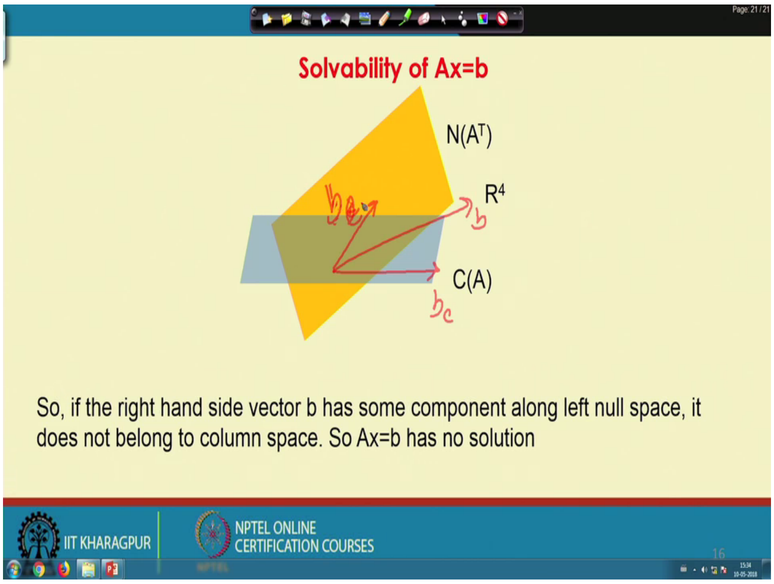

So, look into solvability of A x is equal to b. So, for example, this is the left null space this is orthogonal to column space. So, if we have any vector b say this is a vector b, and it has some component along the left null space this is the, it has some component along the left null space b l. Therefore, the component along the left null space is always orthogonal to the column space. So, b has a component which is away from the column space therefore, b is not on the column space. So, x is equal to b we will have no solution b may have some components bc along column space and right bl along left null space.

But if any vector b has some component along left null space left null space; left null space is orthogonal to column space therefore, b will have some component which is not on the column space b will have a component orthogonal to column space b will. So, if this is the column space, b will come out of the column space because b will have a component orthogonal to the column space. Therefore, b is not on the column space and ax is equal to b does not have a solution.

So, this is a very important observation that right hand side vector b if it has some component along left null space, it does not belong to column space and Ax is equal to b has no solution. So, this is one important observation, but this is made on this particular example, which we have chosen in this case. In a next class we will look for more general expressions and see what happens for a general matrix, what is its left null space column space or null space row space relation, and how are they important in case of solution of A x is equal to b.

Thank you.