Matrix Solvers
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Lecture – 15
Column Space and Null Space of a Matrix

Welcome, in last 2 classes we discussed about Vector Spaces and Subspaces and I assaulted it several times that the motivation behind looking into vector spaces is to look into the solution of x is equal to b systems. And we will try to look into the matrices a and also the solution x called the; so, right hand side vector b these three things and what are the vector spaces associated with this matrices?

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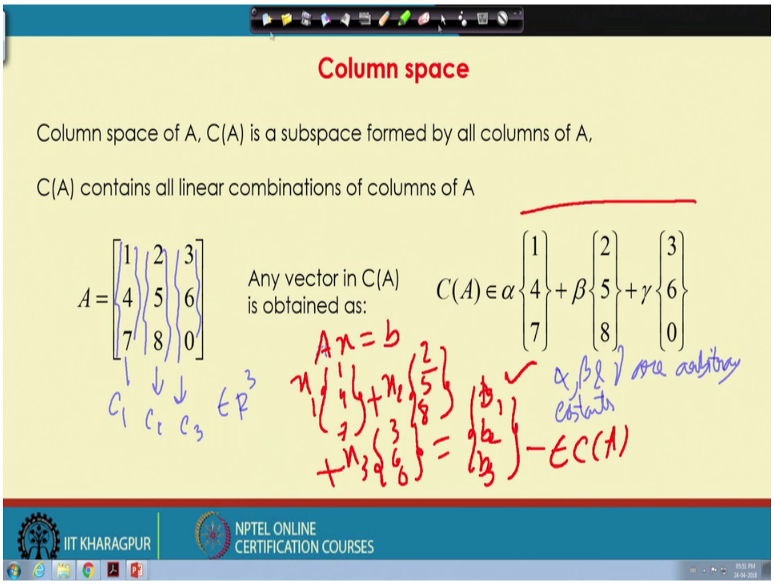

So, the first thing which comes in this discussion is called a column space. If we look column space of A, which is given as C in and bracket A is a subspace formed by all columns of A. It is a subspace because it is a space within the larger space R n if there are total n components in each column or R n if number of rows which are components in each column; it is a smaller space the space contained within R n and this is the space formed by all the columns of ra.

So, if we add the columns of all the columns of A; if we add the columns of A do multiplication by a scalar and find the resultant or from the multiplication and linear combination those vectors also must lie to that particular space which is formed by all the columns of A this is the subspace.

So, this contains all the linear combinations of A of columns of A. So, if I we can look into an example that A matrix is given as 1 2 3 4 5 6 7 8 0. So, the columns are member of each column we can identify the columns this is C 1, C 2, C 3 each column is a vector and they are member of R 3 and. So, any vector which is member of column space can be expressed as vector which is linear combination of these 3 column vectors.

This they may be in all vector thus this can be entire R 3 because R 3 is also subspace of R 3 or this can be a plan or subspace of an of R 3 is either a plane or either R 3 or a plane or a line passing through origin or the origin itself. So, it can be the entire R 3 or it can be smaller subspaces also. But any vector which belongs to column space we must be able to express it as a linear combination of the column vectors where this alpha, beta and gamma are arbitrary constants. That is by changing the value of alpha beta gamma; we can get different vectors and infinite vectors are member of a subspace or a vector space all this vectors you can find out.

Why is it important is that if I think of solving the equation A x or is equal to b; that means, start with the equation A x is equal to b what we will get? x 1 into and this is x 1, x 2, 3 at the solutions 1 4 7 plus x 2 into 2 5 8 plus x 3 into 3 6 0 is equal the matrix b 1, b 2, b 3.

So, b this vector solid vector b 1, b 2, b 3 this b vector forms the follows the rule of the column space of being a member of column space of A; that means, b is also a member of column space of n. Then only we will say that A x is equal to b as a solution; if x is equal to b does not have a; so, any solution then b is not a member of column space of A, we will look into it, but this that the idea of column space is extremely important in terms of solving x is equal to b.

Because the b vector itself is a linear combination of the columns of A; when we write A x is equal to b and b is a member of the column space of A, that is why column space is important to discuss.

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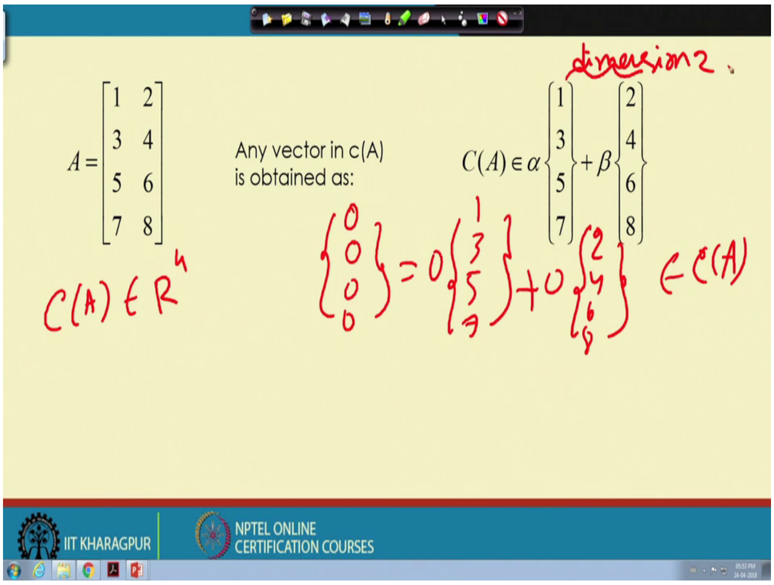

So, we take another vector A 1 3 5 7 2 4 6 8 and the column space becomes any vector which belonging to column space is alpha into 1 3 5 7 plus beta into 3 4 6 8.

Now, there are 2 vectors in 4 dimensional space; using 2 vectors I can think of a plane only. And does origin is origin a member of it; yes origin is the member of it if I use alpha is equal to beta ah, alpha is equal to 0, beta is equal to 0; this gives me origin. So, 0 0 sorry 0 0 0 0 is equal to 0 into 1 3 5 7 plus 0 into 2 4 6 8 here at least in this case. So, this is also a member of column space and column space has 0 vector inside it also.

So, interestingly the vectors of column space C A; they are also member the vectors of any vector in column space is member of R 4 here. Because this is they this any column is a member of real coordinate space R, there are 4 components. However, it has a dimension 2 because with 2 vectors I can think of a plane; so, it should they should belong to a plane.

So, here the column space is a 2 D space, but the vectors are 4 component. So, the vectors are also member of a larger vector space which is R 4 ok.

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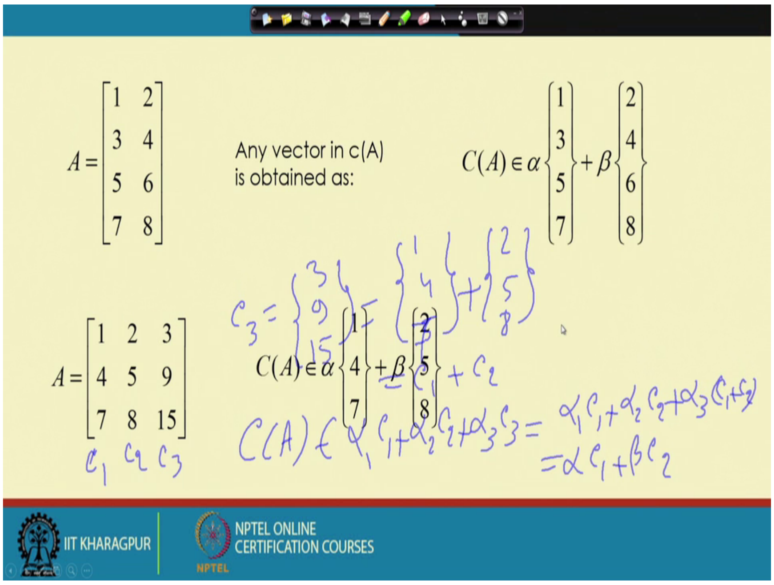

We will see another example like if I see this matrix 1 2 3 4 5 9 and 7 8 15. Now what we can see is that that look into the columns C 1, C 1, C 2, C 3; C 3 is equal to 3, 9, 15 is equal to 1, 4, 5 plus 2, 5, 8 1, 4, 7, 1, 4, 7 plus 2, 5, 8; so, C 1 plus C 2.

So, C 3 can be expressed as a combination of C 1 and C 2; now if I try to write that column space any vector which belongs to column space of A is a combination of say alpha 1; C 1 plus alpha 2 C 2 plus alpha 3; C 3 we should write it as alpha 1 C 1 plus alpha 2 C 2 plus alpha 3; alpha 1, alpha 2, alpha 3 are the multiplier; scalar multipliers C 1 plus C 2. So, this will be nothing, but alpha into C 1 plus beta into C 2. So, if one column is linear combination of 2 other columns; then the column space becomes a space formed by linear combination of those 2 columns which is forming the third column here.

So, we can write that column space of A is if the space in which can be expressed as linear combination of first and second column because third column is also linear combination of first and second columns. So, not necessarily that I have to write all the columns and then get the linear combination; if I see one column is can be formed by linear combination of few other columns, I will take those few other columns only and say that the column space is column space vector can be formed by linearly combining those vectors which are constituting the third vector also.

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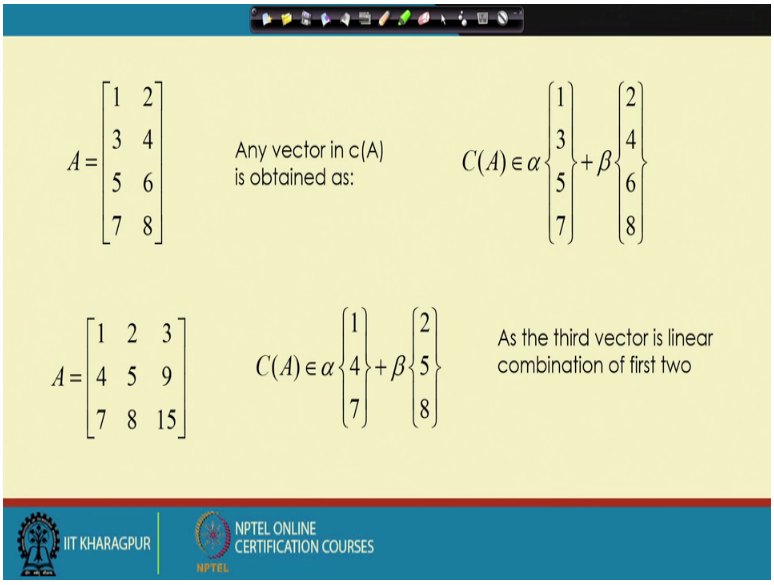

As the third vector is linear combination of first two. So, nevertheless the less a column space is the space; if we go back to the previous restriction column spaces is the space subspace formed by all columns of A. So, in general sense we should be able to write a column space vector as a linear combination of the columns of a particular matrix A. In case one column is linearly com dependent on other 2 columns, I write down with this 2 only.

But in general it is we have to linearly combine all the columns and any vector must lie there and this is a subspace. These are subspace because origin must lie on it, we have seen the adding 0 multiply making alpha, beta, gamma 0 takes me back to the origin also.

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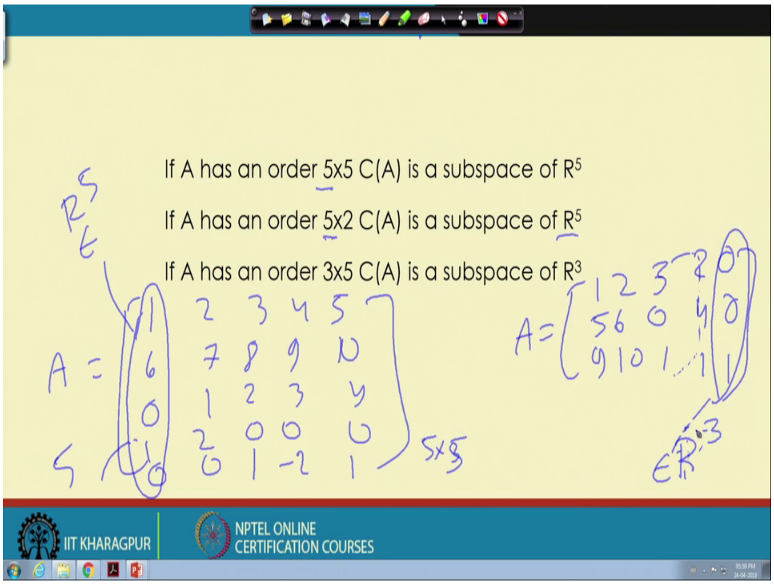

So, if A is 5 into 5; C A is subspace of R 5, why? 5 into 5 means 5 rows and 5 columns. So, if I have A is equal to 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 0, 1, 2, 3, 4, 1, 2, 0, 0, 0, 0, 0, 1 minus 2 one something like this. So, there are 5 rows and 5 columns and each column the number of elements in each column is the number of rows 5. So, if C A is 5 into 5; 5 is the number of components in each column therefore, each column must belong to R 5.

So, column space is a subspace of R 5; if A is a order 5 into 2 still we have 5 rows; that means, each column has 5 components and it is a member of R 5. If A has an order 3 into 5, now my A is say 1, 2, 3, 5, 6, 0, 9, 10, 1, 3 into 5, 3 into 5 I am sorry 3 rows and 5 columns 2 4 1 0 0 1. So, each column here this belongs to R 5; R 3 because there are 3 components in each column vector. So, each column vector is a member of R 3.

So, the number of rows determines how many components each column have and what is the real coordinate space whose subspace is the column space or is the order of the real coordinate space.

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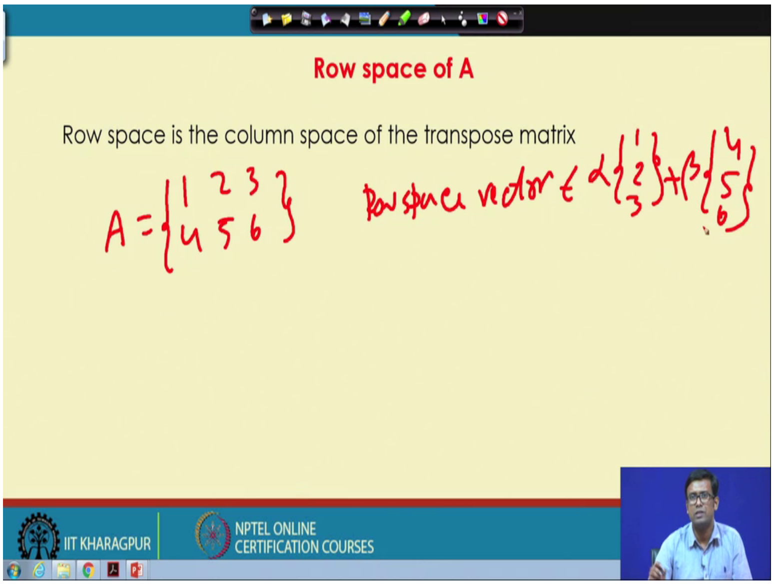

Similar to column space, there is row space; row space is basically the space subspace formed by all the rows of a matrix. For example, if I have a matrix A is equal to 1 2 3 4 5 6; the row space vector any vector belonging to row space will be a member of alpha into 1 2 3 plus beta into 4 5 6; see we are always writing the vectors as a column vector or vertically.

Therefore, we when we discuss about row space we also consider it to be column space and whose column space will be the row space? Row if the row of a matrix is column of its transpose matrix. So, we will say that row space is the column space of the transpose matrix.

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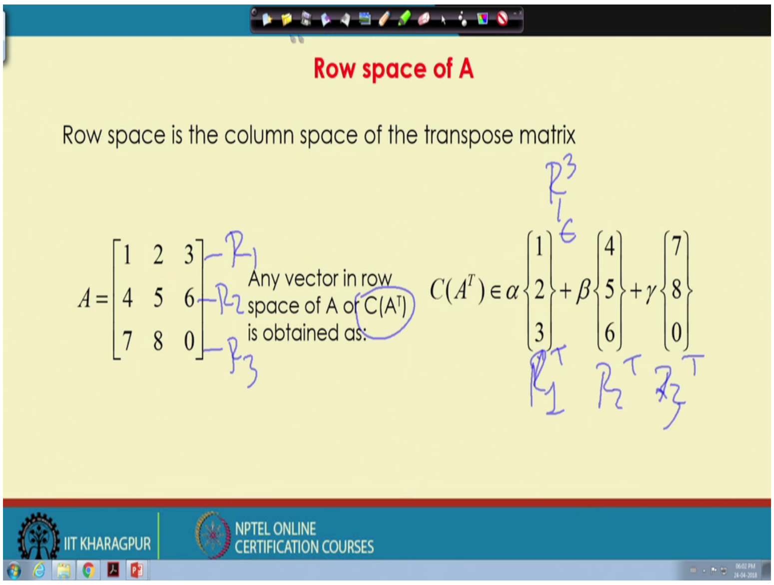

Row space of the of A is column space of A transpose; so, if A is 1, 2, 3, 4, 5, 6, 7, 8, 0 any vector in row space of A and this is be written as exactly as column space of A transpose; that is a notation to write row space also is obtained as column space of A transpose is any vector belonging to column space of A transpose is linear combination of vector row 1; row 1. So, this is R 1, R 2, R 3 they are not row 1, row 2, row 3 this is R 1, R 2, R 3; they are row 1 transpose, row 2 transpose and row 3 transpose because the rows are now written as the columns here.

So, similar to column space there exist also a subspace formed by the rows of one particular matrix. And what should be the dimension of the subspace? That depends on how many rows are linearly independent; that means, if none of the rows can be expressed as linear combination of other rows, then it is a the dimension is actually the number of rows. In other case, we it will be less by the number of rows which is linearly dependent on other rows and which what is the; so all the row vectors belong to R 3 here. So, you also have to see to which real coordinate space this row vector belong to.

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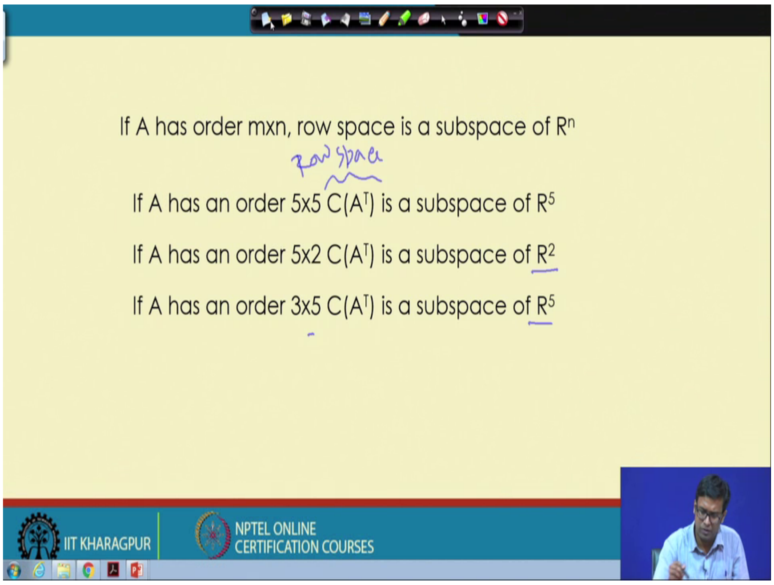

And this will be very similar as the column vector, if A has order m into n row space is a subspace of R n; n is the number of columns that is the number of components in each row. So, a row row space sub subspace there is row vector space or the subspace formed by rows is the subspace of R n or the real dimensional real coordinate space with n components because n is the number of columns here.

If A is 5 into 5; column space of A transpose which is row space this is expression for row space row space is written as column space of A transpose is a subspace of R 5 in 5 into 2 the m m into n n is equal to 2 here. So, 2 row space is subspace of R 2, 3 into 5 n is equal to 2 here row space is a subspace of R 5.

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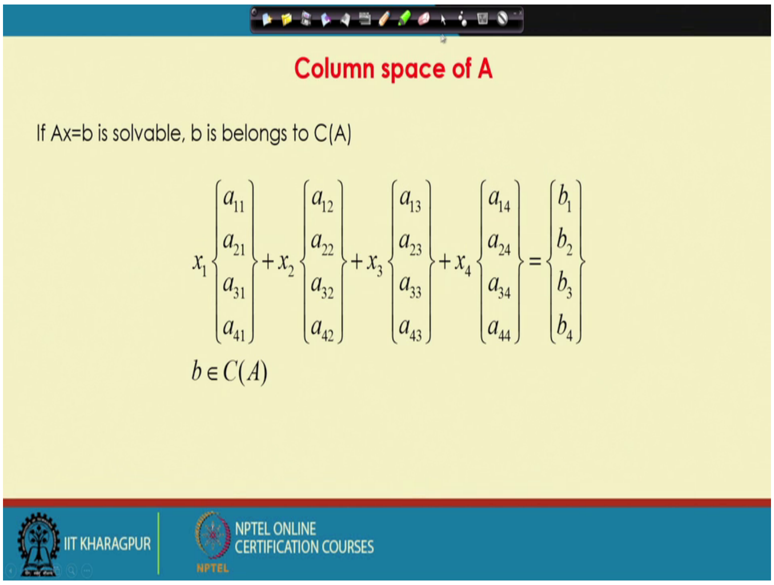

And this becomes the important thing what we are discussing few slides back; that A x is equal to b is only solvable when b lies in the column space of A. The general expression of column space of A is the columns of A, the columns of A multiplied with some arbitrary coefficient will be a column space vector. And if this holds then only A x is equal to b is solvable or if A x is equal to b is solvable, then I might take each column of A and multiply it with on coefficient, which is now my solution and the sum is the b vector.

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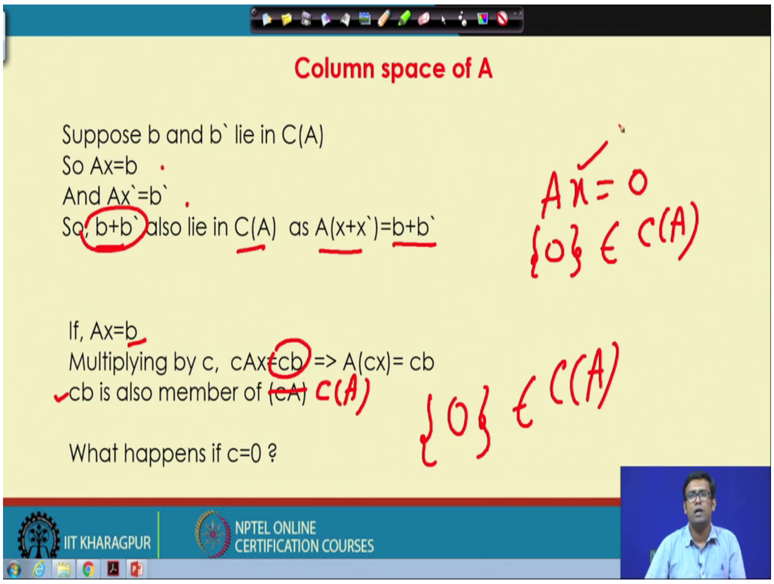

So, b is a member of column space of A and then if we can think of b and b prime; both lie in column space of A. Because column space is a subspace and what is the real coordinate space to which column space is a member is rm; m is the number of rows that is a dimension of the real coordinate space to which column space is a subspace.

Now, column space is a subspace; so, it should have infinite vectors inside it with different choices of alpha beta gamma we should be able to find out different vectors. Now, if we consider column space there is a vector b; x is equal to b, we will get a vector b. With some x prime some other co set of coefficients; so, we will get another column space vector b prime. So, if b and b prime are 2 column space vectors b and b prime both lie in column space of A and we can write A x is equal to b and A x prime is equal to b.

So, b plus b prime must also lie in column space of A and that is the definition of any vector space or subspace; that if we add 2 vectors the resultant must also lie there. Therefore, we if A x is equal to b and x prime is equal to b prime; the A x plus x prime is equal to b plus b prime and b plus b prime must also lie in column space of A. And the other one that if I multiply a column space vector with a, with some constant, it will be again a column space vector. So, that can also be seen that if x is equal to b we multiply these both side by c. So, cAx is equal to cb and cb is also member of column space or we can write a into cx, c can come in this is a constant coming inside a cx is equal to cb.

So, we if b is the member of column space; cb is also member of column space. This is I should write it down this is column space of A becomes here; now what will happen if C is equal to 0? And 0 vector is member of any vector subspace. So, 0 is also a member of column space of A; therefore, when we will think about solving A x is equal to 0, then 0 must be a member of column space of A or A x is equal to 0 must have a solution because 0 is always a member of column space of A.

So, if I write A x is equal to 0; this 0 should always lie to the column space of A. So, we can always find some x by which I can combine the column space vectors and get a 0 vector, it can be all 0s or it can be something else.

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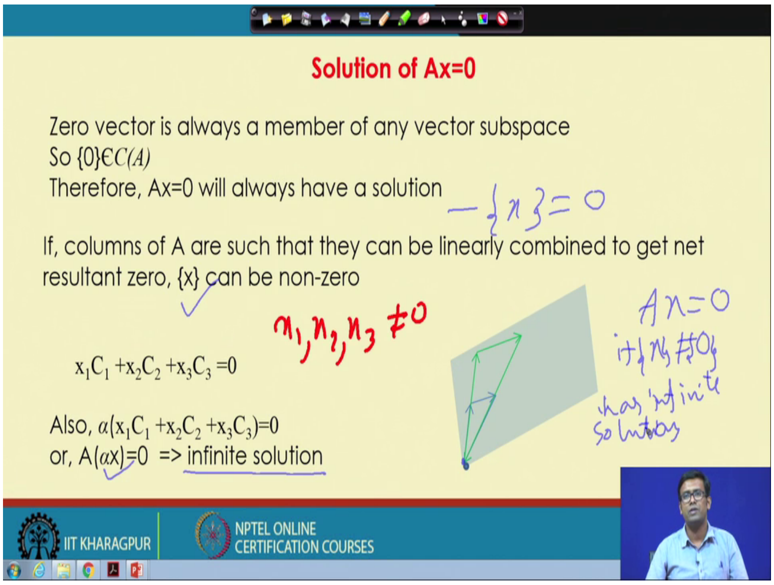

Zero vector is always the member of any vector subspace; so, 0 is a member of column space of A. Therefore, A x is equal to 0 will always have a solution because I will always able to find out some x; x can be all 0 or x can may be it is not all 0.

But some x I will be always able to find out which will give me a solution to x is equal to 0. If column of A are such; columns are such that they cannot be, they can be linearly combined to get a resultant 0; x can be non-zero.

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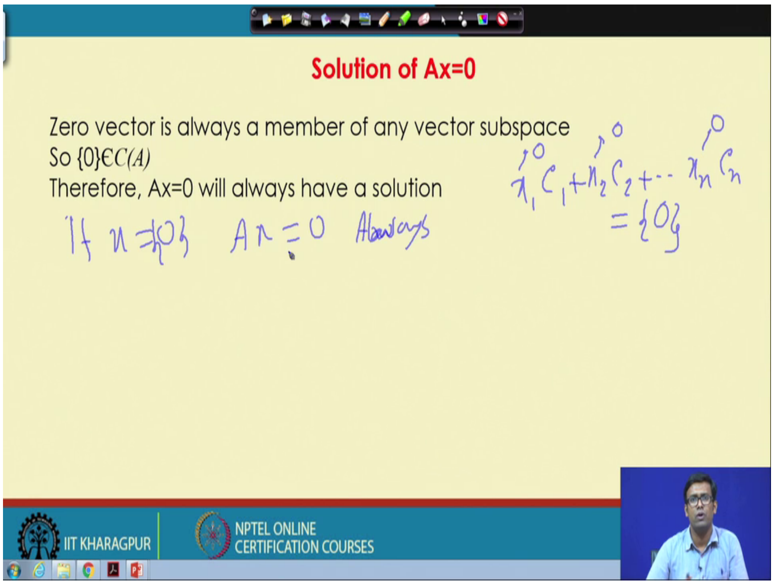

Or if I will probably see that if; if x is equal to 0, A x is equal to 0 always. So, x is equal to 0 is always a solution of x is equal to 0 because I am writing that; if A has the columns x 1 into C 1 plus x 2 into C 2 plus x n into C n, where C are the columns of a matrix is equal to 0.

So, I can always put x x 1 x 2 x 3 up to xn 0 and can get this done. So, x is equal to 0 is always have a solution A x is equal to 0; however, on top of that, if the columns of A are such that they can be linearly combined to get a resultant 0, x can be non-zero; how is it? That x 1 C 1 plus x 2 C 2 plus x 3 C 3 is equal to 0. We can think of x 1, C 1 is equal to 0 the column first column multiplied by some x plus second column multiplied by some x, plus the third column multiplied by some x. And they coming back the resultant is coming back to the origin, the resultant is coming back to the origin.

So, then we can have a non-zero solution; now interestingly I can magnify all the vectors here. The same coming back to 0 can be done; if I multiply some constant with all the in both left hand side or right hand side. Or if I multiply some constant with x 1 x 2 and x 3 I will still have a similar triangle which is finally, coming back to 0. So, now when pictorially this can be a case like this that this is column vector C 1 multiplied with some other coefficient; x 1 into alpha say. Plus column vector C 2 multiplied with some other coefficient plus column vector C 3 is multiplied with some other coefficient will again be 0.

So, if I can find out one set of x 1, x 2, x 3 to which I which are non zero; I can write the condition is that x 1, x 2, x 3 are not all 0 some of them at least 2 of them are nonzero. If I can write this, then I can find out also another x 1 x 2 x 3 just multiplying this by a coefficient which will also give me a nonzero solution. So, if x 1 x 2 x 3 is equal to 0; alpha into x 1 C 1 plus x 2 C 2 plus x 3 C 3 is equal to 0 or A alpha x is equal to 0. So, if x is a solution; if x is a x non nonzero x is a solution; then alpha x is also a solution and for different values of alpha we will get different solutions. So, finally, we can get infinite solutions.

So, whatever getting out of it that there can be one case that A x is equal to 0 has only solution x is equal to 0. The columns cannot be linearly combined to get 0 and multiply 0 with all the column vectors and get back 0. There is another solution that I can multiply some nonzero number with some of the columns and the resultant gives me 0. And then I can multiply something with the multiplier and again the resultant will be 0. So, the multiplied is a solution here; so, I can have different multipliers and get different solutions for which A x is equal to 0 is there; if there is some nonzero x for which A x is equal to 0, so we can get infinite solutions.

In case there is any nonzero solution a solution of A x is equal to 0; A x is equal to 0, if x is not equal to 0 vector; it has infinite solutions. Why is not it for 0 vector? Because if x is a 0 vector, I multiply anything with 0 vector it will still remain a 0 vector; so, it is a single solution only.

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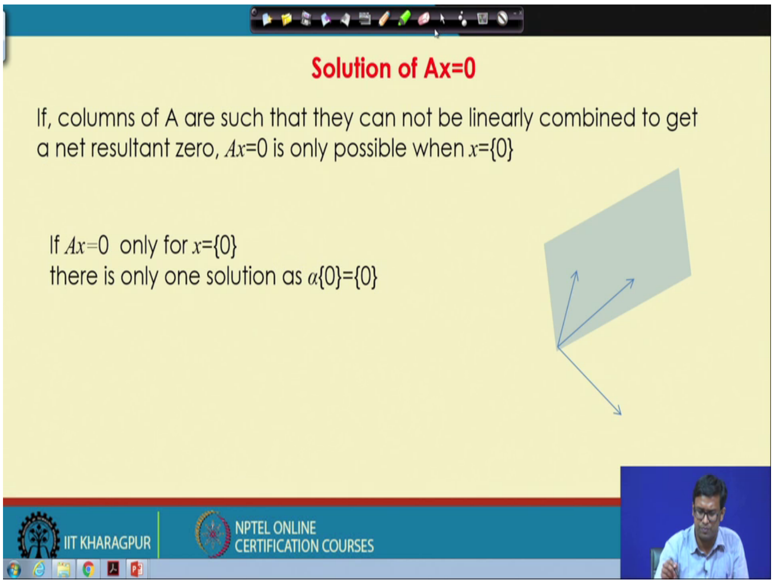

If columns of A are such that they cannot be linearly combined to get a net resultant 0; A x is equal to 0 is only possible when x is equal to 0. If A x is equal to 0 the like I we have see seen here that these are the column space, these are the vector column vector there are these 2 columns and they will combine finding a plane. And the third vec column is out of it; 3 columns are there, 2 columns lie in a space and the third column is out of in a plane and the third column is out of the plane.

So, I whatever combination I take of these 2 columns that will go to a that that will go to a at this like sorry; whatever combinations I take of these 2 columns they 2 this will not cancel this component. So, I have to have some non and I have to multiply this with this with 0 to get a, A x is equal 0 solution as the resultant.

So, in this case there is only one solution because when x is equal to 0 is the solution of it whatever co coefficient I multiply with it will always be alpha into 0 is equal to 0.

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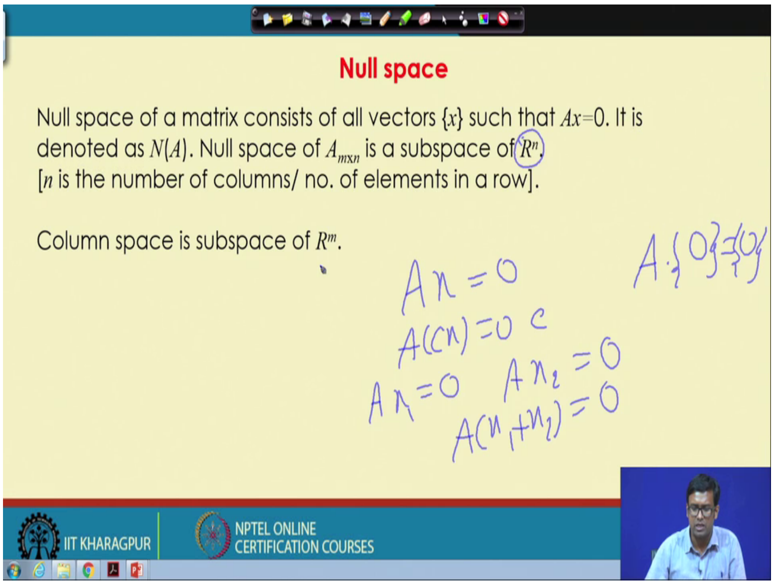

And now we come to an interesting definition called null space; null space is a of a matrix consists of all vector x; such that A x is equal to 0. So, all solutions x of A x is equal to 0 will give me null space, there can be single solution x is equal to 0. Or if the columns can be combined with by multiplying them with certain constants to get a 0 solution then x is not necessarily 0.

However, be it 0 or be it many in solutions; the entire set of the solutions the space that is formed by the solutions of A x is equal to 0 is called the null space and this is denoted by N of A. Null space of m into n matrix is a subspace of R n because the number of the null space vectors are multiplied each component of null space vector is multiplied by one column. So, number of column is equal to number of components in null space vector and therefore, null space becomes a subspace of R n as n is the number of components there; N is the number of compound columns or number of elements in a row sorry.

Column space is a subspace of R m and null space is a subspace of R n that is important to note down that null space is ; you should not confuse null space with column space. Null space is the now from the space inside the matrix, we came out of a subspace outside the matrix that this multiplied with the matrix is giving me 0. And all these solutions A x is equal to 0s; all solution of A x is equal to 0 is also a subspace, why is it a subspace? We can see it a while that if we do A x is equal to 0; if I write if I write A x is equal to 0 and multiply C with x; A into cx is also 0 or if I have A x is equal to 0 and A x 2 is equal to 0; 2 null space solution A x 1 plus x 2 is also 0.

So, x forms a vector subspace and also A into 0 is equal to 0. So, origin is a member of the null space also; so, this is a vector space x now forms a vector subspace. And this subspace has dimension as a mem n or with a components n and they are member of real coordinate space of dimension n. And column space we have seen it earlier is a member of dimension real coordinate space of dimension m.

So, we will explore more details on null space in subsequent classes and finally, we should equip ourself to find out the null space. How does a null space look like? What are the general expression of vectors in null space or if you can find few vectors in null space and linearly combine them, we should find the other vectors in null space.

So, in next class we will see how to find the null space vector or how to find the null space for a for any general matrix A x is equal to 0. And we will also see how to find null space for rectangular matrices because for Gauss elimination or Lu decomposition type of things we restricted ourself to square matrices. But for the cases where number of equations and number of solutions are not same; still all the solution is equal to 0 will give me A x is equal to 0, so null space x is there.

And it is also possible to have nonzero vectors in null space; in case of rectangular matrices. So, we will see null space for any general matrix, how to obtain null space and we will proceed in the course.

Thanks.