Matrix Solvers
Prof. Somnath Roy
Department of Mechanical Engineering
Indian Institute of Technology, Kharagpur

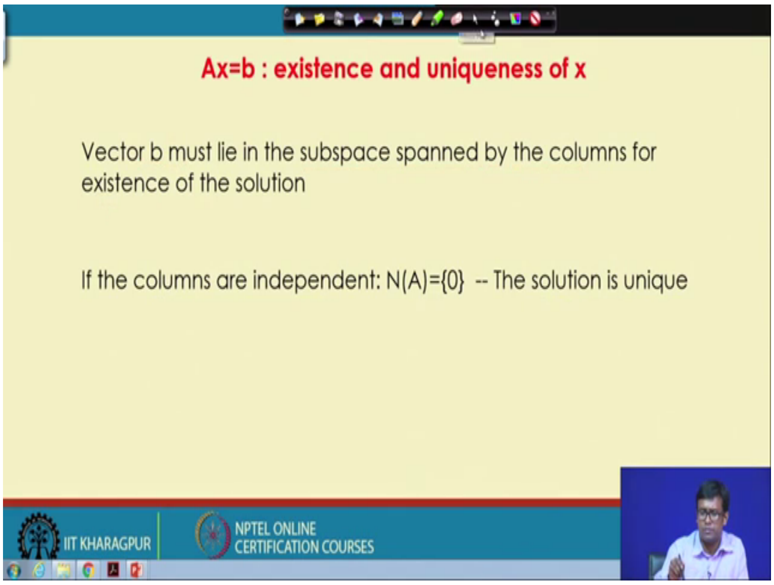
Lecture - 19
Basis and Dimension of a Vector Space

Hello, in the last class, we were discussing about linear independence and spanning; we quickly see an utilization of those ideas in context of solution of A x is equal to b, because, the entire course, we are very focused in understanding how A x is equal to b can be solved for certain cases and at the beginning of this course we have demonstrated, that few direct solvers like Gauss elimination method, Gauss Jordan technique, LUDE composition, TDMA in which we can solve A x is equal to b provided, there is an unique solution of A x is equal to b, a is full rank matrix. Later we came to the case, where is not a full rank matrix and they are infinite or no solution of A x is equal to b, we have seen that A x is equal to 0 will always have a solution, A x is equal to b will have a solution when b lies in the column space of A and we tried to give definition of column space of A.

A x is equal to 0, will always have one solution at least one solution A x is equal to 0 can have multiple solution, if a is not a full rank matrix and we also going through the exercise; where we are finding multiple solutions of A x is equal to 0, we and we told that all these solutions we belong to null space. Now we looked into linear independence and we also looked; how different vector signal space can be combined and we can say that, this is the spanning of this vector span the null space and we get a definition of the null space description of the null space using this spanning vector; which are obtained by getting reduced to a echelon form of the a matrix and looking into the pivots identifying the free variables assigning 1, 2,1 of the free variables and 0 and getting one of the null space vector and.

That way for all free variables will got all null space vectors and using those null space vectors we will allow the null space vectors to span over the space and this space will be a will be defined as null space we have looked into this. So, we also looked into columns space and the column space vectors are the column specters, we which lie in a space; which is spent by the columns of a particular matrix. Column space vector can be different than the column vector, column vectors are given by the particular matrix. Now if we think of a space span by the column vectors, any vector in that space will be a column vector. So, of course, column vector is a column space vector, but there can be other column space vectors then column vector.

(Refer Slide Time: 02:52)



We look into existence and uniqueness of this; of A x is equal to b and solution of solution x, when it will exist and it will be unique. And we come to the definition of spanning and linear independence here. So, vectors b, vector b must lie in the subspace spanned by the columns of columns for the existence of solution. So, if solution b A x is equal to b has to have a solution b, which is existing solution A x which is existing, then b must be spanned by the columns of A. So, b should be we should be able to write b is alpha into first column of A plus, beta into second column of A plus up to and if, A is m into n there will be n columns.

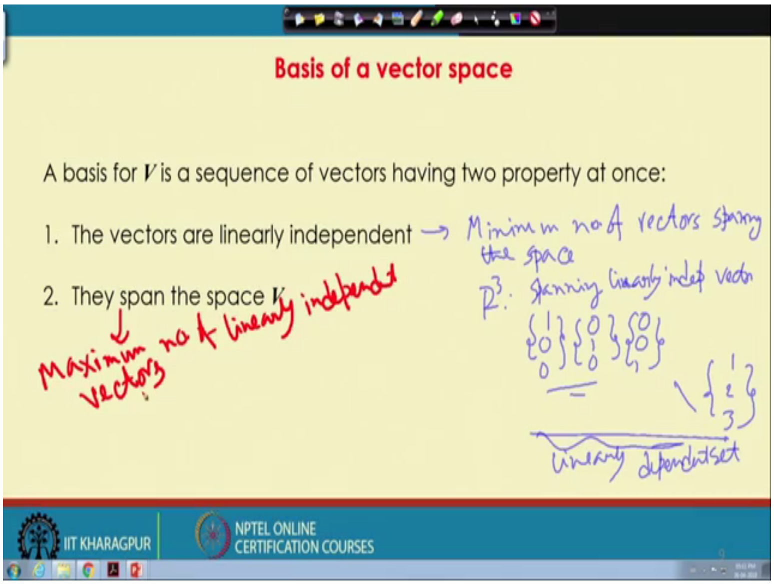
So, eta it will be nu right no eta; eta into nth column of A and each column is basically a 1 1, a 1 2, a 1 n it is basically, a vector containing each term in the row.

So, if I can A x is equal to b solvable, then b must be lying must be in the subspace, which is spanned by the columns of A or b must lie in the column space of a that is not importance of the definition spanning by getting column vectors will define a space which is span by the column vectors and b must lie there for existence of the solution A x is equal to b.

If the columns are independent, then null space of a is equal to 0 and you have seen that, that column is independent means the solution so for example, I have a matrix a 1 1, a 1 2, a 1 3, a 2 1, a 2 2, a 2 3, a 3 1, a 3 2, a 3 3 and x 1, x 2, x 3 is equal to this is the null space equation. If the columns are independent; then we can write x 1 a 1 1, a 2 1, a 3 1 plus x 2 into a 1 2, a 2 2, a 3 2 plus x 3 into a 1 3, a 2 3, a 3 3 is equal to 0 0 0 will imply this will imply that x 1, x 2 x 3 is 0 0 0.

Or null space of A is 0 if the columns are independent then null space of A is 0, but if the null space of A is equal to the solution of unique because, we have again seen that solution can be divided as the null space solution decomposed as null space solution and particular solution the infinite solution possibility of the infinite solution comes from the null space solution, if null space is 0 0 0 or null spacing is the equation is the trivial equation, then there is no infinite solution there is an unique solution. So, independence is linear independence is involved in the definition in the idea of uniqueness of the solution; that is if we have a linearly independent set of columns then the solution is unique, if we have b vector, which is lie in the subspace spanned by the column then the solution exists. So, linear independence and spanning are in that way important in finding out existence and uniqueness of the solutions.

(Refer Slide Time: 07:05)



Now, we come to the definition basis of a vector space, we have earlier seen that there are minimum number of vectors in any real coordinates space are which is needed to span over the entire real coordinate space are n and there can be more vectors, which are spanning over this space, but those vectors will linearly independent. Similarly, if we have that minimum number, but the vectors are linearly dependent then that will not span the space r n. So, this gives the definition of basis of a vector space.

Basis of for vector space V is a sequence of vectors having these two properties once; that they should span the vectors are linearly independent, the vectors will be linearly independent and they should span the entire space V. So, linearly independence means they will be the minimum number of vectors spanning the space.

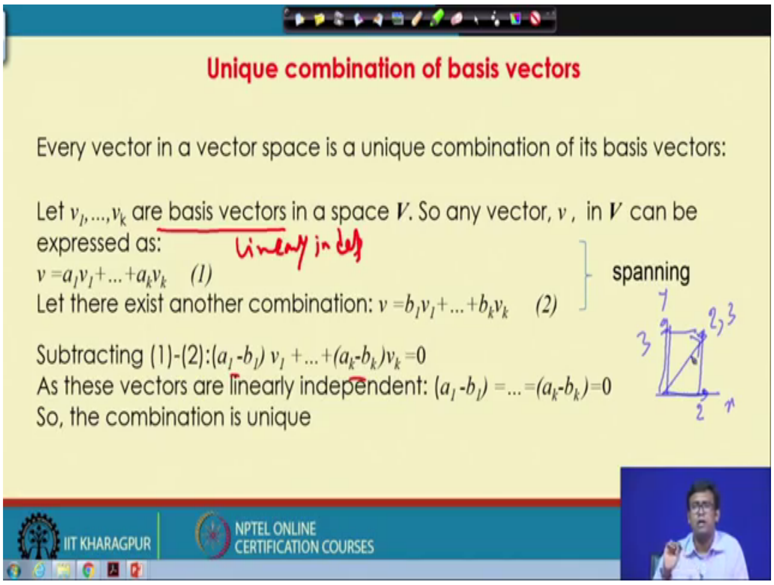
If you add one more vector to that particular set, which is spanning the space that will be linearly dependent with previous vectors, because like for R 3, 3 vectors are 1 0 0, 01 0, 0 0 1, these three vectors are needed to span the entire space. If, I had another vector 1, 2, 3 that will be linearly dependent on these three vectors. So, for example, if we take the case of R 3, we have spanning linearly independent vectors, 1 0 0, 0 1 0, 0 0 1. Now, if I add 1, 2, 3, this entire set will also span the space R 3; however, 1, 2, 3, is linearly dependent on these vectors. So, this will not remain linearly independent, they will be linearly dependent set.

So, minimum number of linearly independent vectors which can span a space is the basis of a vector space. Similarly, if we look into the other criteria, other property that they will span the space and this is the maximum number of linearly independent vectors, which are spanning the space. If we have two linearly independent two linear independent vector in R 3 they will not span the space.

If you have we can have maximum three linearly independent vector in R 3and they will only span the space. So, if we add if we take a space; which is spent the maximum number of linearly independent; if we take a space, which is span by two linearly independent vectors, that will not span the entire vector space unless this is the maximum number of linearly independent vector in that array.

So, never the less the basis of a vector space will be a sequence of vectors a set of vectors which will be linearly independent as well as they will span the space. And therefore, there has to be one particular number of vectors, which will span the entire space and which will be linearly independent.

(Refer Slide Time: 11:14)

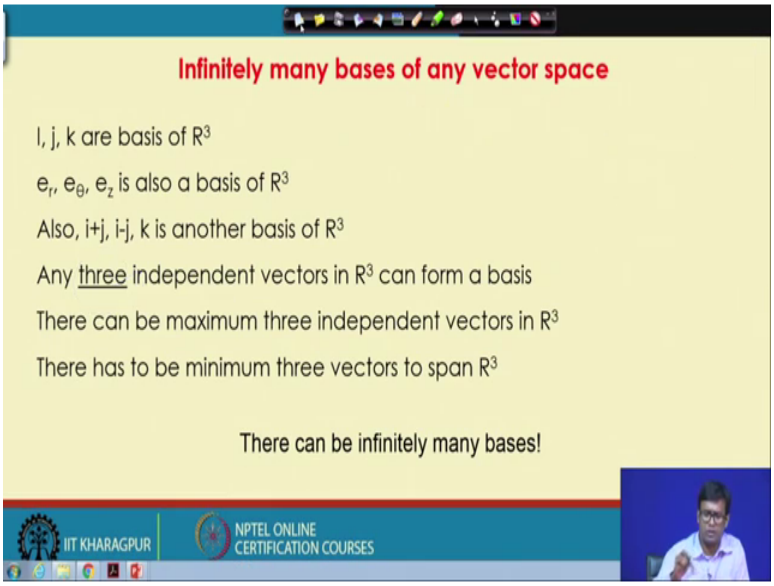


There is another important property of a basis and that any vector in a vector space is obtained by a unique combination of basis vectors. So, let us assume we want V k as basis vector in a space V. So, any vector V, which belongs to this capital V vector space can be expressed as a linear combination of a1 to a1 V 1 to a k V k. Now let there be let us say that this statement is violated, that there is another type of combination of vectors which can express V as b 1 V 1 to b k V k. Now we will subtract 1 minus 2 and this so V 1 to V k is spanning the entire space V. So, any vector V we must be expressed as linear combination of a1 to V a 1 v 1 to a k V k.

Similarly, let us assume there exist another linear combination of the same vectors which can express this vector. So, if I subtract 1 from 2, we get a 1 minus b 1 V 1 to a k minus b k V k is equal to 0. So, V 1toV k are basis vectors therefore, the basis vectors are spanning as well as linearly independent, they linearly independent. If I say any combination of them is 0; that means all the coefficients must also be 0. As these vectors are linearly independent a 1 V 1 to a k V k is equal to 0. So, a 1 is equal to b 1 and a k is equal to b k, this combination is unique and we can think it in terms of the perceivable real coordinates spaces like in R 2 if I have a vector 2, 3, that can only be added by going 2 in the x direction and 3 in the y direction, this only can be obtained.

There is no other combination of x and y through which we can obtain the vector 2, 3 similarly, we can think it in 3 D also. So, that has to be one unique combination of basis vectors, which is giving one particular vector in a vector space.

(Refer Slide Time: 13:39)



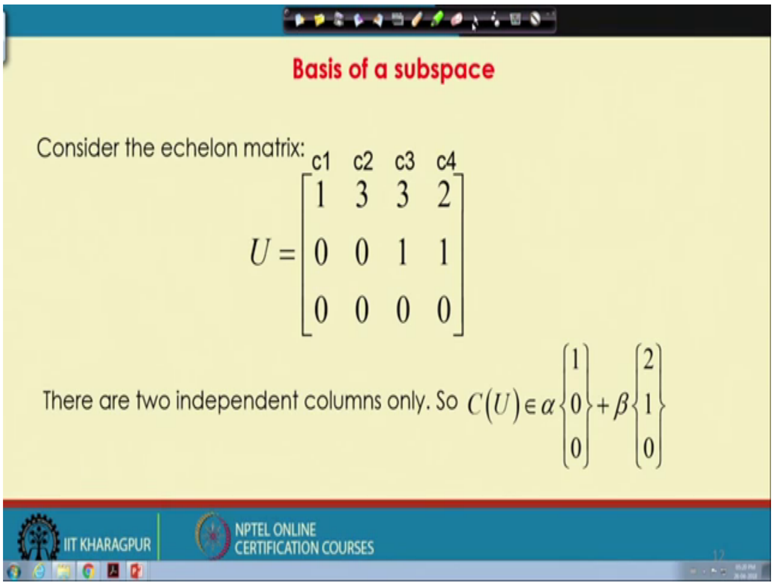
There can be infinitely many bases of any vector space like so bases I will probably clear this idea bases. Bases is plural of basis, if I can form one basis which combines in vector of a vector space R n. In is the real coordinates space R n I can or any vector space for any vector space we I can form one basis, which is n vectors. I can form another basis with n vectors for the same vector space V. And these are very common examples, which we already have encountered in different branches of study for example, i j k vectors one i direction, j direction, and k direction, unique vector in each direction is a basis of R 3. I can explain is any vector in a three dimensional real coordinate space by a i plus b j plus c k.

Also e theta is e r e theta e z. Which are polar coordinate unit vectors are basis of R 3 I can express same vector using polar coordinate. So, I can think of I can express, this is i, this is j, and this is k, I can express any vector by a i plus b j plus c k. Similarly, I can think of the polar coordinate vector space; which is e r e r and then tangential to it e theta, e theta and then e z. So, I can express again this vector by alpha e r plus beta e theta plus gamma e z. So, they are linearly independent separate linearly independent set, which is spanning over the entire set R 3. So, they are also both are basis so there are two basis which you can see.

We can also get further more basis i plus j for example, this one i plus j. So, this is i plus j and maybe this is i minus j and this is k i plus j i minus j k this also can form another basis and so a we can have any number of basis, we can think of basis like this sorry this vector, this vector and this vector, they will any vector in R 3 can be also expressed as linear combination of these vectors. So, any infinite number of basis is possible, infinite bases is possible. Each base each basis it is a basis and infinite combination of vectors is possible or infinite basis or bases are possible for one particular vector space.

However, in R 3 we can see any basis has three independent vectors. So, this number is fixed, that what is the number of vectors in a basis has to be fixed and they have to be linearly independent, then they can span the entire space and they are also linearly independent so they can form a basis. There can be maximum three independent vectors in R 3, seen it earlier there can be minimum three vectors which span R 3 that gives me raise to this number 3, which is needed for which is the needed number of the vectors in a base in an R 3 basis there can be infinitely many basis.

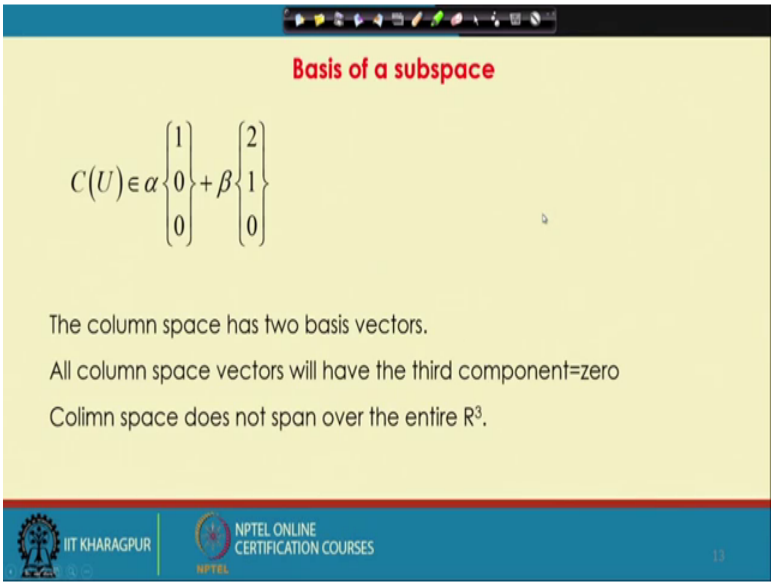
(Refer Slide Time: 18:00)



So, how can we have the basis does the basis of a subspace look like for considered a matrix in the echelon form 1 0 0 1 3 3 2, 0 0 1 1, 0 0 0 0. So, this matrix the column space has like the 3rd column 2nd column is dependent on the 1st column and the 4th column is also in a way dependent on say 3rd column and 1st column 3rd column and 1st column. So, there are two independent columns, which are the 1st column, and 3rd column or 1st column 2nd column or 4th column there can be maximum two independent columns in this vector, though all these columns are members of R 3. So, we can write column space is a linear combination of alpha 1 0 0 and beta 2 1 0 and they are the basis. I can add another vector in because this is the spanning of column space what I am discussing.

I can add another vector I can add gamma into 3 1 0, but 3 1 0 will not be a basis; it will have only two basis. So, what becomes an important idea is that this though all this vectors like C 1, C 2, C 3, C 4, all they belong to real co ordinate space of dimension three the column space, what is the column space look like it is 1 0 0, 2 1 0. So, the entire column space lie here, the column space is a plane column spaces has one dimension less than the 3 D space, though the vectors are member of three dimensional space.

(Refer Slide Time: 19:59)



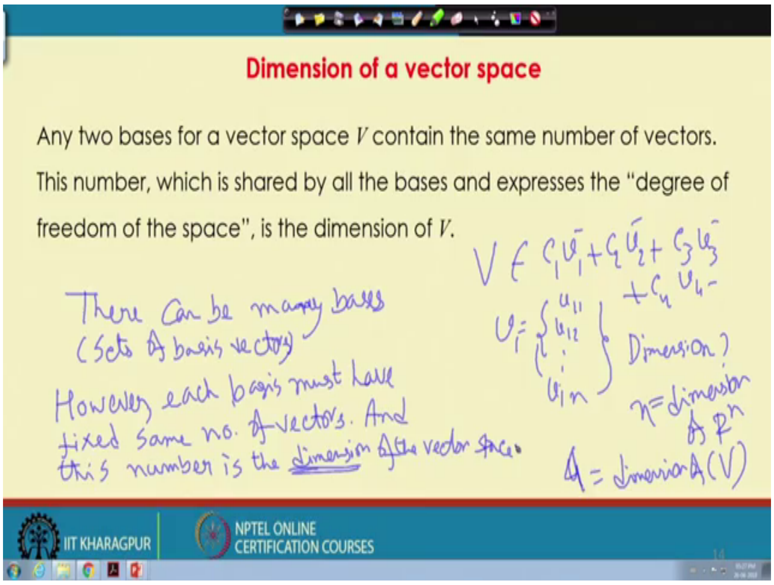
So, the column space has two basis vectors column space vector all column space vectors will have the third component is equal to 0.

So, column space does not span over the entire R 3. If this is R 3 the all the column space vectors this is how the column space look like, column space vectors lie in one particular plane. So, this is a what we will say that column space is a 2 D space in 3 D real coordinate space. The main geometry is the 3 D geometry, but the column space do not span over the entire geometry. Whatever is a column space vector is only restricted to a 2 D geometry.

So, what we are saying the dimension of the column space is not same as the dimension of the real coordinate space and that will give raise to another definition; what is dimension? Dimension of a vector space, when we express our vector space by certain components we say that this vector is a member of a real coordinate space of that dimension.

However, vector space or a subspace is a part of that like here we discussed about the 3 D real coordinates space all vectors are member of a 3 D physical space, but the column space is, all column space vectors are expressed in 3 D coordinate terms. However, the column space are two dimensional space so it has a dimension less than one. And what is important in column space the 3 D R 3 has three basis vectors, but column space found only two basis vectors. So, number of the vectors in one base in the basis will tell us; what is the dimension of the space.

(Refer Slide Time: 22:06)



And we go to the definition, dimension of a vector space; any two bases for a vector space V contain the same number of vectors. We can have many basis vectors, but all bases should all the basis any base there in any basis there should be all the bases rather or in any basis there should be fixed number of vectors.

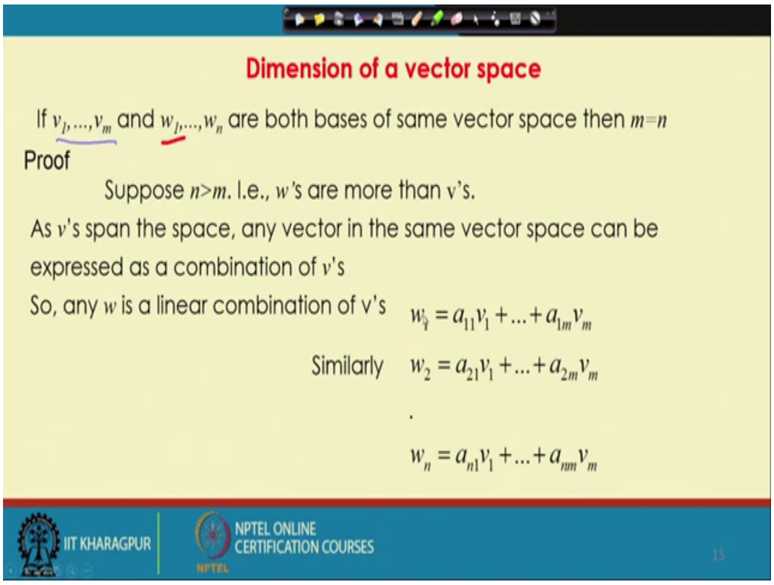
And this number, which is shared by all basses expressed as the degree of freedom of the space, we say it column space is a two dimensional space, any vector in the column space must be within that two dimensional space. It cannot come out of the two dimensional space. So, it defines the degree of freedom of the space and it is called the dimension of the vector space V. So, if we write V belongs to C 1 V 1 plus C 2 V 2 plus C 3 V 3 plus C 4 V 4 and V 1 has or V 1 1, V 1 2,V 1 n. Then dimension then what is the dimension of the vector space V. It is not n, n is the component of vectors, components which is needed to define any vector that. So, n is the dimension n is dimension of real coordinate space R n and 1, 2, 3, 4; 4 is dimension of V.

The number of vectors that is needed independent vectors that it needed to express our span over the entire space V. The number of vectors included in a basis of V is the dimension of V and the number of components in any vector is dimension of the real coordinate space n. They are not necessarily same, there can be cases, when they are same when the vector space or the subspace spans over the entire real coordinate space this dimensions are same, otherwise the dimension is always less than the dimension of the real coordinate space, because we have seen that if real coordinate space has a dimension n, there can be maximum n independent vectors.

So, the number of vectors in a basis can be at most n it can be less than that and what is the number of vectors in the basis that will tell us what is the dimension of the particular vector subspace. And the number of components in the vector will tell me; what is the dimension of the real coordinate space, because we need that many vectors to form a basis of the real coordinate space. So, another important thing is that there can be many bases and bases means; sets of basis vectors.

There can be many sets of basis vectors; each set of basis vector is a set of linearly independent vectors, which can span the entire vector space. However, each basis must have fixed number of fixed same number of vectors and this number is the dimension of the vector space. There can be many bases many sets of basis vectors; however, each basis must have fixed number of vectors and this number is the dimension of the vector space, that what is the number of vectors in one particular basis.

(Refer Slide Time: 27:00)



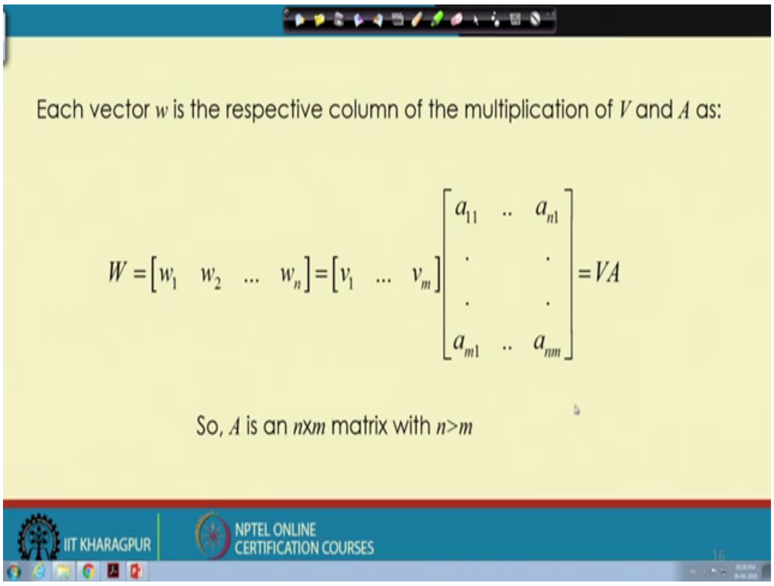
If and we have earlier said, that if there are two bases, we each of this basis will have same number of vectors. So, we think of bases with v 1 to v n, but, there are n basis vectors and another basis with w 1to w n with there are n basis vectors. So, we think that the statement is valid and in matrix algebra, will several time do this what is called second law of thermodynamics type of proofs, that clausius statement is violated and we will see, whether Kelvin, Planck statement holds.

So, we will get one statement violated and we will see whether the proposition still holds violated satisfy some other criteria. So, we will earlier said that all the bases will have same number of vectors, now we are telling that let us consider two basis of same vector space, but they have different number of vectors, but will see that the number will be same. So, if they these are different m is equal to n.

But will try to prove it has m is not is equal to n and then see what is happening. Which is the intuitively, what will happening that linear independence will not be worried, but let us see, suppose n is greater than m ws are more than vs. There are more ws than vs, but they are basis of same vector space, as v's span the space any vector in the same vector space can be expressed as a combination of vs. So, w s and vs are also member of the vector space, basis is also the member of the particular it is lies within the vector space and combination basis will span the entire space. So, v is one basis, all vs are form span the entire space, ws are also in the same space. They are also another basis they are also in the same space. So, any w can be expressed as the linear combination of all the vs.

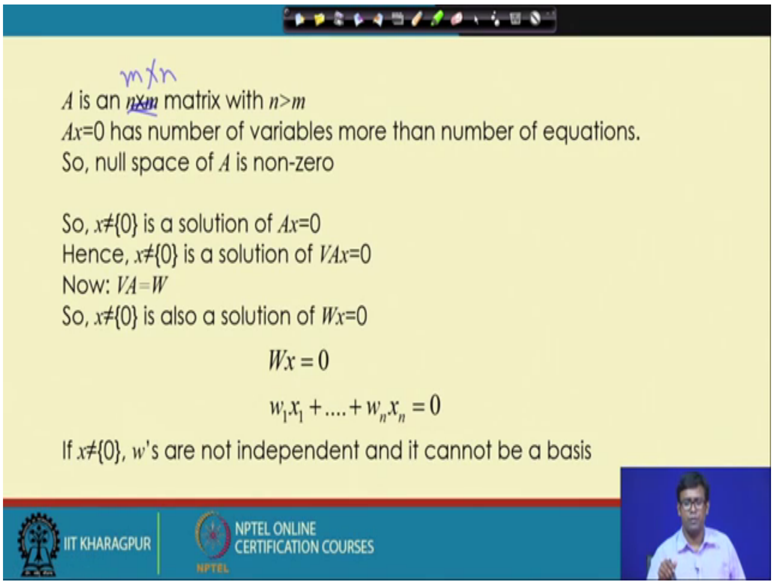
And will see that any w in the is a linear combination vs w is equal to a 1 1 2 v combination of vs similarly, we can write w 2 is linear combination Vs w n is linear combination of vs. So, we can now this w 1 v they are vectors they are there in vectors in R n or R n wherever. So, they have a mod n components in each.

(Refer Slide Time: 29:34)



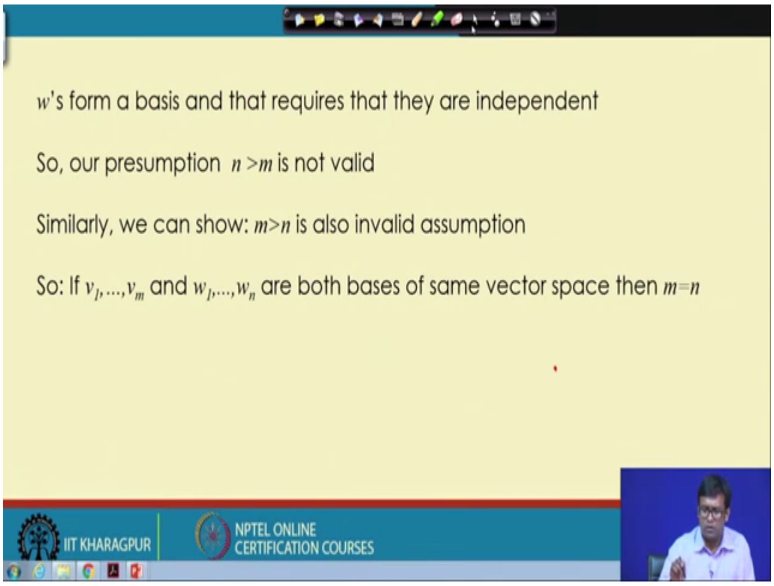
So, we can write, each vector is the as there vector each vector w is respective column of the multiplication of v and w. So, this is a vector multiplied with a another matrix. So, their product is another, this is a all these are vectors. So, product is a matrix where these are each column of the matrix. So, we get w is equal to V A. So, now, this matrix A has n columns, n columns and m rows. A is an n into m, m row and n column. So, A is an sorry m into n matrix m rows and n columns with n greater than m fine. So, you get a rectangular matrix here. Which multiplied with a vector combination of column vector v gives me the combination of column vectors w and v and w are the bases.

(Refer Slide Time: 30:44)



As A is n into m matrix sorry A is m into n matrix, m rows and n columns with n greater than m, there are more variables then the number of equations there are number of variables and more than the number of equations. So, null space is non zero. So, solution of x is non zero, x is non zero as a solution of A x is equal to 0, x is x non zero is a solution of V A x is equal to 0, V A is equal to w. So, x non zero is a solution of w x is equal to 0 and we will get w 1 x 1 to w n x n is equal to 0. And that will tell us the ws are linear not linearly independent and they cannot be a basis.

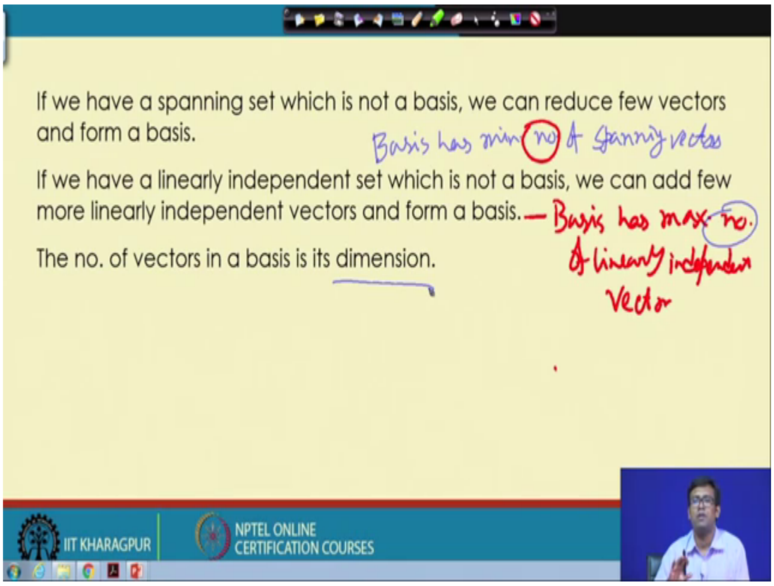
(Refer Slide Time: 31:49)



However, if we start n greater than m we arrived at a point where the; one basis w does not remain linearly independent and they cannot be a basis, so n greater than m is not possible because, ws form a basis and the required that, they are independent so n greater than m is not valid.

Similarly, we can show that m greater than n is not valid then vs will be linearly dependent therefore, m cannot be greater than n cannot be greater than m. So, if the v and w both are bases then the v has a m vectors and w has a n vectors m must be equal to n.

(Refer Slide Time: 32:25)



If we have a spanning set which is not a basis, a spanning set can have many vectors, it is not needed that all have to be linearly independent, then but, there are minimum number of vectors which are linearly independent, which will span the entire set. So, if we have a spanning set which does not have a base which is not a basis, but a spanning says we can reduce few vectors and get a basis.

So, basis has minimum number of spanning vectors similarly, if we have a linearly independent set which is not a basis. That means it does not span over the entire set. So, you have to add few more linearly independent vectors and that will be a basis. So, again this tells us that basis has maximum number of linearly independent vectors. And this number the minimum number of spanning vector, maximum number of linearly independent vector, this number is fixed and this is the number; what is the number of vectors in basis and we call it to be a dimension.

So, number of vectors in a basis is a dimension that is the maximum number of vectors which are linearly independent in that particular space. And this is the minimum number of vectors which can be used to span over the entire space. So, we finished the discussion on linear independence spanning basis and finally, we get what is dimension of a vector space. Now we will use this idea of dimension of a vector space in finding out dimension of column space, null space, row space and other space, will discussion rule left null space and see how they will be important in finding out solution of A x is equal to b in which you are focusing on.

Thanks.