Matrix Solvers

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Lecture -16
Finding Null Space of a Matrix

Welcome. In the last class we have discussed about vector space and vector sub space, we tried to discuss about the definitions and looked into few examples and we also looked into column space and null space. This two spaces are associated with matrix A and solution of matrix Ax is equal to 0 and we observed that these are also vector sub spaces; sub spaces in the sense they follow the rules of vector space within a larger vector space. For column space, it has the larger vector space has a dimension r m where r is the real coordinate space and sorry r n and for null space the dimension is n where dimension of the larger space is n or real coordinate space is n.

So, today we will start looking into the use of the ideas of vector space in solving equations Ax is equal to b, in cases where Ax is equal to b does not have a unique solution. In case they have unique solution, we can use Gauss elimination or l u decomposition or Gauss Jordon the standard method to solve the equation; however, in cases they have infinite solution this methods cannot be utilized

So, in today's class we will see few examples through which we can solve cases where Ax is equal to b can be solved where x as multiple solutions. We will start with finding null space of such matrix A which is A singular matrix so, that Ax is equal to 0 has multiple solutions, we will discuss about this things.

So, the topic for today's class is finding null space of A and solving Ax is equal to b when A is not invertible or A is a singular matrix; so, that we cannot use gauss elimination type of method.

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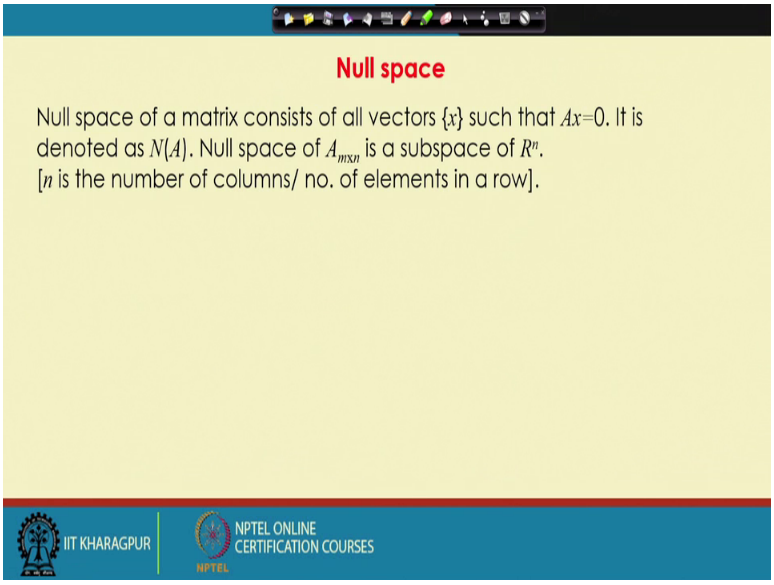

The definition of null space, we tried to give it in the last class, null space of a matrix consists of all vectors x such that Ax is equal to 0. It is denoted as N of A.

So, if I have an equation Ax is equal to 0, if it is not a singular matrix; there is a trivial solution x is equal to 0. In case it is a singular matrix that will infinite x is possible. So, all this solutions before a singular matrix only x is equal to 0 consist the null space for a null singular matrix. For a singular matrix all the solution which satisfy Ax is equal to 0, we will consist the null space. And null space of a matrix a which has a order m into n is a sub space of real coordinate space of dimension n or R n.

So, if n is a the number of column or number of elements in the row, that is the dimension of the real coordinate space R to which null space is a sub space.

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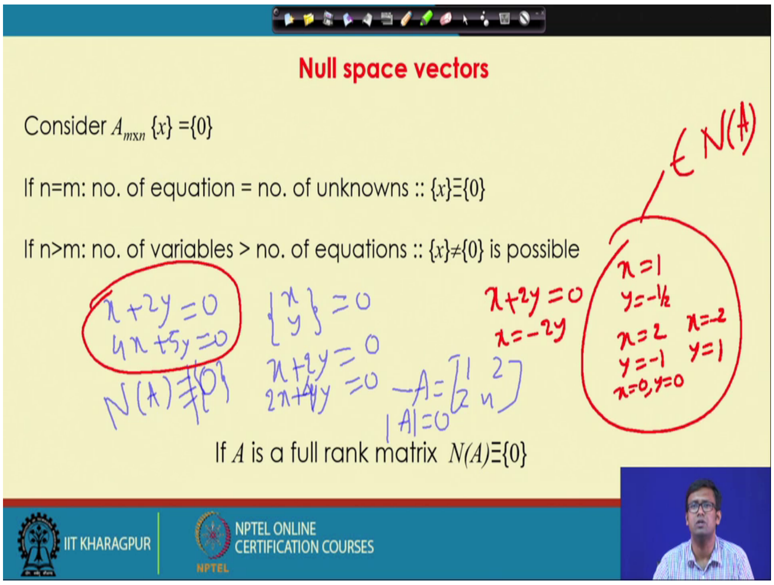

So, we consider the equation Ax is equal to 0; A again is the matrix of order m into n. If n is equal m number of equation is equal to number of unknown and we can write x is equal to 0. We can look into a simple example like x plus 2 y is equal to 0 and may be 4 x plus 5 y is equal to 0 and the only solution possible is x is equal to 0 y is equal to 0, but if we take another example where number of variables is greater than number of equation.

So, I we like only one equation x plus 2 y is equal to 0, then the equation is basically x is equal to minus 2 y. So, all this solution x is equal to 1 y is equal to minus f x is equal to 1 y is equal to minus 1, x is equal to minus 2 y is equal to 1; all of them are solutions to this equation.

So, all this vectors 1 minus 2 2 minus 1 minus 2 and an of course, x is equal to 0 y is equal to 0 is also solution. All this consists what is they belongs to, what is call the null space.

Now, if number of equation is equal to number of unknown null space is usually 0 only that the equation should not be repeating. So, we can have another case like x plus 2 y is equal to 0. The 6 second equation is 2 x plus 4 y is equal to 0.

So, then if I look into the matrix A, this is 1 2, 2 4 and this is the singular matrix determinant of a is equal to 0. It will also have a null space; so, which is not only 0 which has vectors more than the 0 vector, but if the equations are like this that there are two independent equations, then the null space will be only 0. If A is a singular matrix will have null space which is 0 plus something, but if A is a full rank matrix; null space will be only 0.

So, again our intention is to find out null space for singular matrix is where it is 0 plus something and we will see 0 is a member of null space, what are the other members. So, that we can say how the other solutions of nulls of x is equal to 0 or behaving, but how does it look like; if I looking to all solution of x is equal to 0, in case a is a singular matrix.

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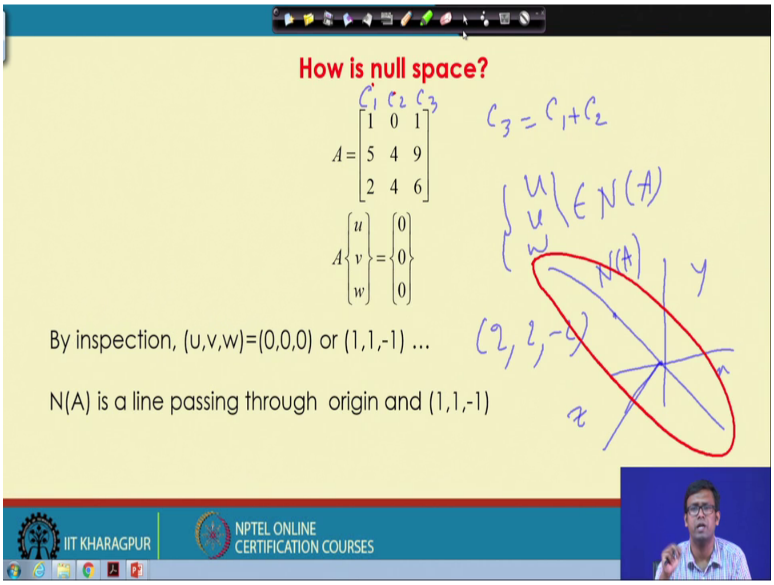

So, we will go through it in next few slides. How is null space? For example, I have a matrix A 1 5 2, 0 4 4, 1 9 6. So, I can make an observation here; if this is column 1, this is column 2 and this is column 3; column 3 is equal to column 1 plus column 2 ok; 1 plus 0 1 5 plus 4 9 2 plus 4 6.

So, this is a singular matrix. If now I have the null space equation A u v w is equal to 0 0 0. So, u v w will belong to null space of A. What will be the solutions? By inspection we can see one solution is 0 0 0 that will be definitely one solution and another solution is 1 1 minus 1.

So, first column into one because third column is sum of first and second column. So, first column into 1 plus second column into 1 and third column into minus 1 is another solution. Similarly another solution can be of sorry 2 2 minus 2 2 into 1 5 2 plus 2 into 0 4 4 minus 2 into 1 9 6 is equal to 0.

So, all this will be a solution and null space will be finally, a line which passes through origin and 1 minus 1. So, if I think of a 3 D geometry and this is the point say x y and z this is my point 1 this is 1 and this is the point somewhere 1 minus 1, the null space will be this. Any vector pointing to any point on this line will be a null space vector. However, this is a simple matrix and we can find the null space by inspection only that first and second column can be summed up to get third column.

So, any vector which any solution which multiplies 1 with first column 1 with second column and minus 1 with third column or multiplied of that with the solution. My question becomes how to find out null space for a larger matrix or for any general matrix. So, we cannot do am the make simple inspection based discussion on solution in as that simple as we done here.

The next important thing is that we can see that the null space here is a line, straight line. So, we will know that though x is equal to 0 has a infinite solutions; all this solutions will fall on that particular line. And if someone is interested on solving some case where probably only first 2 columns are available. So, I have 3 variables and 2 equations; I can still say that all this 3 variables will fall on that particular line. This information's can be important in several applications, where we cannot form the entered equation set for all the variables still we have an idea how the variables are behaving based on few are equations.

However, we will now try to look, how the null space behaves and what is this importance in case we try to solve Ax is equal to b from Ax is equal to 0, which is a null space equation, if we try to for solve Ax is equal to b and which is a not a null space x does not belong to null space when we are solving x is equal to b. The importance is that if A is a singular matrix x is equal to 0 will have infinite solution also x is equal to b will have infinite solution.

So, if I try to find out how infinite solutions of x is equal to b look like I can probably utilize the fact that I have an already have made an idea of x is equal to 0. So, we will try to find out null space first and then we will see how we can utilize this to find out Ax is equal to; solution of Ax is equal to b.

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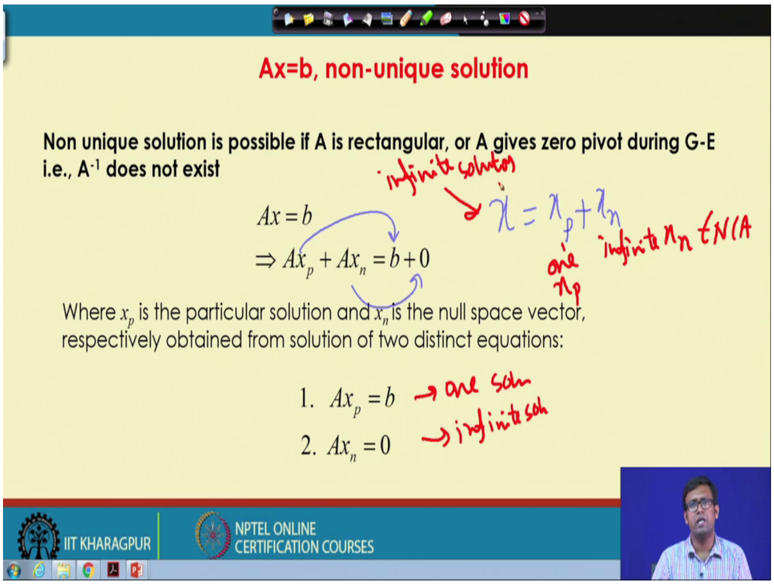

Non unique solution of Ax is equal to b is possible when A is rectangular; that means, we do not have enough equations to find out all the variables uniquely; what we are looking here. Now non unique solution is possible, if a is rectangular or a gives 0 pivot during gauss elimination or A inverse does not exist.

So, this three conditions; if I see 2 conditions rather if you see that A is rectangular, we do not have enough equations to find out all the variables uniquely like; earlier this one equation was dependent on others we did not have enough equation to find out all the variables uniquely or the matrix is itself singular. If I again the, if you look into the previous matrix, if I try to form a gauss elimination after two three steps, we will start getting 0 pivot which cannot be eliminated.

So, or both cases A inverse cannot be found out. So, that I cannot uniquely find out x is equal to A inverse b because A inverse cannot be found out here. In this particular case where Ax is equal to b has a non unique solution a inverse cannot be found out here because a is a singular matrix or a rectangular matrix.

So, then if I try to solve Ax is equal to b, we can divide Ax is equal to b as Ax p is plus Ax n is equal b or we will write x is equal to x p plus x n and Ax p can be equated with b Ax n can be equated with 0.

So, Ax n is equal to 0 will be a null space equation and Ax p is equal to b will be something else not a null space equation and we can therefore, break when Ax p will call it is a particular solution. It is a solution of Ax p is equal to b does not contain the null space and Ax n is the null space vector. So, Ax n is equal to 0 is a is the solution is the equation, which we are solving to find Ax n.

So, this 2 we can break down Ax is equal to b into this 2 equations and can distinctly solve Ax equal to p Ax p is equal to b the particular equation solution in a 1 equation and the null space equation Ax n is equal to 0 in another as the another equation. The advantage of doing that is Ax n is equal to 0, will give me how this infinite solutions will behave. And we will get one solution of Ax p is equal to b right. We cannot directly gave it through a technique like gauss elimination, we have seen it earlier because A inverse does not exist and it will gives 0 pivot. Still we will see some method through which we can get one particular solution of Ax p is equal to b.

So, this will give me one solution and this will give me infinite solutions. So, this there is 1 x p and there is infinite x n all belonging to null space of a. Therefore, as Ax will get infinite solutions; that is the goal of this excise that I have an Ax is equal to b where A is rectangular or a give 0 pivot a is a is not a full rank matrix; A inverse does not exist.

So, x is equal to b we will have infinite solutions. So, what should I do here? I will try to find out infinite solutions of Ax is equal to b. I will take Ax n is equal to 0 in the null space equation from which I will find out infinite Ax n infinite type types of Ax n will all belonging to null space. And I will solve Ax p is equal to b which will give me one particular solution for certain condition; I will gave one solution which is satisfying Ax p is equal to b and then I will add those things and say that Ax is equal to Ax p plus Ax n and this gives me this infinite solution in definitional equations.

We often use this technique we find the solution to homogenous equation and we will find out the solution to particular equation and we say that this is the final solution which contains both the homogenous part. This is null space part here and the particular integral part which is particular solution here and we will follow these steps in next few slides.

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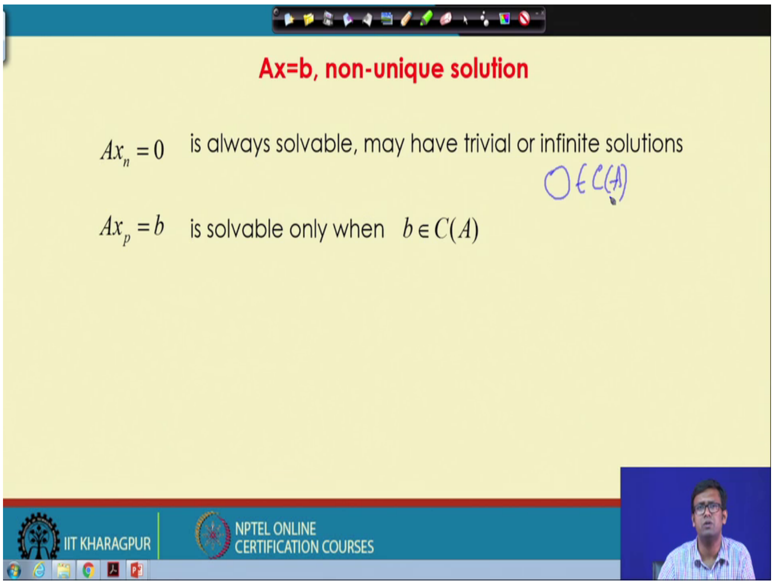

Ax n Ax is equal to b, non unique solution; this case once we try to find out the solution of Ax n is equal to 0. This is always solvable it might have trivial solution, infinite solution and why is it always solvable because 0 all is all sub space 0 belongs to any sub space.

So, 0 vector is also a sub space of column space, member of columns sub space 0 vector is a member of column space. Any sub space or any vector space must contain 0 vector. Therefore, 0 must belongs to the must belong to the column space and Ax n is equal to 0. We will always have a solution. Ax p is equal to b is critical, it will only have a solution or multiple solutions when b belongs to column space of A. If b does not belong to column space of A, this cannot be solved; then we will look into an example where we can solve Ax n is equal to 0, but we cannot we will not able to solve Ax p is equal to b and we will stop that this system. We will not have any solution.

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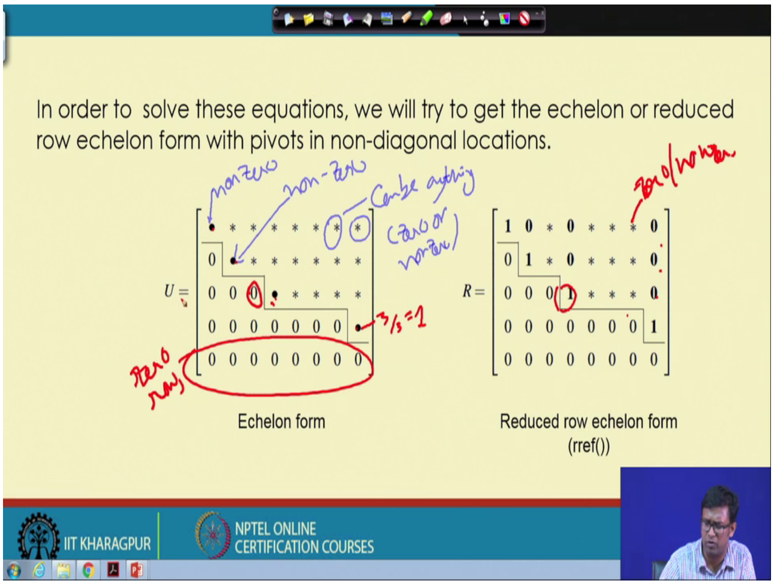

In order to solve this equation Ax p is equal to b or Ax n is equal to b y and I do not have a full rank matrix A; A is a singular matrix, A is determinant is 0 We will try to get echelon or reduced to echelon form with pivots in non-diagonal locations.

So, what is it? If we see the echelon form it is like a step; from the idea echelon, we discussed about reduce to echelon or echelon matrix when discussing Gauss Jordon steps. Gauss Jordon steps takes a matrix a and gives it is reduce to echelon form. So, echelon the terminology echelon came out from how to arrange soldiers in a battle field and it is a step like arrangement. There will be zeros arranged in steps; 0's arranged in steps and above 0 in any step, there will be non-zero numbers.

So, this is a non-zero number this is a non-zero number, all this black dots are non-zero numbers and this stars can be anything. It can be a 0 it can be a non-zero.

So, what is this matrix? Actually if you think of getting an upper triangular form through a pivoting, we started this is a one pivot point. Let us we have some U matrix and started gauss elimination step, this is one pivot point. Here the pivot becomes 0.

So, the next point becomes a pivot here; the diagonal element is not a pivot because it is not a full rank matrix and we will in standard your triangulation or gauss elimination technique, we are getting a 0 pivot here. The next term is probably non-zero. Similarly we get next one 0 next few term 0 and the non-zero here. And we will of course, because the pivots do not exist the diagonal location, they will finish before the last; they will finish before the last row and we will have few rows which are completely 0, 0 rows.

And this matrix can be converted into a reduced to echelon form like for example, if I divide this by say, this as a value 3 if I divide this matrix by this row, the matrix by 3 of this row by 3 this will be 1 and we get a reduced form. And then we will subtract this one it is multiplied by suitable multiplied with all these terms and we will get all this terms to b 0. There can be in the non-pivoted columns; there can be some non-zero terms.

So, this can be 0 or non-zero any term. And then we will get a pivot, which is one because it has been divided by the value its value in echelon form matrix and the above terms are 0.

So, this what we will call a reduce to echelon form and the idea of reduce to echelon form is that there the first term coming from left in any row is one; the first term is one. And the terms in one particular column above this 1 are always 0.

So, we will find then one which has above term 0 and then left term 0 at any column row position and then there will be a step like arrangement. And this called a reduced to echelon form, we have seen that in mat lab or in any many computer programs. There are functions which we call rref, which we will take a matrix and give it is reduced to echelon form.

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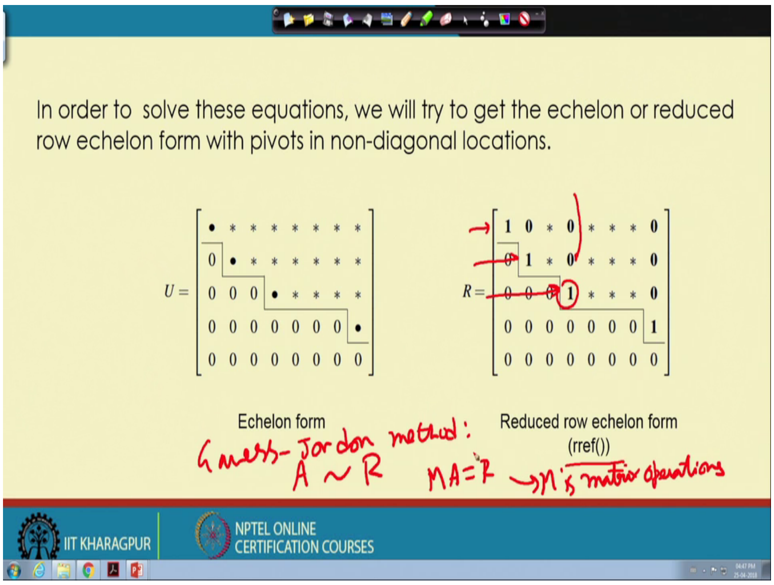

And A if we can remember of Gauss Jordon method, it takes any a and converts it is to convert it is to R or reduce to echelon form through matrix exercise or we have written that MA is equal to R, where M is matrix operation matrix operations, n is a matrix combining different matrix operation like subtracting a row from other multiplying a row by a constant, dividing a row by a constant etcetera.

So, if I multiply n with A we will get R, R or reduced to echelon form and we have seen it in Gauss Jordon method. Why is it important? We will come in a while because this gives us matrix, which are like a pivoted matrix, but with 0 pivots and if I have a singular matrix and I try to form follow Gauss Jordon steps or gauss elimination steps. I will get a matrix like this; gauss elimination step we will fail here, because it will see 0 at the pivot locations.

So, the gauss elimination method will failure. However, this is what we will get out of gauss elimination steps in a singular matrix and we will use this to get the null space solutions. So, we will take the example 1 3 3 2, 2 6 9 7 1, minus 3 3 4, u v w y 1 5 5 is equal to 1 5 5; this is a rectangular matrix. So, it is not invertible and we will we cannot utilize gauss elimination step here to get solutions.

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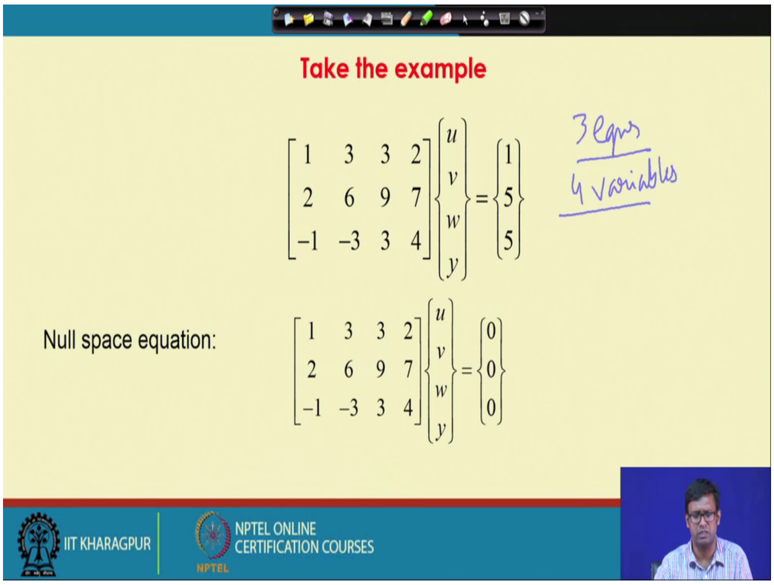

And there are we can see in total 3 equations and 4 variables.

So, it has to have infinite solutions. So, let us see how to find out the solution of this equation and we will first look into null space equation. We will find out the null space solutions first and then we will find out the particular solution and at them and say this is the final solution.

So, the null space solution is 1 3 3 2, 2 6 9 7, minus 1 3 3 4, u v w y is 0 0 0.

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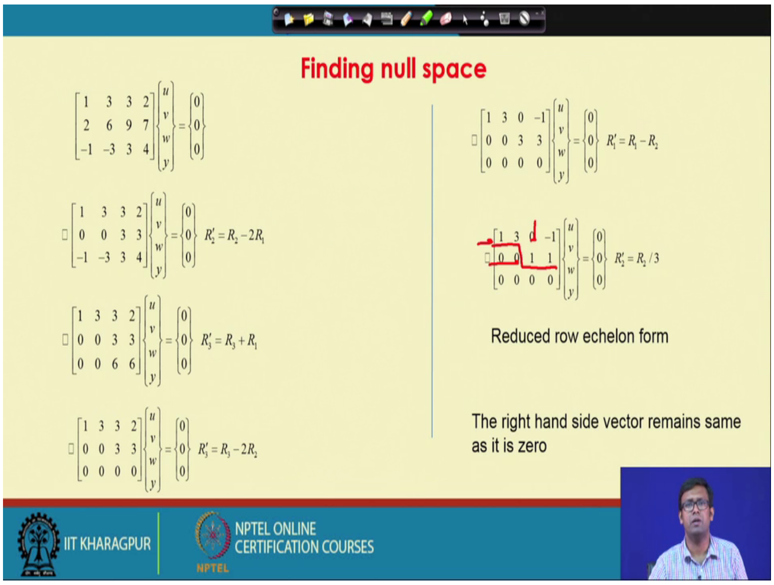

And we will see how can we find the null space. So, this is the, my null space equation; I do a low operation to eliminate the first term of the second row because I my idea is to get a reduce to echelon form here. So, the there will be a 1 1 and the if this is a 1 here, the terms that they will be another one here and the terms left to it will be 0 something like that. So, I will try to eliminate the first column term from the second row

So, and this is very similar as gauss elimination steps. Second row is first row second row will be second row minus twice a first row. So, these terms are eliminated.

The next step is third row third row is third row plus first row. So, first term and the second term is also eliminated. See when we are trying to eliminate the first term of second row, the second term of second row which is ideally a pivot term in a gauss elimination step is eliminated ok. We will we are not trying to get a upper triangular matrix, we are trying to get a reduced to echelon from matrix. So, this will work.

So, the next step is what will be the next I will try to eliminate the third term of third row or I will try to see a how many 0's I get eliminating in the third row. So, twice of third row, a second row will be eliminated from third row and we will get a 0 here.

Now, I got something close to a reduce to echelon form rather I got what is called an echelon form. I got an echelon form of matrix. This is not a step form of the matrix, this is not a reduced to echelon form. So, what I have to do? I have to divide this 3 by 3. So, this is 1 this row by 3 and this row is already the pivot is 1 here. So, the next step is well yeah I have to also eliminate the, if this is the pivot term the above terms will be eliminated.

So, first row is eliminated by first row is formed as first row minus second row and this type is 0. So, this now can be a pivot in a reduce tri echelon form matrix. I divide by second row by 3 and first row is already the first term is 1. So, this is what I get as a reduced row echelon form. This is a step like form echelon form; the first term in any row first non-zero term in any row is 1. If I have a non-zero one term pivot term in any row the above term is also 0. So, this is the reduced to echelon form.

The right hand side vector remains same as it is 0. We have done many matrix operations because, but this as there is a null space equation the right hand side always remain 0.

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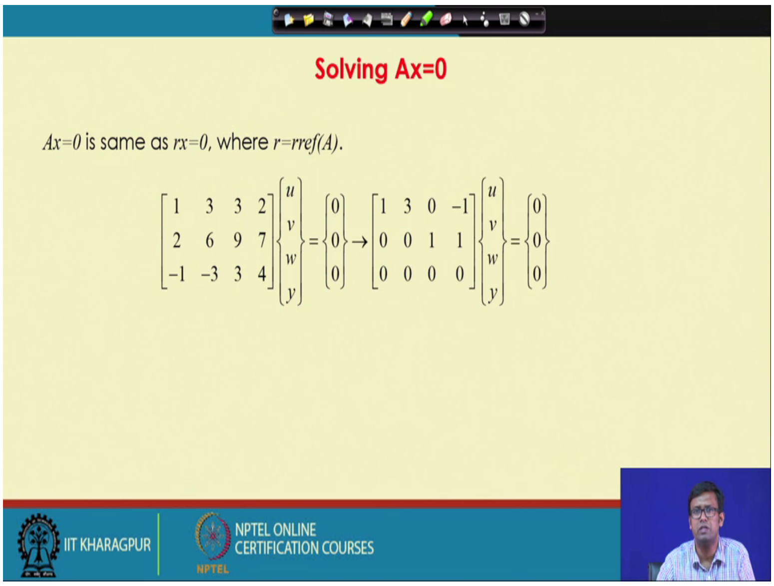

So, Ax is equal to 0 is same as rx is equal to 0, where r is the reduce to echelon form of A. This was the initial Ax is equal to 0 and this is what we are getting in the new reduce tri echelon form matrix rx is equal to 0. And now we will see how we can find out u v w y or the null space solutions by through this reduce to echelon form of matrix and we look into it in the next class.

Thank you.