Matrix Solvers
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Lecture - 18
Linear Independence and Spanning of a Subspace

Welcome in last few classes I tried to give you the definitions of vector space, subspace of a vector space what is the real coordinate space. And we also discussed on subspaces associated with matrix A in context of solving Ax is equal to b. We discussed about column space and in context of solving the homogeneous equation Ax is equal to 0; we discussed about null space. What we have seen there is that for example, the column space in case of a A matrix which has 3 columns can be expressed as combination of linear combination of this 3 columns or any vector belonging to the column space can be given as a linear combination of 3 columns.

In case one of these 3 columns is dependent on other 2; any column space vector can be given as a linear combination of first 2 columns, the third column is also linear combination of first 2 columns in that case. We have also seen for null space we find out what are the free variables and what are the pivot variables in reduced row echelon form. And depending on the number of free variables we get null space vectors and we told that the null space is again a linear combination of these null space vectors. And we have seen that in case of 2 null space vector, null space can be thought of a plane or linear combination of 2 vectors in case of one null space vector.

Null space null space can be thought of a line in case there is zero vector in null space null space is only the origin or the only a point. Now, what are the number of vectors or minimum number of vectors which can be utilized to define a particular subspace? In some of the examples, we have seen 1 or 2 null space vectors and needed to define the internal space or all column vectors or few column vectors are needed to define the entire column space. So, how is this number of vector determined and how does it determines the shape of the space? There is one point of interest.

So, in today's class we will start looking to few definitions which will probably answer these questions. And in the next class or next to next class will depending on the phase of the instruction; we will see we will discuss on how this definitions of what are the minimum number of vectors to define a space is relevant for a particular matrix and also for a matrix equation. So, we will come to today's topic which is linear independence basis and dimension in terms of a vector space.

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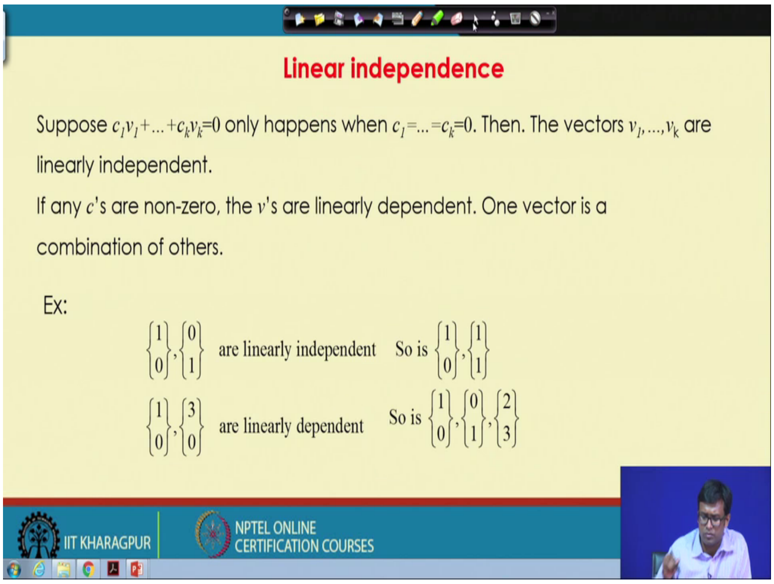

Linear independence is given as suppose there are k vectors v 1 v 2 up to v k. And these vectors are added as a linear combination all any each vector is multiplied with a constant and then they are added and if this addition is 0 c 1 v 1 plus c 2 v 2 plus c k v k if this is 0. And if this happens only when c 1 is equal to c 2 is equal to c k all the coefficients which are multiplying it is 0; then this vectors are called linearly independent; that means, I cannot lie it any vector v i as a linear combination of a 1, v 1 plus a 2, v 2 a k; v k except the ith term.

I cannot write express one particular vector v i as a linear combination of other vectors. And then I say that none of the vectors and resultant of other vectors and this is a linearly independent set. The converse is a linearly dependent set, when? One of these vectors or one or more of the vectors are resultant of few vectors here. And that is given as that if any of the c's are non-zero then this v's are linearly dependent one vector is a combination of other.

So, if all the c's are 0 we cannot take some of this in the right hand side and divided by another cv because everything is 0 and it will give a 0 by 0 form. But if any of this c's are non-zero then we will get a dependent set of vectors and we call them to be a linearly dependent set. What are the examples? One is 1 0 0 1 they are linearly independent if I think of a 2 D space x and y this is say 1, 0 and this is 0, 1; one vector here and another vector here. Obviously, the vector along y is not a result anyway is not related with vector along it is; it cannot be it cannot be expressed in that way; so they are linearly dependent. Similarly if I think of a vector 1, 1 that is also linearly dependent on 1 0 because I; there is an y component and 1 0 of 1; 1 1, 0 does not have an y component; so I cannot combine them.

In when we case they will be linearly dependent, if there is 2 vector they are linearly dependent means one vector is just multiply; multiplied of the other vector. So, if 1 0 and 3 0 are there; so one vector is 1 0 and the other vector is along the same line 3 0 which is just multiplication of that. So, we can write 1 0; 1 into 1 0 plus minus 3 into sorry minus 1 by 3 plus minus 1 by 3 minus of that. So, where is it 1 3 plus minus of 1 by 3 of 3 0 this is; 0 0; so, they are not linearly independent they are linearly dependent.

Similarly, if we think 3 vectors all in are 1 0 0 1 2 3; anything 2 3 2 3 any vector 2 3 can be expressed as a linear combination of 1 1 and 0 1; i plus 2, i plus 3; so this is also linearly independent set. So, in a nut shell if the one of the vector can be expressed as linear combination of any other vectors or at least multiplication of any of the other vectors, then it is a linearly independent then it is a linearly dependent set.

In case we have a zero vector; for example I take this case 1 2 3 4 5 0 and 0 0 0 that is always a linearly independent set. Because I will multiply anything 0 with this plus 0 with this plus any number say 5 with this; this will give me 0 0 0. So, is there is any zero vector 0 is anyway linearly dependent on other vector; I can have any vector and combine them, multiply them with something add them and subtract similar vector from them and get 0.

So, any set of vectors with zero vector is always a linearly independent set because I will put a non-zero number in front of 0 and we will get 0. So, any set of vectors with a or not with in which includes which includes a zero vector is always linearly dependent. So, anywhere I can have a zero vector in the entire set I end up in a linearly dependent set.

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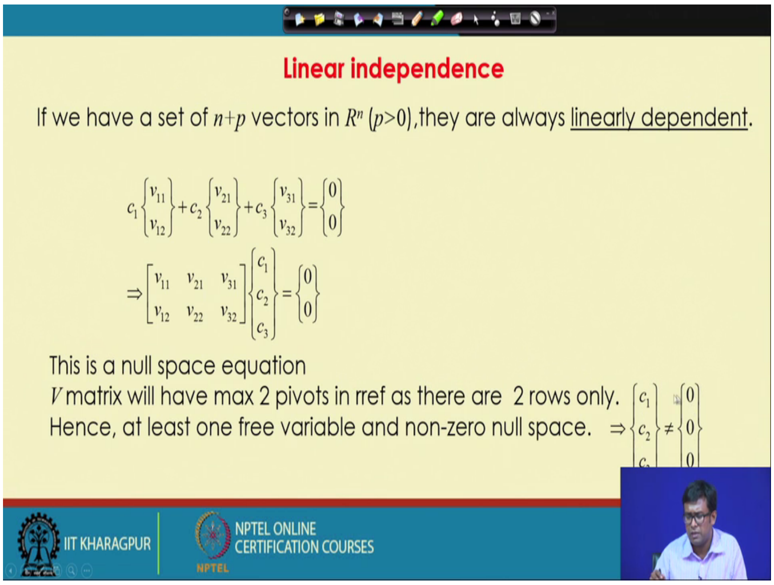

If we have a set of n plus p vectors in a real co ordinate space R n where p is greater than n; so, if we have vectors in R n and we have more than n vectors in R n, they will be always linearly independent. So, we have a case like that we have 3 vectors and of course, the third vector which can be very easily inspect and tell it can be expressed as linear combination of first 2 vectors because all these vectors are belonging to r 2 or all these vectors have 3 components 2 components.

So, if we write the equation that c 1 into first vector plus c 2 into second vector plus c 3 into third vector is 0 0. And when we write it this in the matrix from this is nothing, but a null space equation, this becomes a null space equation of the matrix v. And there are number of columns which is greater than number of rows. So, we will end up in a reduced row echelon form; remember we have done it in the previous class which will be say something like 1 0; some non-zero number here or maybe a 0 number here; then 0 1 and maybe a another non-zero number here into c 1 c 2 c 3.

So, c 3 will become a free variable; these are the pivots free variable and will give me at least 1 in this case no non-zero null space vector. So, we will get a non-zero null space and non-zero null space means c 1 c 2 c 3 is not equal to 0; therefore, this set becomes an linearly becomes a linearly independent set; V matrix will have maximum 2 pivots in rref as there are 2 rows only; hence at least one free variable will be there and there will be non-zero null space.

So, linear independence and linear dependence is related with the null space also. If we can express this vector combine all these vectors and write a matrix and the null space of the matrix becomes 0, then the vectors are linearly dependent. If there is any non-zero null space of that matrix combine which consists combination of all the vectors all the column vectors as the columns of the matrix; then we have a linear non-zero null space will mean that there is the linearly independent set, these vectors are linearly not linearly independent non-zero null space will said that these vectors linearly dependent.

So, it becomes important for us to look into null space vectors. So, hence c 1 c 2 hence at least one of the free variables are ah; there is at least one free variable and one non-zero null space and the c 1 c 2 c 3 becomes non zero.

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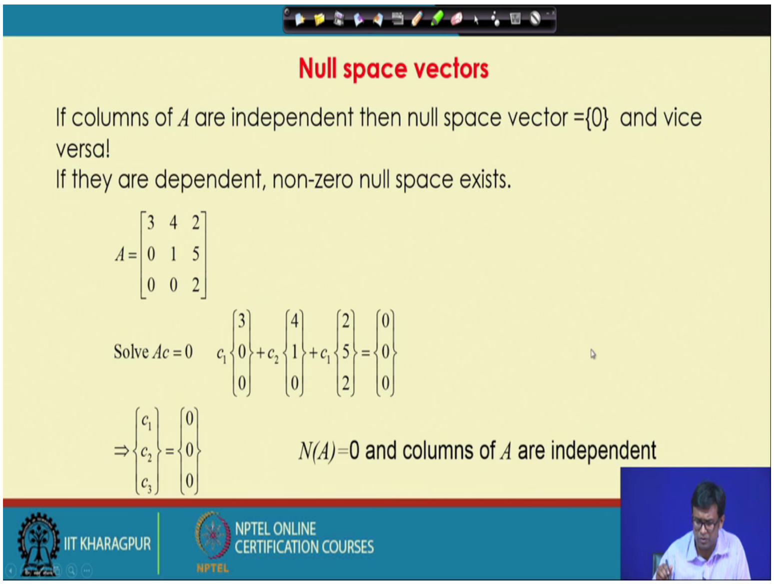

So it becomes important for us to look into small space vector; if columns of A are independent; then null space vector is 0. Columns of A are independent then null space vector is 0, if they are dependent then non-zero null space exist and the vice versa. So, we look into a an example a is 3 0 0 4 1 0 1 5 2 and we try to solve Ac is equal to 0.

And if the null space solution is 0 only there is only one solution which is 0, then we can write the columns as a function of c 1 as c 1 into first column plus c 2 into second column plus c 3 into third column is 0 and c 1 c 2 c 3 will be 0, v is an null space solution. So, depending on the null space we can determine whether the columns are linearly dependent or independent. If we have the null space vector is equal to 0 only one null space vector which is 0 then it is a linearly independent set.

If there is no non-zero at least one non-zero null space vector then it is a linearly independent set. For example, if we take one vector with one matrix with dependent columns 2 1 4 2; the columns are dependent on each other. And write c 1 c 2 is equal to 0 0 and we can clearly see that c 1 is equal to minus c 2 is one answer. So, c 1 c 2 minus c 1 is equal to minus 2 c 1 is equal to c 1; c 1 is 1 and c 2 right. So, c 1 will be c 1 is equal to minus 2 c 2; minus 2 c 2. So, c 1 c 2 is equal to minus 2 and 1, that is one answer c 1 c 2 is equal to minus 6 and 3 that is another answer; however, they are at non-zero c 1 c 2 for which this satisfies therefore, we can say 2 1 and 4 2 are linearly dependent.

So, we go to next slide and see what happens with linear dependence and independence. If the columns are linearly dependent before going to next slide, you can probably discuss here. If the columns are linearly dependent then one column can be expressed as linear combination of other column or when we will try to find out determinant we will find the determinant is 0. So, for linearly dependent columns the matrix will have a 0 determinant and we will see it in the next slide.

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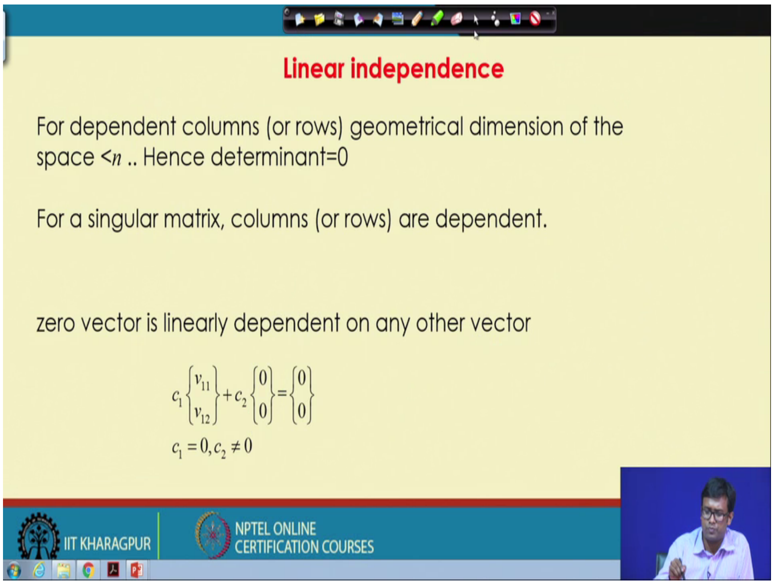

For dependent columns or rows the geometric dimension of the space becomes less than n, where n is the n is the dimension of the real coordinate space.

For example if I have linear and the determinant is volume of the space contained within the geometry. So, we will write that determinant if we recall the definition of determinant; it is volume of the space contained by the vectors. So, if we have 3 independent vectors for example, I have 1 1 0 0 now I 0 1 0 0 0 1, these 3 independent vectors. The volume we can find it out that will be the determinant the volume will give us the determinant which is determinant will be 1.

If 2 vectors are linearly dependent; so the third vector 1 1 vector is linearly dependent on the third vector. So, it will having will same have this 3 dimensional geometry R 3, this is one vector, this is the other vector say this is vector a, this is vector b and this is vector c. So, the space contained by them is a plane and it has volume 0. So, when will remember getting dot product we took volume created by the space; enclosed by the space that is dot product of the area vector with the with one of the vector. So, dot product of this area vector with the vector c which is again in this area will be 0; so, determinant will be 0 and the matrix will be singular.

So, what we can say for a singular matrix the columns or the rows are dependent. And we have seen it in the discussion on determinants also that if we make one column or one row combination of another column; another columns or other rows then we will get a 0 determinant. Zero vector is linearly dependent on any other vector and we have discussed earlier that any set of vectors where I have one zero vector must be a linearly dependent set. Because we have c 1 into 1 vector plus c 2 into another vector is 0 and we make c 2 non-zero makes c 1 0 again we get a 0 0 combination.

So, 0 0 is all if we have this vector 0 0; we will always end up v 1 1, v 1 2; 0 0 they are linearly. So, you get any set and also if I have a 0 0 here and we will look into the determinants that I have in x y coordinate; x y which is R 2; I have one vector say v 1 1, v 1 2 and I have another vector which is origin 0 0. So, what is the area contained by this? Area will be 0 because it is a vector ended point, it does not we cannot in compass in area by drawing a parallelogram with these 2 vectors; area will be 0 that is determinant is 0 therefore, this will be 0 is also linearly dependent vector.

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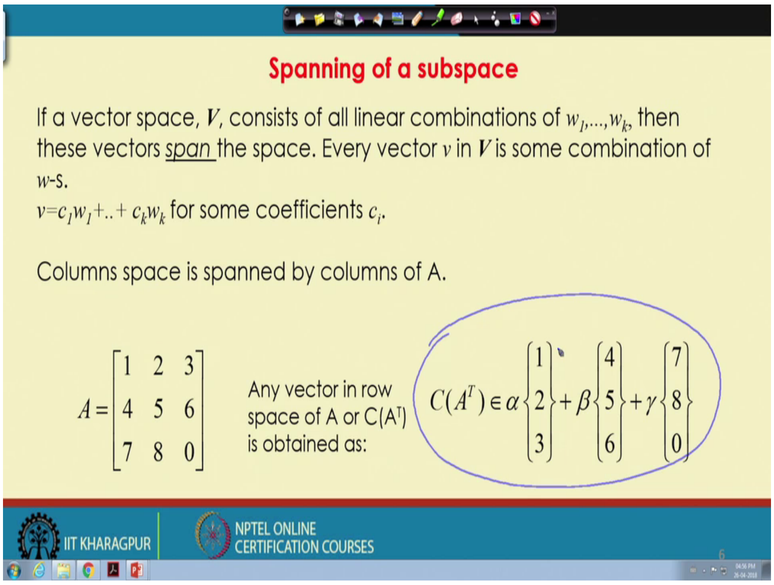

Now, we come to another definition what is spanning and that is an important definition regarding column space and null space. Because when we are showing the form of any vector in column space or any vector in null space, we are writing it as the any null space vector is combination of 2 null space vector that came out using 2 free variables in the first example we did that. So, we say that these 2 null space vectors which came out using the 2 free variables; span over the entire null space. If you know these 2 vectors, we can linearly combine them and then we can get the get any vector in the null space entire null space.

So, if a vector space V, consists of all linear combinations of w 1 to w k then these vectors span the space. Every vector in v vector v in the space capital V is some linear combination of w-s. So, we can write any vector v we can write any vector v which is belonging to capital V as the linear combination of w-s and then we will set the w span the entire space, the set of w vectors. That is if we can write v is equal to c 1 w 1 plus c 2 c k w k for some of the coefficients c i and then we will say that v is capital V is space which is spanned by w provided small v exists in capital V.

So, can all the cs be 0 yes; zero vector is very much part of a any vector space. So, if you look into capital V vector space; zero vector is a part of that. So, we will put all c 1 2 c k everything 0 and we will get zero vector which is also part of that. So, c can have any value any c can have any real value and we will get one particular vector for one particular set of real values of c, we will get one particular vector in v and that is why with infinite values of the combinations c 1 to c k; we can get infinite vectors in v and we will get a vector space.

So, except one important thing is that except the zero vector if consider zero vector to be a 0 dimensional vector space except; that except the zero vector any vector space also must have infinite vectors, that is one important definition out of it. Because it will be spanned by some vectors spanning means those vectors which are spanning the space will linearly combined, we will get one vector in the vector space. And this linearly com linear combination can be done in infinite ways assuming any arbitrary value of c 1 to c k. So, you should get infinite vectors in the space V.

Column now we look into the column space; column space is spanned by columns of A. For example, we have the vector matrix A 1 4 7 2 5 6 and 3 6 0 are the column vectors; any vector in sorry yeah any vector in rows space this is given in terms of rows space, but in rows space is also formed. So, you can also write row space is spent by rows of; so, we look into the row space row space of A, which is linear combination of the rows of A. So, if we make any arbitrary choice of alpha, beta and gamma you should get one vector which is lying on the row space.

So, when you looked into the null space this is important that spanning of null space; vectors found using free variables, span the null space. So, we took one or the free variables 1 and remaining 0 we get one vector we take another free variable to be 1 and remaining 0; we get another null space vector. And so the that way we get all the dependent if there are n free vectors; n free variables will get in null space vectors and these n vectors span the entire null space. So, the number of free variables is equal to number of vectors spanning the null space.

And if we use; we have to find out all the free variables and vectors associated with obtained using all the free variables. This vectors the number of 3 variables will determine the number of these vectors and they will say to be spanning the null space. And that just also justify as lighting a vector space using this expression because if we know the vectors which are spanning the space; now we know about all the vectors, location of all the vectors in the real coordinate space which will be the orientation all location of the particular vector space.

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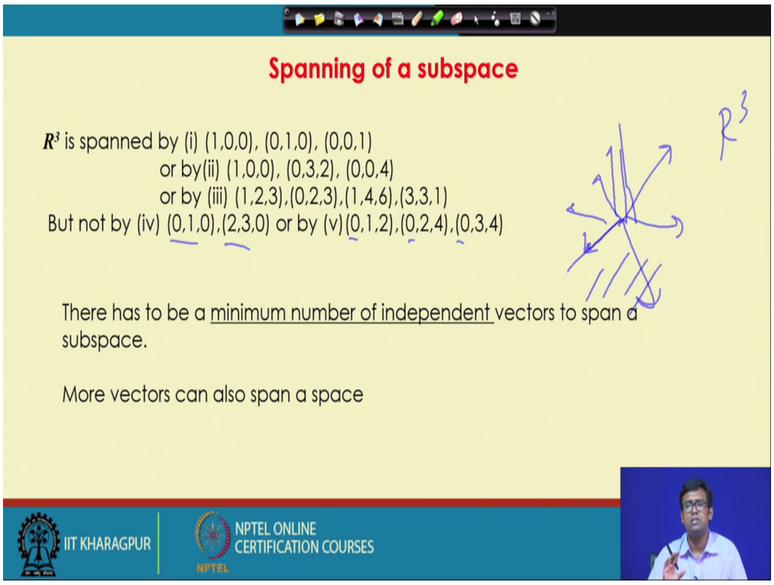

R 3 is spanned by vectors 1, 0, 0; 0, 1, 0 and 0, 0, 1 R 3 can be also spanned by vectors 1, 0, 0, 0, 3, 2, 0, 0, 0, 4 or even by 4 vectors 1, 2, 3, 0, 2, 3, 1, 4, 6, 3, 3, 1. In any case, we can come linearly combine vectors using any arbitrary constants and get anyway tell belonging to real coordinate space R 3. However, so this is like I can use this axis and linearly combine them to get any vector here, I can also use the axis like 1, 0, 0 this is fine 0, 3, 2; so, some vector here.

And 0, 0, 4 some vector here at this 3 and again can form this vector by linearly combine these vector. I can access this and add one more vector and again linearly combine this will be a redundant vector, but still I can have the linear combination and find any vector in r three; however, there has to be minimum 3 vectors to define any vector in R 3 to give the spanning of R 3. In case we have 0, 1, 0 or 2, 3, 0 will be able to; so we have same R 3 with the vector 0, 1, 0 which is say here and 2, 3, 0; I can only define any vector belonging to this particular plane, I cannot obtain a vector by linearly vector like this which is has a component in the z direction by combining these 2 vectors; so, they do not span R 3.

Similarly 0, 1, 2, 0, 2, 4, 0, 3, 4 though they are 3 vectors; all these vectors have a 0 in the x direction. So, all this vectors lie in this plane they do not span R 3. So, what do you need? We have to have 3 vectors to span R 3; minimum 3 vectors we can have more vectors, but these 3 vectors like here this will they are linearly dependent. These 3 vectors cannot be linearly dependent, so there has to be a minimum number of independent vectors to span a subspace. If we have less than that will many vectors we cannot span the subspace, we can have same number of vectors, but they are not linearly independent then also you cannot span the subspace. More vectors can also span a space think linearly independent vector can span R 3, but for vectors they also they are also spanning R 3. But the minimum number is there which is determined by the number of independent vectors which can span the entire space.

So, we will continue with the discussion of spanning and see that linearly independent and spanning when both are satisfied; we get a new set of vectors which is called basis for a vector space; we will discuss about this in the next class.

Thank you.