Matrix Solvers
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Lecture – 17
Solving Ax=b When A is Singular

Hello in the last class we are looking into solution of Ax is equal to 0, where A is a singular matrix. And we started with the A matrix and got it reduced row echelon form and we observed that if I think of going through typical Gauss elimination that will fail because we are getting 0 pivots.

Also we took a case where number of equations are less than number of unknowns; therefore, you should have infinite solution that that is confirmed for the equation system we have taken.

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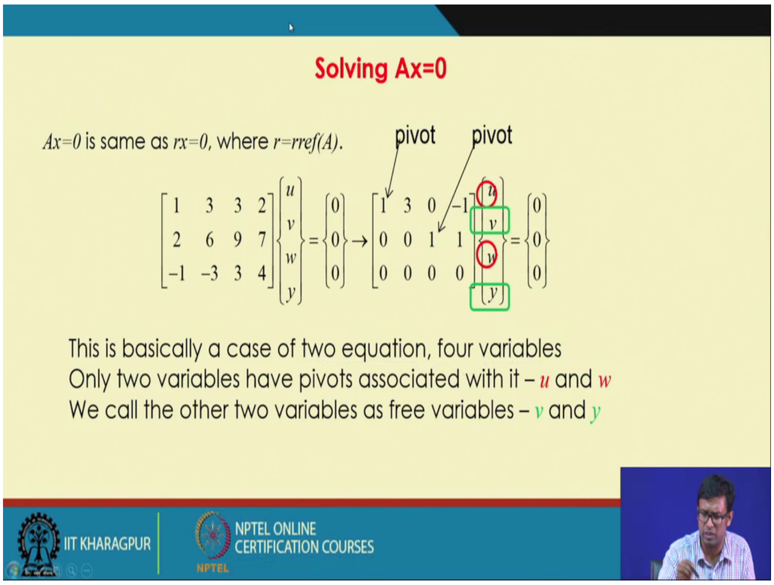

And we started with Ax is equal to 0 and through some matrix operation we ended up in the reduced row echelon form rx is equal to 0. So, this was Ax is equal to 0 which is converted to rx is equal to 0.

Observe that because this is the right hand side vector is a 0 vector. So, through matrix operation it did not change it is same only the A matrix has been converted to r transformed to r matrix through this matrix operation. Now if we look into the rx is equal to 0 form; what we can observe? That there are pivots which are one we will call them as pivots. And before the pivots there are rows where it should be a pivot, but this is 0; so these rows has no pivot, this similarly this row has no pivot.

The first row first column term is 1 pivot in the second row the second column is does not give me does not give me a pivot. So, second column has no pivot the third column gives me a pivot. So, third column has a pivot and again fourth column, column does not have any pivot in; even in third row it is; everything is 0; so, there is no third pivot there are only 2 pivots. It is very important to find out the null space solutions. So, we move forward and identify the pivots 1 and 1 these are the pivots and this pivots are basically the third one; if I see the third one is 0 is equal to 0 which is not a equation. So, there are basically 2 equation and 4 variables.

Only two variables have associated pivot with it because this pivot will be multiplied with u and this pivot will be multiplied with u and this pivot will be multiplied with w. So, v and y does not have any pivot to multiply with them and we call; so, these are the pivots u and v they are the pivoted variables. We will call the 2 other variables as free variables v and y; they if I try to look into this equation like the third; second equation w has a pivot term and the second equation is w plus y is equal to 0, w has is a pivot term. So, we will try to find we can find out w what is w? W is minus y why we do not have any value?

We will assume some value of w and for each value of some value of y and for each value of y we will get one value of w. So, y is a free variable now similarly v is a free variable now; we will assume values of v and y and we will start getting different values of u and w and that is the idea of getting infinite solutions. So, to identify the pivots the pivots are the ones in the first one in any row in the reduced row echelon form matrix and with which term the pivot is being multiplied and that we will call as the pivot variables u and w.

And we will see the terms which are not the components of solution vector, the terms in the solution vector we cannot multiply it with any pivot and we will call them as free variables. Then this is important in a sense I basically have 2 equation and 4 variables. So, I will allow these 2 free variables to move freely, I can give in any value on the free variable and I will get values of the pivoted variables.

So, and that is how we will say we will; we will get a space where there are 2 degree of freedom v and y can be anything; however, u and w will be dependent on v and y. Therefore, with these two variables, we can define the entire null space and that is what we will follow in the next few steps.

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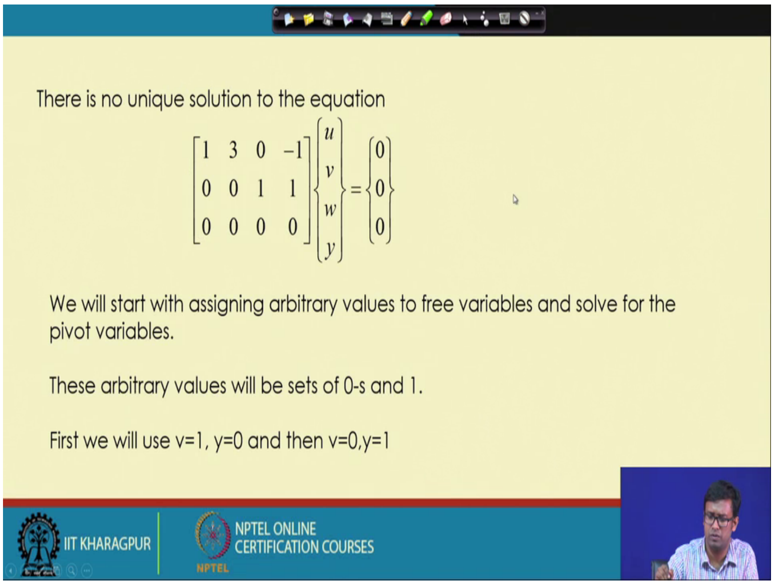

There is no unique solution to the equation because there are 2 equation and 4 variables. We will start as with assigning arbitrary values to the free variables and then we will solve for the pivot variables. We know that there is no unique solution there are infinite solutions; however, this infinite solution are not arbitrary solutions; they have certain nature certain nature.

For example, if you remember last class we have looked into a problem where the null space is a line passing through 0 0 0 origin and 1 1 minus 1; so, any point on this line is a solution. So, infinite solution does not means any solution, infinite solutions means some solutions which has a particular functional form or particular vector space, subspace they are forming.

And this subspace is formed by the pivoted variables as they follow their relation with the free variables. So, we will assign arbitrary value with the free variable; so infinite values can be assigned with the free variable. And how the pivoted variables will be varying that will tell me how is how does the space look like. These arbitrary values will (Refer Time: 06:13) sets of 0 and 1; in a sense we will make one free variable 0 or if there are multiple free variables, we will make more all the free variables 0 except 1 and this variable we will assign a value 1 and solve this equation.

So, if I can put v equal to 0 and y is equal to 1; u and w are unknown there 2 equations. So, there is 2 v and 0 y is known to us, there are 2 equation 2 variable; so, we can solve for u and w. And this solution will be dependent on the value of y is equal to 1. So, we will multiply this solution with any value of y and get a different solution here. Similarly we will take y is equal to 0 and now make v is equal to 1 and get one another set of solution or another null space vector.

If there are multiple vectors like u v w x z something like that, then we will make v is equal to 1 and y and x is equal to 0 and get one set of solution, v is equal to 0, y is equal to 1, x is equal to 0; get another set of solution; v and y is equal to 0, x is equal to 1 and get the third set of solution.

So, we will start making one that is we will put this arbitrary value as sets of 0 and 1; that is one free variable will be set as 1, the others 0. And repeat this for all free variables that set one of them 1 and other 0. Now set another one and remaining other 0 and repeat it for all free variables and get different null space solutions. So, the next first we will use v is equal to 1 and y is equal to 0 and then we will use v is equal to 0 and y is equal to 1 and get 2 null space solution.

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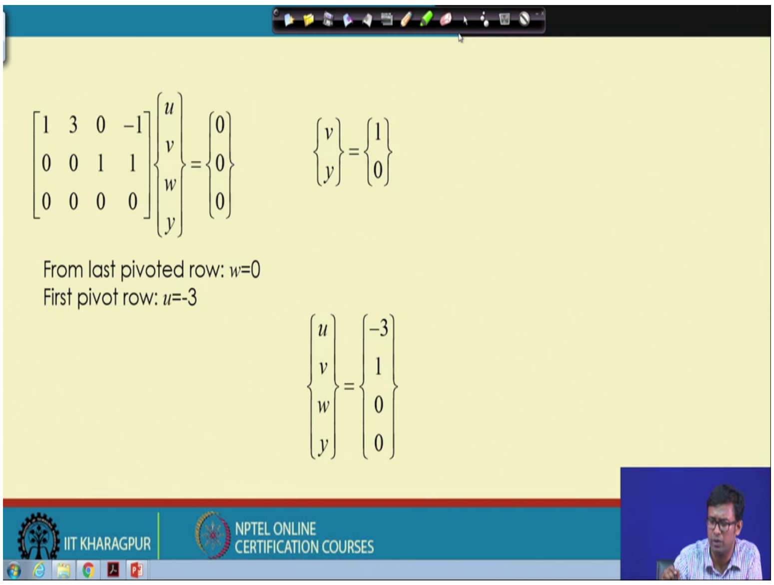

So, we will put v is equal to 1 and y is equal to 0 and the equations will become 1 plus 3 sorry; u, the equation will be u plus 3 into 0 because v is equal to 0 plus 0 into w plus minus 1 into again y is equal to 0 here.

So, basically u is equal to v is equal to sorry I am sorry I am sorry v is equal to 1. So, v is equal to 1; so, this is 1 u plus 3 v minus 0 into w minus 1 into 0; so u is equal to minus 3. And the next equation will be v plus w is equal to 0 and v is put as sorry v plus in next equation is sorry w plus y is equal to 0 and y is 0 here; so, w is equal to 0.

So, we will get one null space solution as w is equal to 0 and u is equal to minus 3. So, u v w y is minus 3, v is 1, w is equal to 0 and y is equal to 0. Similarly we can find out the other null space solution by making this as 0 and 1; v is equal to 0 and y is equal to 1.

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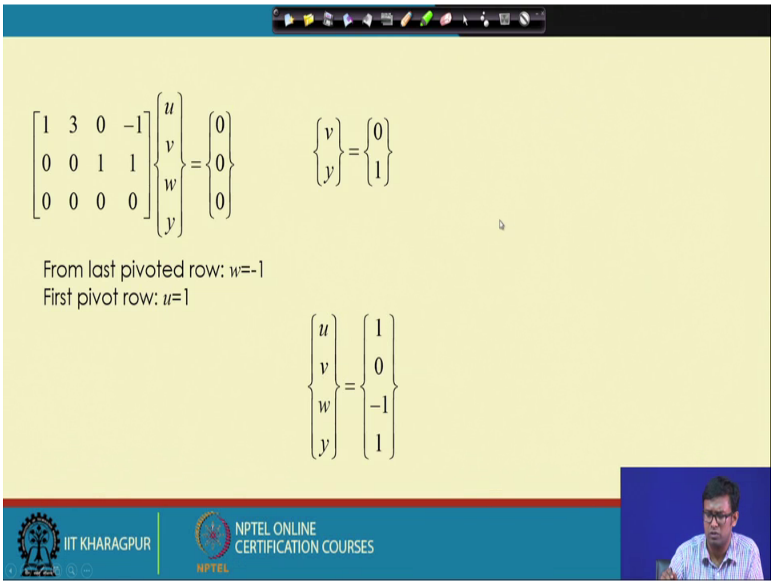

So, another solution is we will put v is equal to 0 and y is equal to 1. So, the first equation will become u plus 3 into 0 plus 0 into w plus minus 1 into 1 is equal to 0 which is u is equal to 1 and w plus y is equal to 0, but is w is equal to minus y is equal to minus 1, the solution will be w is equal to minus 1 and u is equal to 1.

And actually we can go as a back substitution step we will find out last pivot for pivot row first solve for the last pivoted row first and then substitute and get the first pivoted row very very similar like that. So, this will be the another null space solution.

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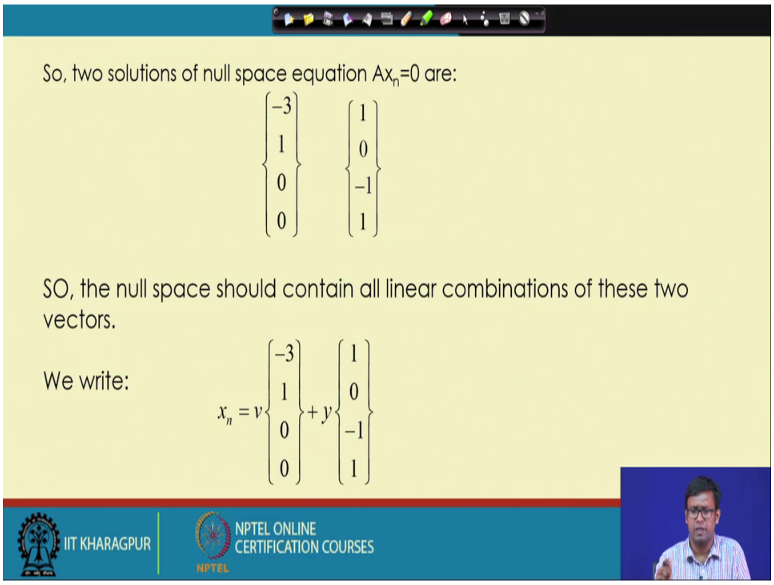

So, we finally got 2 null space solutions; 2 solutions of null space equation minus 3 1 0 0 1 0 minus 1 0. Now this solutions are obtained the first solution is obtained for v is equal to 1 and the second solution is obtained for y is equal to 1; here y is equal to 0 v is equal to 0.

Instead of v is equal to 1 if I had put v is equal to 2; I would put minus got the solution minus 6 2; you can confirm it. I will put y is equal to 3 I would have got the solution 3 0 minus 3 and 3. So, this solutions scale with this solutions scale with the value of u and v and y. So, we can assume any value of v and can get any all different solution, we can assume any value of y and can get a different solution.

So, the finally, we can write that the solution space contain all linear combinations of these 2 vectors. And therefore, the null space can be expressed as we will write x n is equal to v minus 3 1 0 0 plus 1 0 minus 1 1; this is the final null space solution. So, if we summarize the steps how through which we find out null space.

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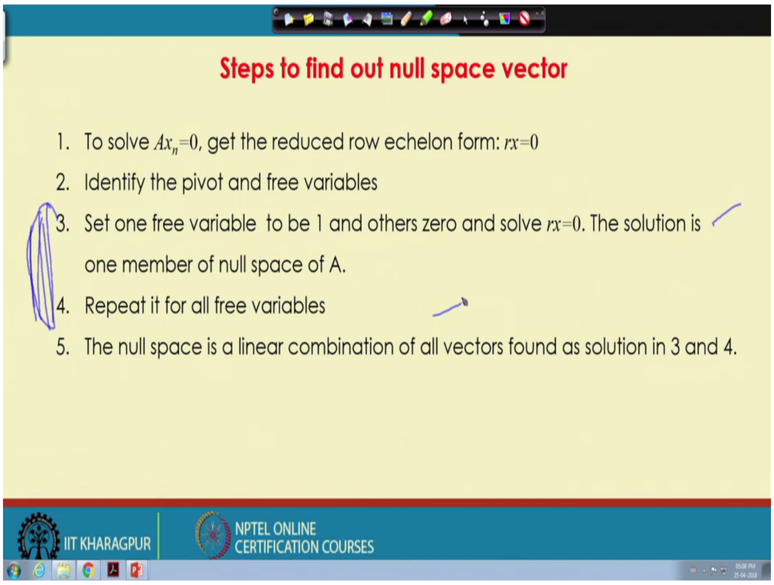

We have to solve Ax n is equal to 0, we will get the reduced row form r x is equal to 0. Identify the pivot and free variables, set one free variable to be 1 and others 0 and solve r x is equal to 0. So, anyhow we identified the free variables and set the value for the free variables. Now I have exactly the unknowns which are equal to the number of pivots and the pivots is a each pivot is implying one particular equation. So, number of equation is equal to number of unknown I can directly solve it.

So, for one particular set of free variable 1 and setting other 0; we can get one member of null space and then we will repeat it for all free variables. We will set the other free variable 1 and remaining all 0, get another member of null space so on. Finally, the null space is a linear combination of the vectors which are found through the solutions by setting one free variable 1 and other 0 through the processes 3 and 4; through this processes. So, this is a very standard technique; the reduced row echelon form can be found out following something like a Gauss Jordan step and then identify free and pivot variables.

So, the number of free variables was I will have, I will get that many members of null space. And then this members will identify through this particular step; then the final null space is a linear combination of all the members. So, any other vector which is a linear combination of all the vectors found through step 3 and step 4 can be my final null space solution. So, how many vectors or how many free variables we will identify that will tell me how many null space vectors I can find through the step 3 and 4; whose linear combination will give me the final subspace through which my null space solutions should belong.

Though they are infinite solutions there are infinite solutions all of them belong to one particular subspace. And to which substance do they belong? That will be identified by the vector sum finding out by solution of the null space equation following the steps 3 and 4 setting one free variable to be 1 and other 0. So number of free variables are giving me the number of vectors, whose linear combination will be null space to which my solution will exist. So, it will be important to see what are the number of free variables.

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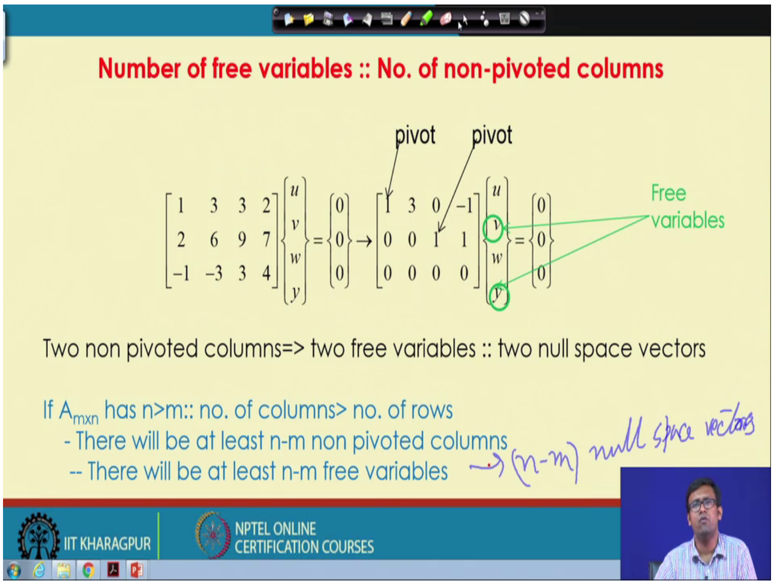

So, number of free variables is equal to number of non pivoted columns; we have seen in the previous case that if there are 2 pivots in a 4 variable system, if the number of column is 4; n is equal to 4 if there is 2 v for the free variable is n minus 2. And this free variables will give me the non pivoted columns will give me the free variables. And these free variables will give me different solutions of null space which will be linearly combined finally, to get the final expression for the null space.

Then 2 non pivoted columns imply that 2 non free variables and 2 null space vectors we will find out. For each free variable we can find out one null space if the free variable is 1 and rest are 0; find one null space vector it repeats. If A m into n has n greater than m; number of columns greater than number of rows.

There will be at least in n minus m non pivoted columns because each row will have a pivot. So, maximum number of pivot is maximum number of row which is m; m is the number of row. So, there will be n minus m non pivoted columns, there can be less than that. Like here there are 3 rows, but there is only 2 pivots because one row is linearly dependent on other; ones we do follow these steps we get one row to be 0 because this is this row is can be express as linear combination of these 2 rows.

So, in this case we get only 2 pivots, but if this row is different not linear combination of these 2 rows we would have got maximum 3 pivots. So, where the maximum number of non pivoted column; the minimum number of non pivoted column is n minus m that should be there. So, there will be at least n minus m free variables and this will give me n minus m null space vectors. And this n minus m vectors will be then linearly combined to get the final null space.

Now we will try to see how we can utilize this to solve as is equal to b when; A is a similar non singular matrix. So, when we are solving Ax is equal to b; I assume that we have solved Ax is equal to 0 and found out null space. Null space has infinite solution depending on non pivoted rows which are which will give us the free variables, depending on the number of free variables we got the number of null space vectors which can be linearly combined in infinite way to get a null space in which all the infinite solutions of Ax is equal to 0 exist.

And x is equal to 0 is also a member of that, but there are more solutions at x is equal to 0. Now we are solving the equation Ax is equal to b 1 part is the homogeneous part which is A xn is equal to 0. Another part is a particular solution where Ax p is equal to b and we will try to find out one solution of Ax p is equal to b and this one solution when combined with all the null space solution. So, then finally, give me infinite solution of Ax is equal to b.

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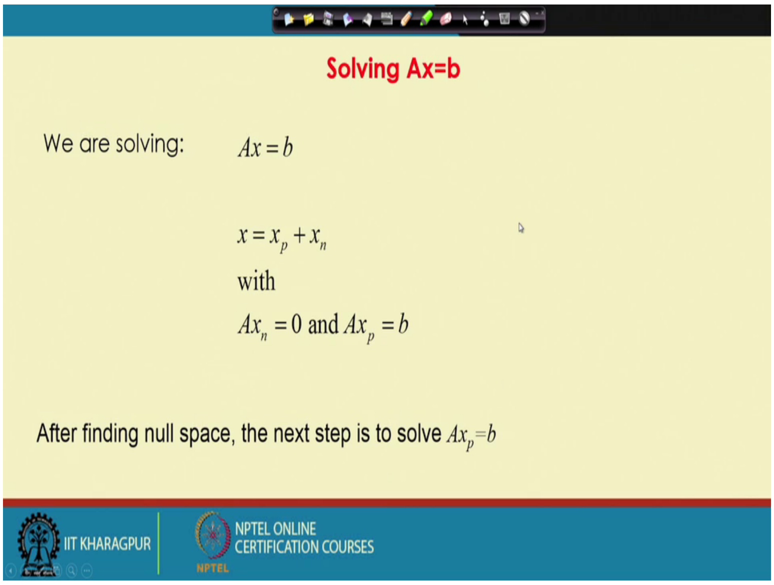

So, Ax is equal to b; x we decompose it as x p and x n x p is the sorry particular solution and x n is a null space vector.

So, to get Ax n; we will solve Ax n is equal to 0, to get x p we have to solve Ax p is equal to b. Now we have to solve only one particular solutions, we can make certain assumption we will probably remove the free variables where free variables are anyway represented in the null space, we can get infinite values of free variable solutions for the null space. So, we remove the free variables and solve for the pivoted variables only get one particular solution all free variables are 0.

So, it is; it is possible to have the null space there all free variables are 0; what then what is the solution of Ax is equal to b? That is one particular solution where it does not depend on free variables, it is I will say it is independent of null space and then we will add it with null space and get the infinite solution.

So, we will follow the steps after finding null space the next step we will be solving Ax is equal to b.

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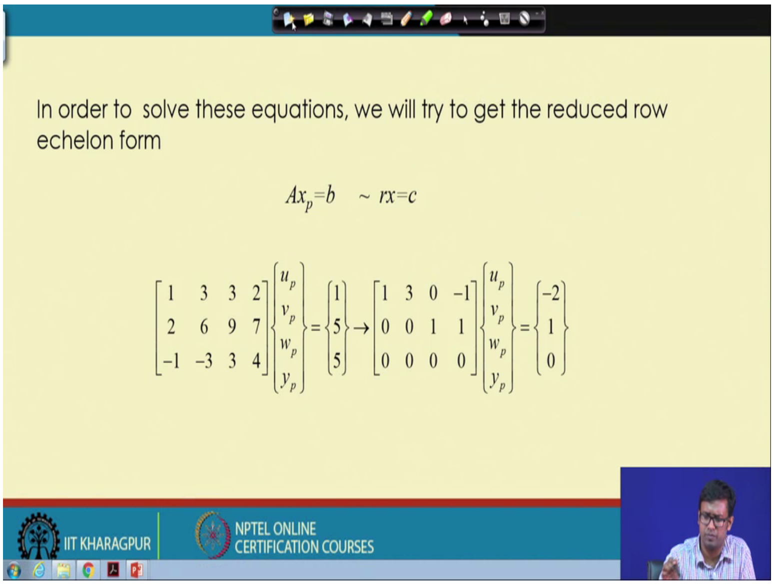

In order to solve the solve the equation we will try to get the reduced row echelon form. And Ax p equal to b will be rx is equal to c; remember b will changed to some other column c; some other vector c because we it will go through the operations to which A will change; as A will change to transform to reduced row echelon form r, b will transform to a another matrix c because column row operations will be done on A as well as on b.

Earlier when we are solving Ax n is equal to 0 the right hand side is 0. So, even after row operation it was it was always remaining 0; as this is not 0 after row operation it will change to some other vector c fine. So, and this happens like if we have a equation 1 3 2 2 6 9 7 minus 1 3 3 4 with the particular solution 1 5 5 which is the b actually. If we transform the matrix to reduce row echelon form the right hand side vector changes to minus 2 1 0.

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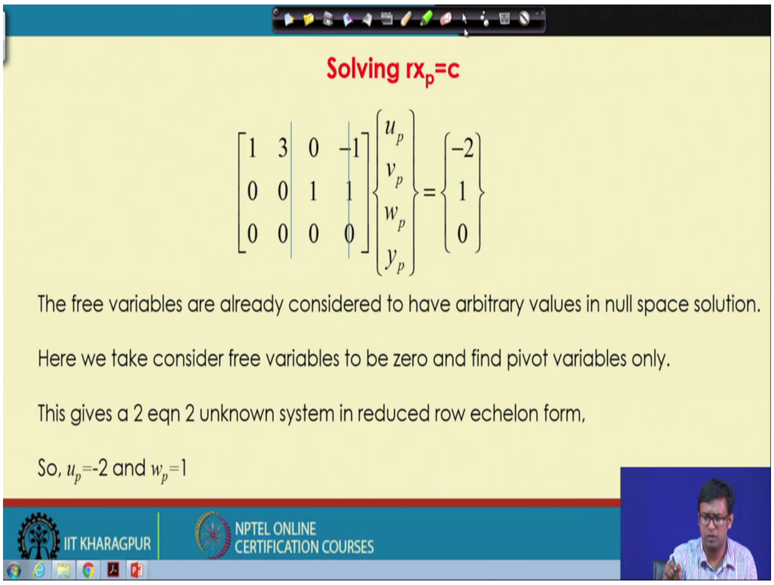

So, we have to solve rx p is equal to c; the free variables are already considered in to have arbitrary values in null space solution. Why? Because null space is written as if I go back to the expression for the null space vector, it is written as x n is equal to v into minus 3 1 0 0 plus y into 1 0 minus 1 0 minus 1 sorry minus 1 1 1.

So,; so this v and y are already included in null space, we are trying to find out x is equal to x n plus x p and this has infinite solutions and infinite solutions of v and y can be included in x n. So, x in finding x p we can remove v and y, we can remove v and y in finding x p or rather I should not remove v from here; this is fine this is v and this is y. I can remove v and y from this equation and I can remove the columns with which they are multiplied because they are already there in null space.

So, particular solution with v is equal to 0, y is equal to 0 these 2 columns are not multiplied. And this is a 2 equation 2 unknown system, which can be solved that is the idea. So; however, we take we consider free variables to be 0 and find pivot variables only. And this is take is not here because we will consider free variables to be 0 and find the pivot variables only. Because this is a particular solution we only need to see that whether the solution satisfies the equation rx p is equal to c or Ax p is equal to b.

And with any assumption if I can get a solution which satisfies it we are done we will add it with null space and say that these are the final set of infinite solution. So, we will make the free variables to be 0 and only considered the pivot variables. So, free variables are not considered the column associated with free variables are not considered, they are equated free variables are equated with 0.

So, we left with the 2 equation 2 unknown system which is the last equation is also of no use. So, we can delete the last equation; it is basically 0 is equal to 0. So, we have 1 0; 0 1 u p; w p is equal minus 2 1 and as this is a reduced row echelon form the truncated equation is basically an identity matrix into u p v p is equal to minus 2. And we can write that the solution is u p is equal to minus 2, w p and this comes with v p is equal to 0 and y p is equal to 0; this is one particular solution. And so, substitute this in any of rows row equations we will get the equation to be satisfied; so, this is one particular solution.

And this there is a 2 equation 2 unknown system in reduced row echelon form and we will get u p is equal to minus 2 and w p is equal to 1.

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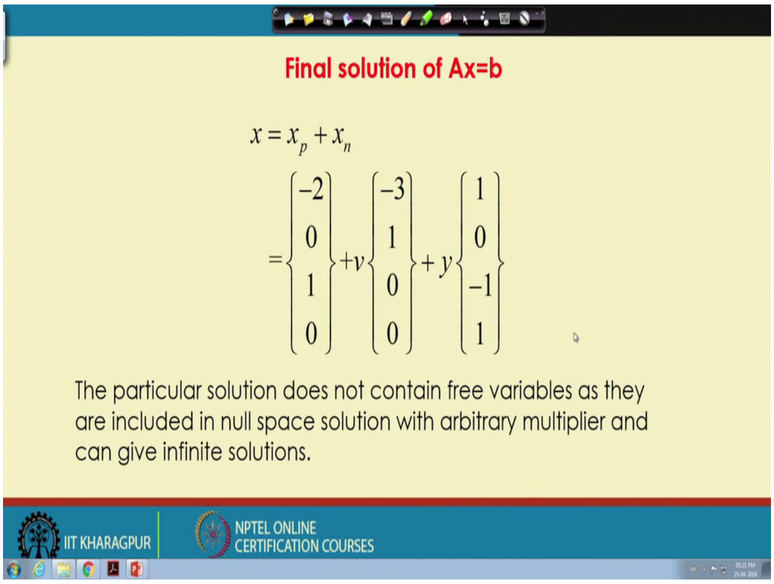

And the final solution will be the particular solution plus the null space solution x p plus x n. So, minus 2 0 1 0 plus v into minus 3 1 0 0 plus 1 0 minus 1 1 will be the final solution. The particular solution does not contain any free variable as they are included in null space solution with arbitrary multiplier; we entire solution depends on different values of v and y. So, we do not need in particular equation in we need to find out v and y, we will get any value of v and y and can get a solution which is the solution of the actual equation also.

And this v and y through which are basically the multiplier for the null space vectors will drive my infinite; the nature of my infinite solution I will get different v and y and get the solution. So, finally we arrived into a solution which shows how the infinite solutions of Ax is equal to b; when is a rectangular matrix with 0 pivots is singular matrix or a rectangular matrix can be solved and Ax is equal to b is solvable; even we find out the null space.

The null space contain the free variables the particular solution is found neglecting the free variables because free variables are already in will be active in null space in determining the different infinite solutions. This infinite solution is added to one particular solution and we will again get infinite solutions of x ok.

So, now what we will quickly go through is few examples how this technique can be applied for some other problems.

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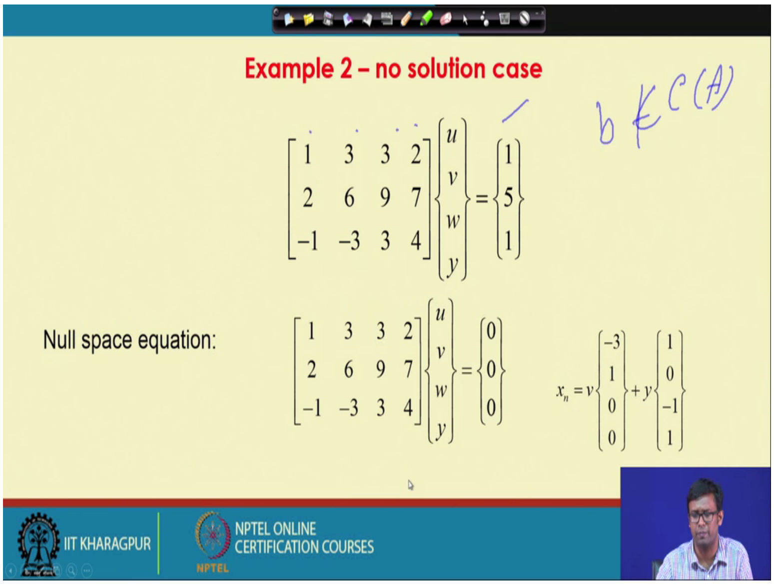

Example 2, we will see a case where there is no solution; no solution in a sense this equation has been formed in a way that this column 1 5 1 cannot be found out by combining any of c 1 c 2 c 3 c 4; any of the column or b does not belong to column space of A; therefore, there should not be any solution.

But we will follow the same steps; let us try to find out the null space equation we, this is the null space equation. Null space we will always have a solution; this the same equation the earlier null space is the same is the same, A is the same only b vector change a is the same already A that we have solved in the case. So, null space will have same equation, null space will always have a solution. Now when we will try to see particular solution we will see that there is no particular because b is not a member of column space.

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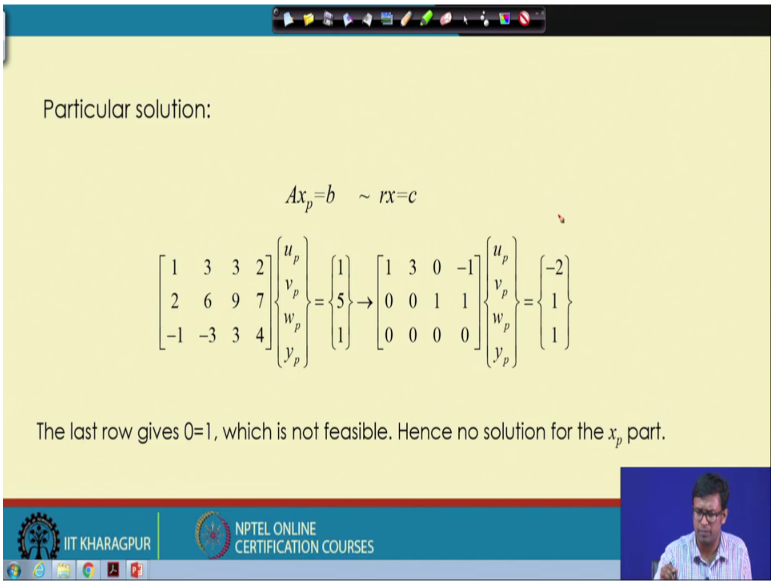

The particular solution is rx is equal to c which is the right hand side is minus 2 1 1 and this will give me an equation 0 is equal to 1. This equation is basically the last row will basically give me a equation 0 is equal to 1 which is not possible.

If we see the last example the in the last row this term was 0. So, 0 is equal to 0 was an equation of no use we discarded it, here equation is actually not of no use 0 equal to 0 has to be there in last (Refer Time: 28:49); right hand side is left hand side is 0, right hand side has to be 0; as the right side is 1, this is a absurd system we cannot solve this equations. So, we will infer that the last row gives 0 is equal to 1 which is not feasible, hence no solution for the x p part. So, though x n has a solution there is no solution of x p therefore, x is equal to finally, we have x is equal to x p plus x n.

Now, this x n has no solution therefore, x will also have no solution, there is no possible feasible value of x though x n exists. And that is the importance of particular solution; it will only tell us whether it has no solution or not. If in particular solution we get an absurd equation we will say that the solution is not feasible; we will look into another example before finishing this class.

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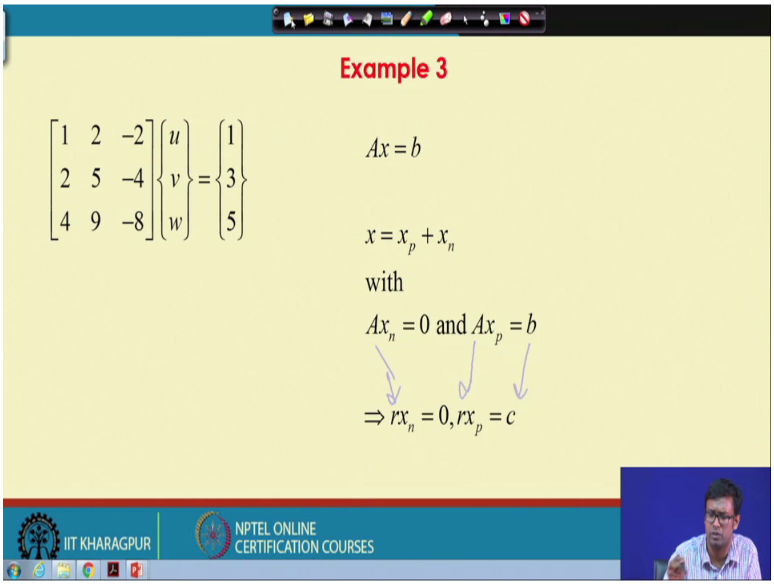

This is a 3 by 3 equation system and let us go through it. So, these equations though this is a 3 by 3 equation system, there are should be infinite solutions because the third row is formed as twice of first row plus second row is equal to third row.

So, Ax is equal to b we have x is equal to x p plus x n; null space Ax n is equal to 0 Ax p is equal to b; in reduced row form we will get rx n is equal to 0 and rx p is equal to c. And we will try to solve the reduced row echelon form rx n is equal to 0 and then we will also solve a particular solution here. So, this is the equation we will let us do one thing; we first get the rx p is equal to 0 form. So, the same r will be utilized for getting rx n is equal to 0. So, only idea is you get A to r that can be done in both sides, here I will get transform A to r here; I will I am also transforming A to r b is being transform to c

So, we first transform Ax p is equal to b to rx p is equal to c and then we solve rx n is equal to 0 also which is the null space equation.

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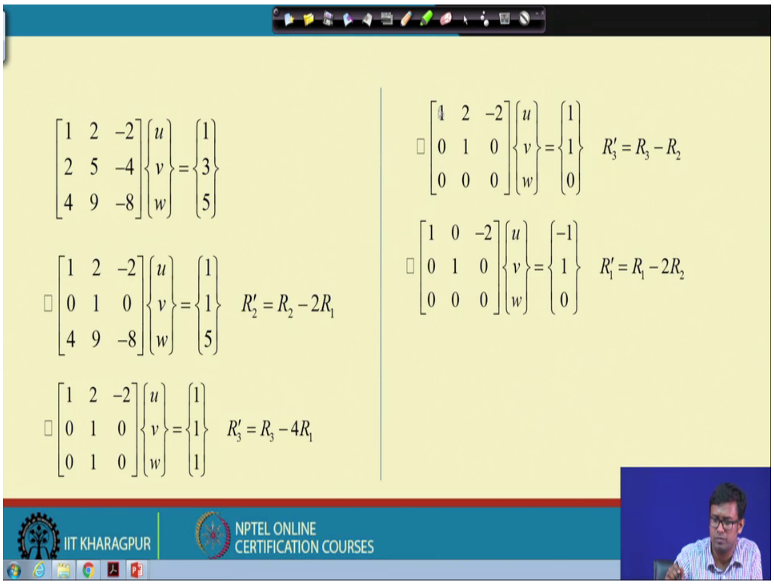

So, this is the Ax is equal to b rather; Ax p is equal to b equation, we subtract second first row into 2 from second row to make this term 0. And we do the 4 times of first row from third row to get this term is equal to 0.

Then we subtract second row from first row this entire row becomes 0. So, this will have a solution because we will get 0 is equal to 0 here; b should lie in the column space of A. So, and then we get try to get the reduced row echelon form of the matrix; this is the reduced row sorry, this is rx p is equal to c. This is the reduced row echelon form of A; r is equal to r of A reduced row echelon form.

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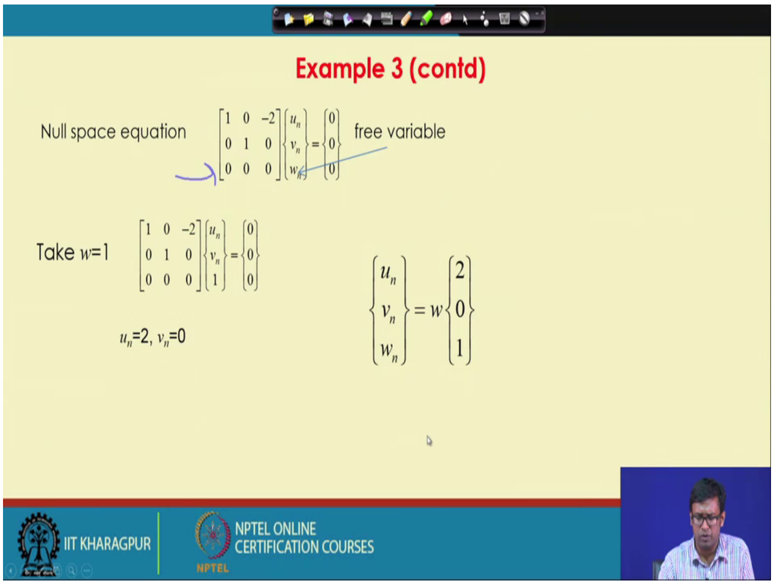

So, this is my null space equation this is though this is a 3 by 3 equation system; the matrix is not a full rank matrix. So, one row has been convert transformed to entire 0 and there is 0 pivot there is no pivot 0 (Refer Time: 32:25) at the pivot.

So, therefore, w will be my free variable also because there is no pivot in the third row, the third variable becomes free variable. So, we will take w is equal to 1 and there is only one free variables; only one null space vector and we will get u n is equal to 2, v n is equal to 0. So, null space is u n, v n, w n 1 0 0; 2 0 1 sorry.

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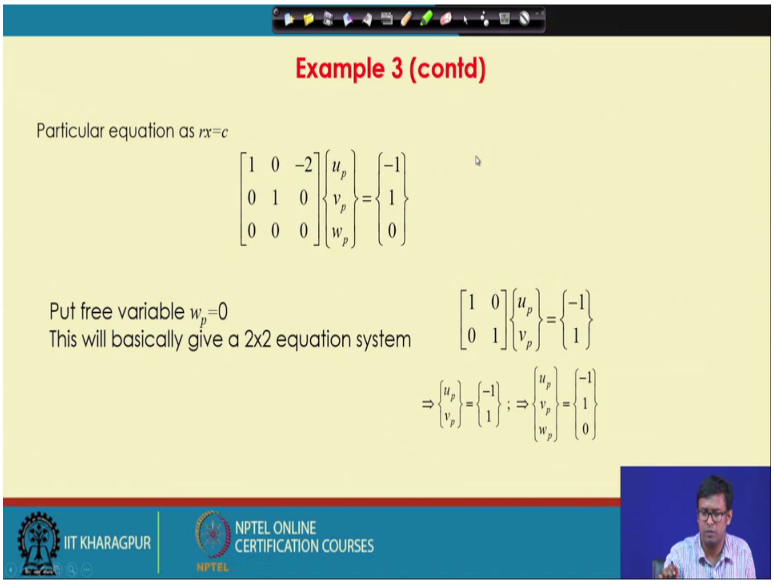

And we now try to find out the rx is equal to c particular integral form; we will discard w, put w equal to 0. So, we will discard w; so with discarding w the third column w is basically multiplied with the third column. So, once we write w is equal to 0; we can neglect the third column also. So, what we will get? A 2 by 2 equation system 1 1 0 0 1; u p, v p is equal to minus 1 1; and we can directly solve u p is equal to minus 1; w v p is equal to 1 and w p is equal to 0; so this is my final x p.

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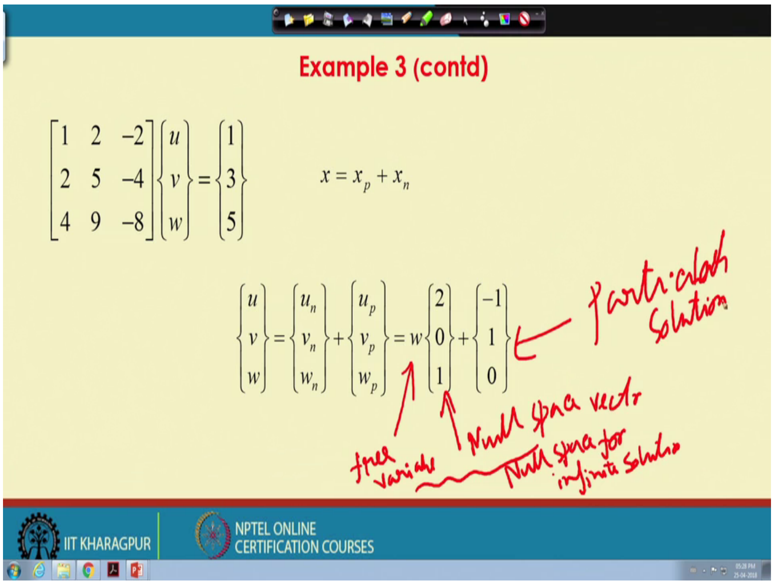

And final solution will be x is equal to x p plus x n. So, u v w is u n v n w n plus u p v p w p is w into 2 0 1 which is my null space vector. I am sorry which is the null space vector and this is multiplied with the free variable. So, they in total gives; give me the null space for infinite solutions and this become the particular solution which is added with the null space.

So, that in total finally, gives me the solution; find out null space now identify the free variable and the free variable we have to identify it while finding null space. Make the free variable to be 0 for your particular solution get an identity matrix into pivot variables is equal to the left hand side vector.

And get the particular solution and add them with null space; null space is the null space vector multiplied by a coefficient which is basically the free variable. And can give you the entire space spanning over the null space, add it with particular solutions we will get infinite null space solutions plus one particular solution which in turns give you infinite solutions for this particular Ax is equal to b, where A is a singular or a rectangular matrix.

So, in next class we will try to look into few definitions like what is spanning of a space or what should be basis vectors we; through which we can more precisely define the behavior of null space. And the properties of that or properties of infinite solutions as a vector subspace, we will look into basis dimension linear independence spanning these 4 definitions in the next class.

Thanks.