Matrix Solvers

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Lecture – 20
Four Fundamental Subspaces of a Matrix

Welcome. In last class last two classes we discussed about the definitions of basis dimension linear independence spanning and the classes before we discussed the definitions of few spaces like column space and null space. Now we will see; how what are the spaces associated with a particular matrix A in the context of solving A x is equal to b. And what are their dimensions is there, if we can work with independent vectors they are etcetera. So, we will look into Four Fundamental Subspaces associated with A x is equal to b and basically, we look with four fundamental subspaces which can be formed from the matrix A.

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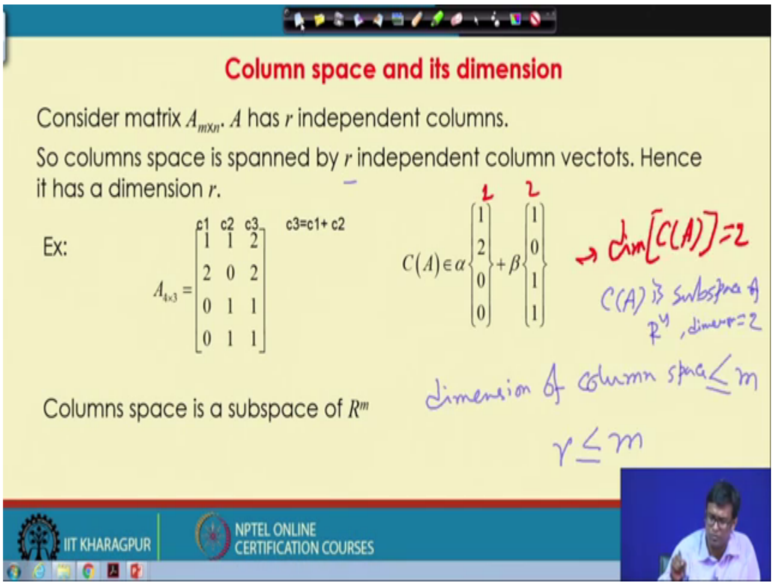

So, it let us consider matrix A, which has order m into n, and let us assume that A has r independent columns. So, column space is spanned by r independent columns, because the columns will columns of the matrix or the vectors; which is span the column space. Hence, the column space has a dimension r because these are the vectors these are vectors will span the entire column space.

So, in think of an example, A 4 3, which has columns 1 2 0 0, 1 0 1 1, 2 2 2 1. And what we can see here is that, so there are three column vectors, column space can span maximum three dimensional space. However, the each vector is a member of R 4 the real coordinate space is R 4 for each vector. So, what we see that the 3rd column can be expressed as combining 1st and 2nd column. So, 3rd column is linearly dependent on 1st and 2nd column, 1st 2 are the linearly independent columns c 3 is equal to c 1 into c 2.

So, the column space can be expressed by minimum three spanning minimum 2 spanning vectors or maximum 2 linearly independent vectors, which are the dimension of column space is 2, we can write here dimension of column space is equal to 2 because, there are 2 vectors which can 2 linearly independent vectors which span over the entire column space.

So, column space is also a subspace of R m n, R m because m is the number of rows that is number of vector terms or number of components in each column vector. So, column space is a subspace of R m, but it its dimension can be so dimension I am sorry; we can also write dimension of column space is less than equal to m, m is the dimension of the real coordinate space in the to which column space is the subspace. So, that is a maximum dimension of a column space.

Column space is has a dimension we wrote r. So, r is always less than equal to m. Here we can see that column space is subspace of R 4 but, dimension is 2. So, the dimension is always less than the dimension always the equal to the dimension of the real coordinate space.

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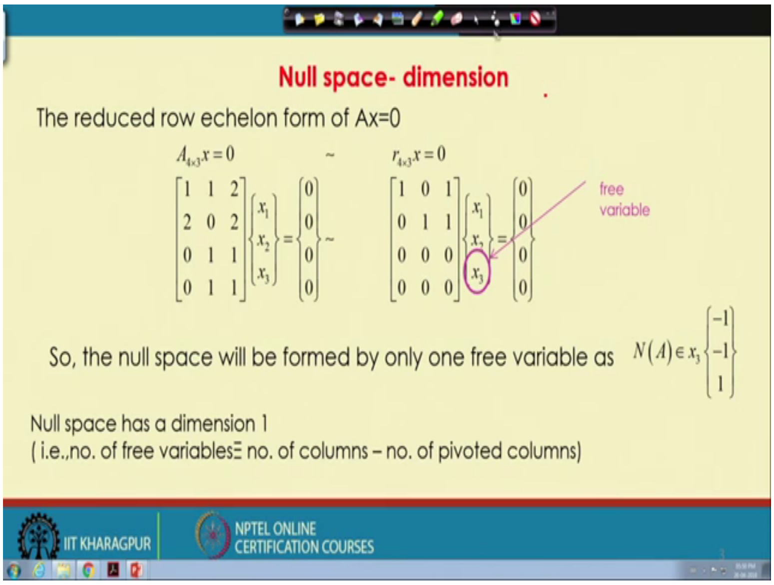

Now, what is the dimension of the null space look into the reduced row echelon form? A x is equal to 0. A is a 4 by 3 matrix here, when we; get the reduced row echelon form of A x is equal to 0, we get the 2 pivoted rows or 2 pivoted columns and 2 non pivoted rows or 1 non pivoted column.

So, the this 2 pivoted rows will give us 2 pivot variables and the 3rd one will be a free variable. So, based on the free variable the null space will be formed so, there is one free variable therefore, there will be only one null space vector and null space will be a line joining the which is passing through the center and this point minus 1 minus 11 on any vector along that line will be a null space vector. So, null space is spanned by minus 1 minus 1 1.

If I multiply 0 with this gives origin which is also a point on the null space vector and any point on the null space vector can be found there. So, null space has a dimension 1, that is number of free variables will give me the dimension of the null space. What is number of free variables? Number of free variable is number of columns because, we have 3 columns and only 2 columns we have 3 columns and only 2 pivot elements. So, number of columns minus the number of pivot pivoted columns or number of pivots. That will give me the number of free variables and null dimension of null space dimension of null space is equal to number of free variables of reduced row echelon form of the matrix A.

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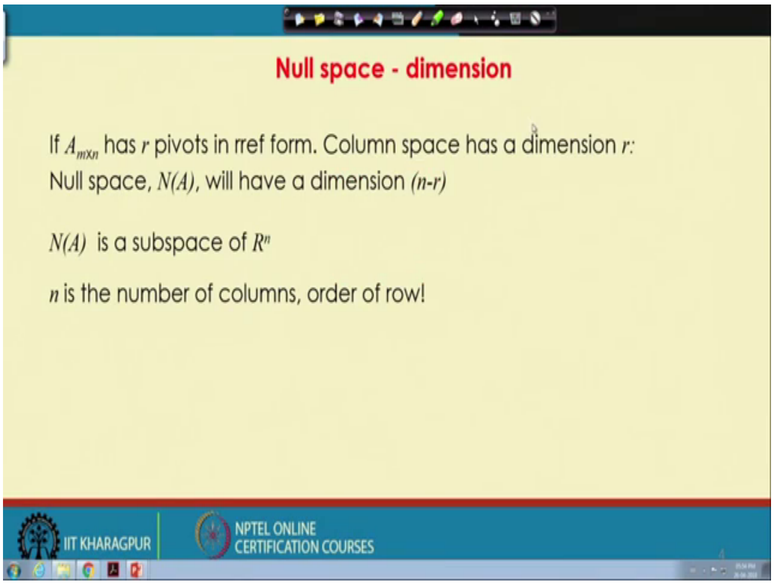

So, we explore it little more if A m n has r pivots in reduced row echelon form column space has a dimension r, null space will have a dimension n minus r n is the number of columns.

Null space is also not a not necessarily a member of the real coordinate space to which column space exist, because number of columns will give me maximum number of vectors in the maximum dimension of null space or number of vectors in a null space; that is define the number of columns.

So, dimension of null space is number of columns minus dimension of column space. And null space is a subspace of R n, column space is a subspace of R m and n I am sorry; n is the number of columns or the order of row. So, if I have this equation, that is 11 2, 2 0 2, 0 1 1, 0 1 1, each row has an order 3 into 1, each row has an order 3. So, n number of column is the order of row and each row therefore, has three components. So, we can say that null space and row space both are subspaces of R n, where n is the number of columns or number of components in row.

And however, dimension of null space is number of column n minus number of independent columns. In case all the columns are independent R and it is a in case all the columns are independent n columns are independent, the dimension of null space will be 0. If so, this is this is important to discuss if all columns are independent. So, the number of vectors which span over the column space will be exactly the number of columns number of independent vector that span over the columns will be number of exactly the number of columns or column space has dimension n.

So, dimension of null space will be of null space what will it be? N now that means, r is equal to n n minus r so, n minus n is equal to 0. That means, N A is only a 0 vector, 0 vector has no dimension it is only a point. So, null space is only a point in case all the columns are independent. And why will it be? Because all the columns will have 1 pivot therefore there will be no free variable. So, null space will not have anything rather than 0 0 0, it will be uni identically the 0 vector.

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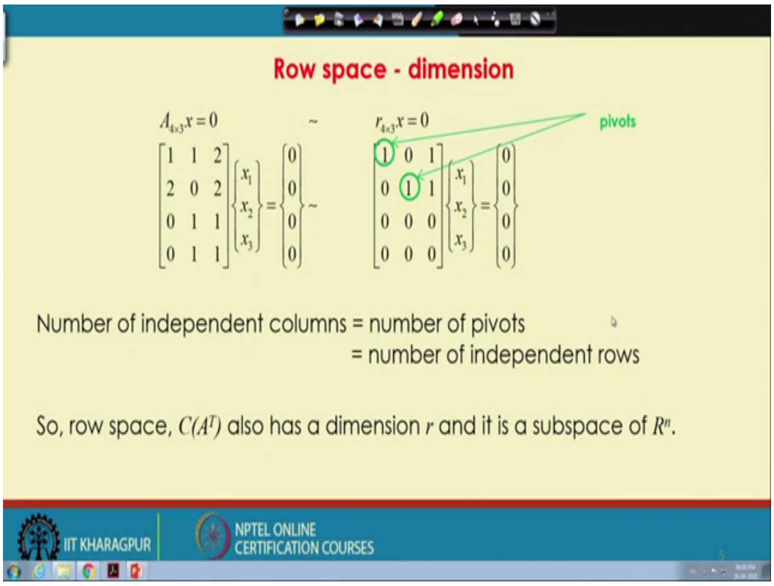

So, now we come to the concept row space and the dimension of the row space; this is A x is equal to 0 and in the reduced row echelon form this is r x is equal to 0. A and r both are 4 by 3 matrix 4 rows and 3 columns. So, if we look into the reduced row echelon form matrix, both the matrices are one matrix is basically reduced echelon form matrix is basically transformed form of a matrix. So, both have similar dimension, similar independent and columns and rows etcetera only there determinant is different.

So, they have the we discussed about this transformation when we are discussing about determinants and different matrix transformation and so, what is the effect on the determinant as well as what is the effect on the solution of A x is equal to b, when we transform one equation to other. Essentially the one matrix to other matrix essentially, these matrices remain very similar in many cases the determinant remains same in some case, the determinant is scaled by certain factor by which the rows are divided specially from up to echelon from the determinant will be same and reduced row echelon form, when you are dividing a row by a constant the determinant is divided by that constant. However, the number of independent rows columns remain same.

So, we will see, what is the dimension of the row space or what are the number of independent rows, why because row space is the space spanned by the rows of the particular matrix. So, all are all rows are spanning gives me the spanning set then what is the number of independent rows that will me give that will only give me the number of independent spanning vectors. All rows when you span the vector space so, number of independent rows will be the dimension. Similar for the column space the all columns span the entire column space, number of independent columns are only important, which will give me the number of independent spanning vectors of all the column space or the basis vector, basis of this column space or the dimension of the column space.

Similarly, number of independent rows will give me dimension of the row space. So, we can convert it A x is equal to 0 r x is equal to 0 and when we look into the r matrix we see that there are 2 pivot terms. The 1st and 2nd one these are the 2 pivot terms and the remaining rows have 0 only. So, there are 2 pivoted rows, there are two independent columns because 2 columns have pivot term, 2 rows also have pivot terms therefore, there are 2 independent rows.

So, number of independent columns will give me number of pivots and number of independent rows. Therefore, the row space will have same number of independent column row as the number of independent columns in the column space. So, the dimension of row space will be same as the dimension of column space which is r. However, column space like here the column space was subspace of R 4 and row space row has three element which is written as transpose A transpose column space, because row is basically transpose of A should be discussing later I should have discuss it.

So, row is the transpose column of A transpose is a row of A. So, column row space of A is written as column space of A transpose. So, this is the row space is a subspace of R 3 because each row has 1-3 components. So, row space row space which is given as column space of a transpose because, column of a transpose is row of A. So, using only one notation for the spaces that is column space C, C of A is column space C of A transpose is a row space. So, row space has same dimension as the column space r.

However, its subspace of a different real coordinate matrix, real coordinate space which is R n n and we can write that, this n is order of row vectors the number of components, that the row vectors have or number of columns. The number of elements in a row vector is basically, 1 2 3 the number of columns. So, n is the number of order of row vectors and number of components in the row vectors or the number of columns. So, row space has same dimension, dimension of row space is same as dimension of column space.

However, the order of row space is different of than the dimension of column space because column space is of R m column space belong to the real coordinate space m row space belong to real coordinate space of order n. So, number of independent columns is equal to number of independent rows; which is apparent from this example and that gives rise to a very important theorem in linear algebra and matrix algebra.

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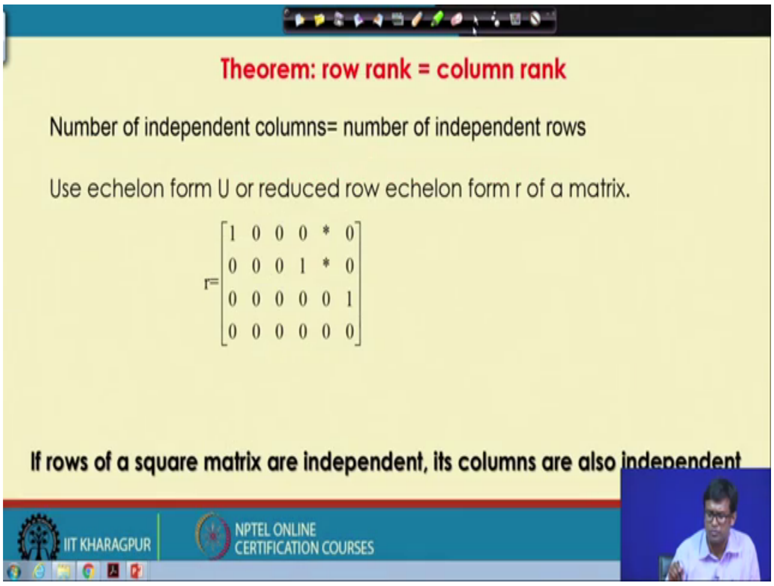

That row rank is equal to column rank number of independent columns is equal to number of independent rows.

So, if we and this is evident from the example we saw in the last slide and similar thing will see here, that if we take echelon from or reduced row echelon form of a matrix. Reduced row echelon form means, there will be 1 1st non zero term in any row will be 1, coming from left everything will be 0 and the 1s term will be 1 and coming from top any column will have the 1st non zero term, which is 1 and remaining terms can can be anything however, each row will have a one term. So terms below the one in one particular column will must be 0 because, the next row will have a 1 at the 1st non zero term before the terms before that will be 0, these are reduced row echelon form.

But, this terms can be non 0. So, what you see from a reduced row echelon form this this is transformed from of the matrix A. So, it will have same number of independent rows and columns as matrix A. These are the 3 independent rows, the 4th row is a 0 anyway is a independent row, but any of the pivoted rows, because it has the 1st non zero term here and the this term is 0 here this term is 0 here. So, this is this cannot be expressed as linear independence or linear combination of 1st and 3rd row.

Similarly, 3rd row cannot be linear in combination of 1st and 2nd row. So, these three are the linearly independent rows. And if you look from the column perspective they are the 3 independent columns. So, what we know earlier is that rank of a matrix, remember this we discussed when we are discussing about determinants is equal to number of independent rows or columns.

Therefore, number of independent rows and number of independent columns in a matrix must be same, which is a very important theorem. So, if we think of an n by n matrix if A is an n by n matrix, all rows are independent, all rows independent, that will imply that all columns are independent. So, a square matrix with independent rows must have independent column and sorry.

If rows of a square matrix are independent its columns must also be independent number because number of independent rows is equal to number of independent columns. If there are there is n into n matrix, which is always independent there are n independent rows therefore, there should be n independent columns. So, and in that case we call this to be a full rank matrix and this case we call it to be a if it is all rows are independent columns are independent rank is equal to n and this is called a full rank matrix, that is number of row and number of column is equal to the number of independent rows or columns; that is equal to the rank and the this is the only case the matrix is nonsingular and it has a non zero determinant.

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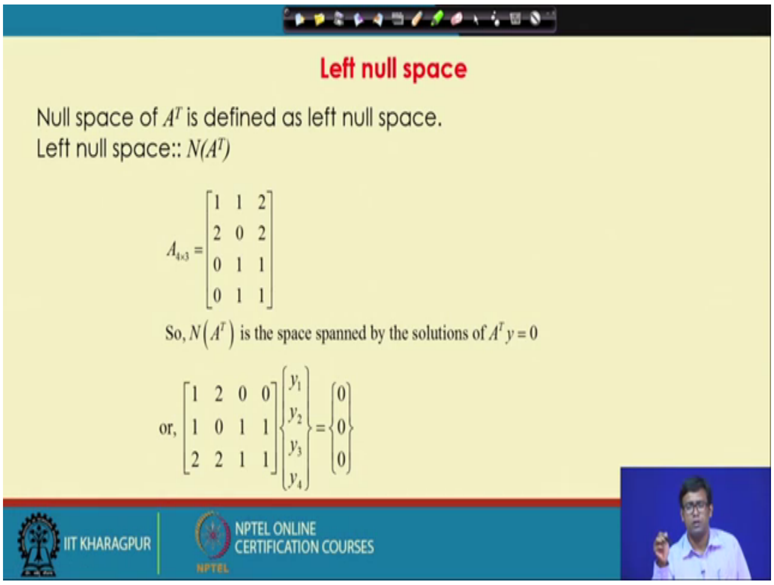

Now, there is another important space, which is called left null space: null space of a transpose that is defined as left null space. Left null space is given as N of A transpose and null space of A transpose. If A is a 4 by 3 matrix, null space will be the space spanned by the solution of A transpose y is equal to 0, which is all row transpose of a all rows are written as columns and multiplied by y 1, y 2, y 3, y 4. So, left null space will be a subspace of if A is m into n, m is number of row and n is number of column.

Left null space, null space is a subspace of N A is a subspace of R n and left null space will be a subspace of R m because, we see that there are 4 rows and left null space has 4 components. So, it is a subspace of R m. So, this is after row space column space, null space, row space, left null space, is also a fundamental space of associated with the matrix A.

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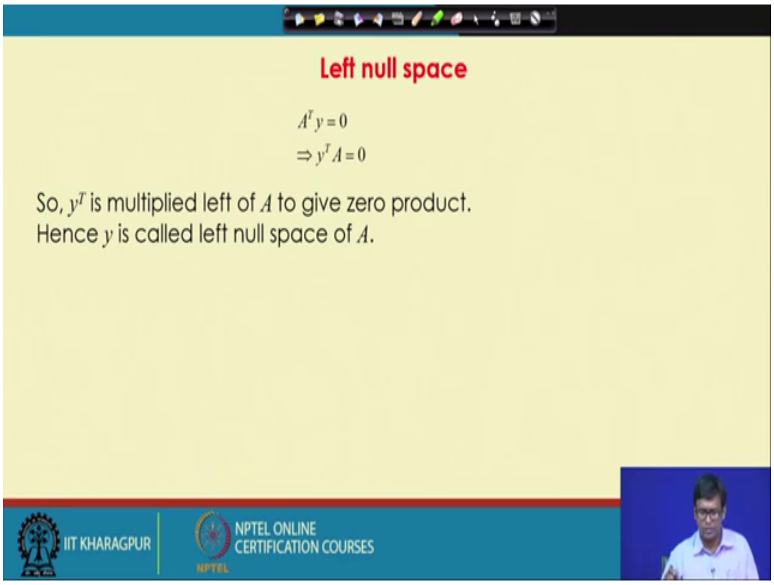

And if A transpose y is equal to 0, we can write y transpose A is equal to 0. So, this is y is in its transpose form is multiplied with A and gives us 0 product and therefore, y is called y comes in the left of A so it is called that left null space.

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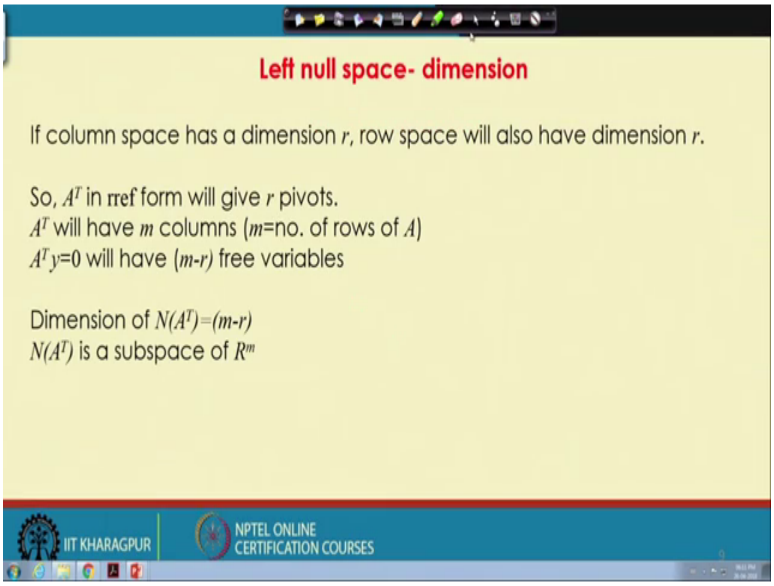

So, if columns what is the dimension of left null space? If column space has a dimension r, row space will also have the same dimension, dimension of row space will also be r. So, A transpose in rref form or in reduced to echelon form will have same number of pivots, the number of pivots number of independent rows is equal columns gives me the number of pivots, which is equal to number of independent rows. Now number of rows or columns of A transpose. So, number of independent columns in A transpose is equal to number of independent column rows in A is equal to number of independent columns of A.

So, they should have same number of pivots, A if A has a pivot r, A has dimension r or r independent columns, A in rref form, A has dimension r that will make that A in reduced row echelon form of A will have r pivots. A transpose in reduced to echelon form will also have r pivots. So, A transpose will have m columns m is number of rows in A, A transpose will have m columns and r pivot. And so, null space of A transpose will have m minus r free variables. A transpose is has m columns among which, added pivoted columns. So, m minus r are the free variables.

Therefore dimension of null space of A transpose will be m minus r which is the number of free variables. So, free variables consists null space. So, null space will be spanned by so we can probably also write it N of A transpose will be spanned by m minus r null space. So, the dimension of null space of A transpose will be m minus r and null space of A transpose is a subspace of r m.

So, when question comes, that we are telling that the number of vectors which are spanning the null space is the dimension of null space; that means, these vectors are independent vectors, through that they are independent vectors. However, I will ask you to ponder over this question, that we make 1 free variable to be 1 and other 0 and getting one independent vector. We will make another free variable to be 1 and make other 0 and get another null space vector linear independent vector null space vector, that way we will get a set of null space vectors.

Each null space vector will have 1 in the position of that particular free variable and 0 for other free variables. Are this vectors linearly independent? Yes they are linearly independent, but why are they linearly independent? I will ask you to ponder over this question, if you write down the form of like take any example and write all the null space vectors you will see that they are coming to be independent vectors, but look into it probably at the later stage or during discussion of in the discussion forms we can discuss over this. I will try to address it in some of the later lectures, but right now ponder over this question.

So, we got dimension of null space, left null space is m minus r and it is subspace of R m; it is not a it is a subspace of left null space is a subspace of R m.

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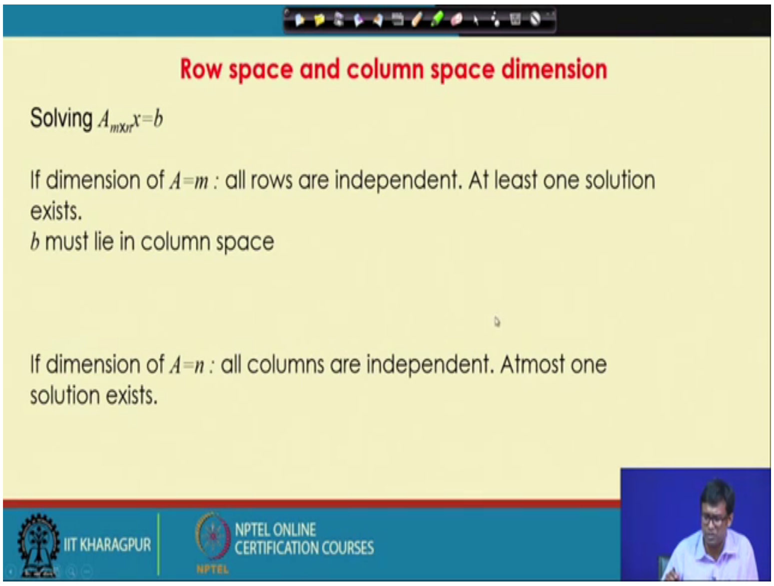

So, the utilization of this idea in solving A x is equal to b. If dimension of A is equal to m; that means, all rows are independent so, we have the equations which are independent to each other. Therefore, there should be at least one solution to these equations; the equations are not one equation cannot be expressed as linear combination of others. The equations are independent to each other there is at least one solution and which ensures that b must lie in column space.

So, what you are getting the rank rows are independent and that ensures that b is in the column space. So, row columns are in that way very much interlinked. And if we see the other case that the dimension is of A is n, all columns are independent, then maximum one solution will exist there may be no solution, but if rows are independent there will be solution.

So, max there has to be if all the columns are independent there can be maximum one solution so, this two combine tells us about existence and uniqueness of the A x is equal to b equation.

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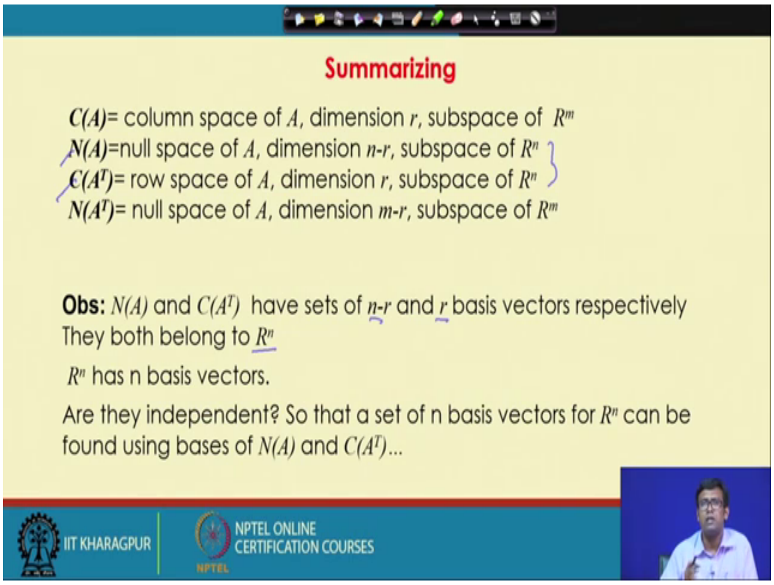

So, if we summarize this column space of A is has a dimension R and it is a subspace of R m given A has an order m into n, m rows n columns and r independent row or column rows, rows or columns are independent rows or columns. So, we have m rows and n columns so column space is a member of R m is a subspace of R m real coordinate space m and dimension r.

Null space has A dimension n minus r number of columns minus r and it is a subspace of R n. Row space or column space of A transpose is it has dimension same as column space R it is a subspace of R n, left null space has a dimension m minus r it is a subspace of R m. The interesting observation is that: null space and row space they are member of same real they are subspace of same real coordinate space R n.

R n will have maximum n number of basis vector R n will have n number of basis vector not maximum n number of basis vectors. Null space is n minus r basis vectors row spaces r basis vectors. If they are linearly independent the vector basis vectors of null space and basis vectors of row space.

Then we can probably say that, row space and null space will span the entire real coordinate space R n. Are they independent? So, that n basis vectors from R n can be found using the basis of null space and row space and is it same for column space or left null space. And we will look into this question in discuss this answer of this question in next few class, whether row space and null space are linearly independent. So, that if something belongs to null space that means, there is some it does not belong to row space, that is an it is basically, there is a dependency between the row space.

And same for left null space and column space very important thing will come from left null space and column space is b has to lie in column space for a solution of A x is equal to b. If something already lies in left null space, because dimension of left null space and dimension of column space add up to the dimension of the real coordinate space they belong to. So, they can you say that they are complementary to each other they might be an overlap between them.

If there is no overlap they are complementary to each other. So, if A x is equal to bs b does not lie in the left null space, it will lie in the column space and we can solve it. If it lies in the left null space for the part of b, which is lying in the left null space that is not solvable. If I try to have approximate solution of A x is equal to b, the part which is lying in the left null spaces to be subtracted.

So, it is important to look into the question that row space, null space are they linearly independent? Vectors in one vector in row space is linearly independent that any other vector in null space or left null space and column space are they linearly independent. Can we form a basis for the larger real coordinate space; using the basis of row space null space or using the basis of left null space column space? This is a very interesting question, where we arrive and in few next few lectures, I will try to address this question.

Thank you.