Matrix Solvers
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Lecture – 14
Vector Subspace

Welcome. So, last class, we were discussing about Vector Space. I tried to give you a brief introduction about what is a vector space and mainly discussed on the definition of a vector space and tried to explain it in with a prospective of a real coordinate space and we saw that vector spaces has certain dimension which the or the real coordinate space has certain dimension which is given as R n. Where, n is the number of components of a vector in that particular space, but R n or a real coordinate space spends the entire space with that particular dimension.

Now, we are interested in sub spaces or the smaller spaces inside that real coordinate space in which our solution of x is equal to b will lie in case there are infinite solutions; that is the main motivation for us to look into the de particular linear algebra chapter called Vector space. So, we came into the definition of a Subspace of a Vector space.

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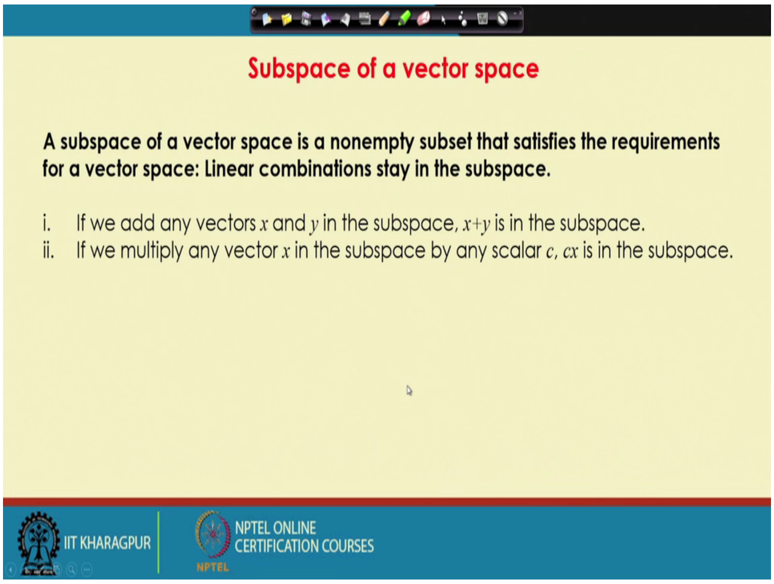

Subspace is a smaller space inside a vector space which satisfies the requirements of a vector space that is if we take two vectors in that particular subspace, we add them; they will be in this in that particular subspace. If we multiply a vector with some constant that will also lie on that particular subspace.

So, the definition can be followed up in that way that if we add any vector x with another vector, y in both are lying in the subspace x plus y should also lie in the subspace we have to see how does it happen? But it happens.

And if we multiply any vector x with any scalar c; c x will also be in the subspace. So, it becomes probably little more restrictive than the vector space definition because vector space or the real coordinate space definition because real coordinate space ensures that any vector with number of components same as the dimension of real vector; coordinate space must be a member of real coordinate space or the largest vector space we can think of in that particular dimension.

But sub when we are talking to subspace in that real coordinate space, we have narrowed down a particular regime and say telling that all vectors belonging to that regime can be linearly combined with each other; in a sense they can be added to each other or can be multiplied by a constant. One vector can be multiplied by a constant or verified and this linear combinations will also stay on that particular subspace. So, we will we will we will see how does it work?

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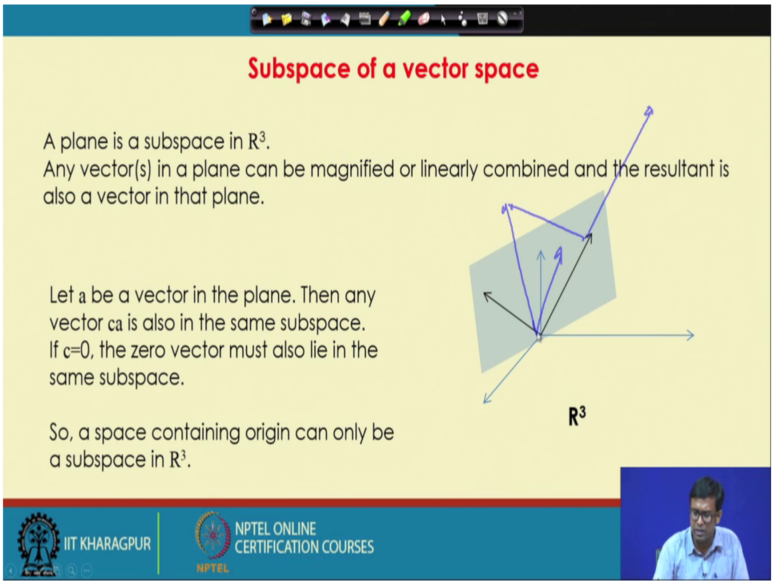

For example, if we think of a plane in a real coordinate space of dimension 3 R 3. This is a subspace; this means like probably simpler that a plane is a subspace of a 3 d real coordinate space. Why? I have 2 vectors in that plane with these two are coplanar vectors. If we add them the resultant, sorry if we add them the resultant will also be a coplanar vector or I should add it properly.

Say if I add it the resultant must lie in that particular plane that that comes from the law of vector addition umm or law of planarity and if I multiply a vector with one constant, this should also lie in that particular plane. So, this plane becomes a subspace. Any vectors in a plane can be magnified like multiplied with a particular constant or linearly combined and the resultant is also vector which is on that plane only.

So, this becomes a subspace; that means, it satisfies the requirement of a vector space, also it says that in the sense that the we started with a restriction that this the vector lie in a particular plane and the resultant is also lying on that particular plane. So, this restriction is already even after we have done linear combination to get the resultant.

So, it is now not the large infinitely spanning 3 dimensional space, it is only a plane which at least in one dimension, it cannot span anywhere. It is a if we think of the 3 dimensional geometric it is a plane. So, it is not going anywhere probably in this particular da direction. So, we have narrowed down the space.

Let a be a vector in this plane. Then ca is also on the subspace that comes from the definition of a subspace. Any vector can be in a particular subspace can be multiplied with a constant and the resultant will and the multiplication, it should also be a member of that particular subspace.

So, now if a is a vector if this for example, if this is a vector in the sub place space if I multiply it by some constant c that will also lie in that particular subspace. Now if what will happen if c is equal to 0? If c is equal to 0, this will be the origin. Sorry, if c is equal to 0, the multiplication will come back to the origin only.

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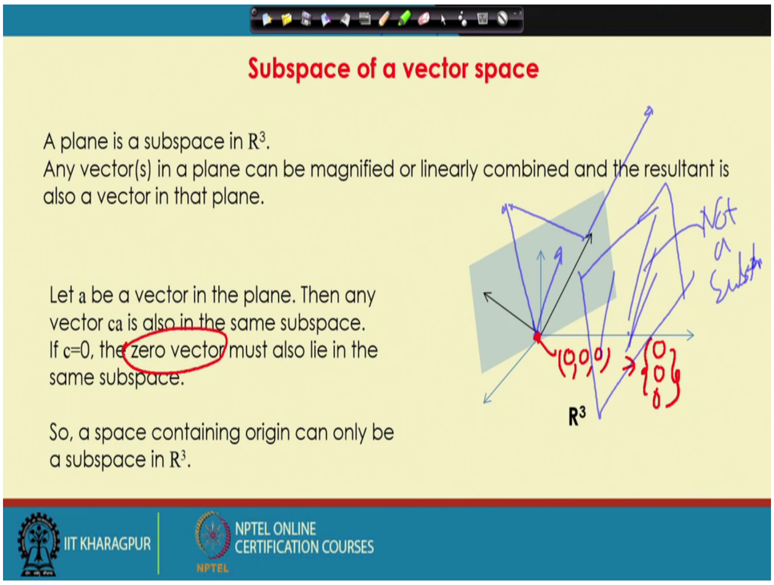

Or this will be a vector 0 0 0 or if we write in a column vector format, this will be written 0 0 0.

So, what we call typically a zero vector. Zero vector must lie on the sub same subspace. Therefore, if we call a plane to be a subspace, it must contain the zero vector or the origin or only a pay plane passing through the origin is a subspace.

Any subspace has to cos contain the origin with the because I can multiply any vector with 0 and get the zero vector. Zero vector must be contain within a sub plane sub subspace.

So, if I think of a plane like this, if I think of a plane like this, this is not a subspace because zero vector does not lie in that particular plane. So, subspace must contain zero vector which is another requirement of subspace which it comes out from the definitions only.

And this is not only subspace any vector space must contain as it thus it a vector because anyway that can be multiplied with constant 0 and we can get the original zero vector. So, it has to be any vector space has to have origin within it and that is the original coordinate spaces are vector space because they start from origin. Origin is very much within them.

So, a space containing origin can only be a subspace in our theory in any R n. Any space we which can be called a subspace mass counting origin and this is extremely important when looking into the subspaces. So, three things we have got. Subspace, it is a smaller space than the vector space ideally.

It must contain origin and all this things is due to the fact that if we add two vectors they in a subspace, they must lie on the subspace; if they do not lie in the subspace then initial two vectors we cannot call them that they were also member of the subspace. The subspace do not exist; if we say that this is the def this is the criteria say x is greater than this y is greater than this I will get an x x y which will be a subspace.

This is only will be acceptable if we add two vectors of the subspace or if we multiply any constant with a particular vector on the subspace and the resultant lies in the same subspace that oh that tells us that origin has to be there. So, x is greater than 1; y is greater than 1. Any vector x y cannot be a subspace because origin is outside the origin is 0 0 which is less than what in both cases ok.

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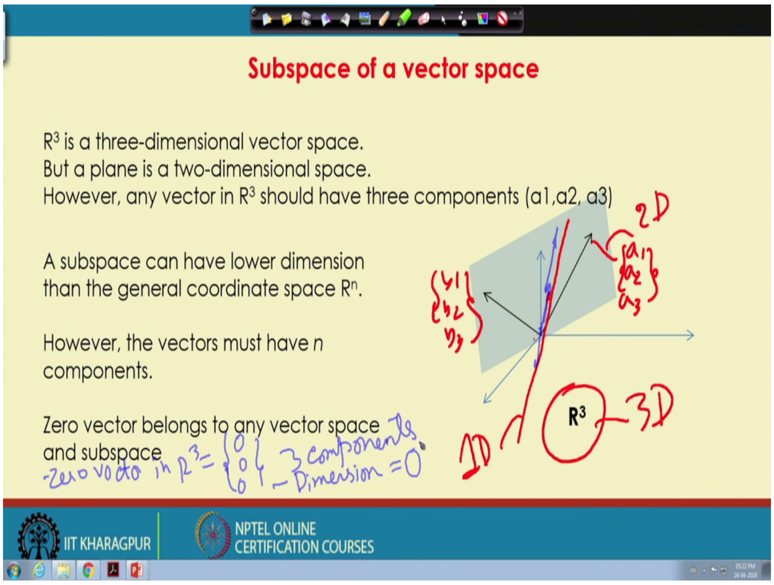

Lets as go to the next slide. So, if R 3 is a three- dimensional vector space, but a plane is a two-dimensional space. So, a subspace can has le can have lesser dimension than the actual space; then not actual space than the real coordinate space. Real coordinate space can be up to R infinity.

A subspace can have a lesser dimension that like here we discussing about a real coordinate space we goes to R 3 which is R 3, it is a three-dimensional real coordinate space, but the subspace we what we discussing is the plane, if we look into the slide this subspace is the plane and this plane is a 2 D geometry plane is a two-dimensional geometry.

However, any vector in this plane must have three components because it is in R 3. So, this vector is a1, a2, a3 and this vector is b1, b2, b3. Though this is a two-dimensional space, but the vectors of 3 components because vectors are member of R 3 or three-dimensional real coordinate space.

A subspace can have lower dimension lo lower dimensions than the general coordinate space R n. So, R n R 3 has a is a three-dimensional coordinate general coordinate space three-dimensional vector space, but they apply in this particular vector space is two-dimensional vector space.

We can also think of line passing through the origin. Any vector along this particular line think of a line passing through the origin. So, any vector along this particular line can be added with another vector on this particular line and the resultant will be in this line.

Any vector can be multiplied and the resultant will be also along this line; only thing the line has to go through origin. It also become becomes a subspace of R 3 that a line passing through origin.

However, this does not have three-dimensions a line has a single dimension only. So, a subspace is a smaller space within a particular vector space; where, the vectors have components same as which is the number of components of the vector is same as the dimension of the real coordinate space.

However, the subspace we have a lesser dimension the than the number of components in the vector. Like here we got look into about two subspaces; one is the plane which is two-dimensional space, another is the line which is a three-dimensional space sorry which is a one-dimensional space and R 3 is a three-dimensional space.

So, three-dimensional real coordinate spaces 1 D subspace, 2 D subspace; it can have larger dimensional real coordinate space also can have smaller dimensional subspaces.

However, vector number of components in the vector is the same as the dimension of the real coordinate space. The ma vectors must have n components.

Zero vector belongs to any vector space and subspace and we have seen that that if we multiply any vector by 0 it will be 0. So, in the by definition of vector space any vector multiplied by with another vector must be a member of that vector space or the subspace. Therefore, zero vector has to be member of the vector space or subspace any vector space or subspace cannot be formed by pressing the origin.

And interestingly zero vectors may have if we talk about R 3. So, zero vector zero vector in R 3 has 3 components. This is 0 0 0; three components. It is different than a zero vector in R 1 or R 2. It is a 3 components zero vector because it is a member of R 3. However, the dimension of zero vector has is always 0 because it is a point only; it has no length, no orientation nothing.

So, dimension of a vector and dimension of a vector space is and dimension of a subspace is different. Dimension of a real coordinate space is the number of components we need to express that vector and that will stay with the vector if we think of a subspace also.

Subspace may have lesser dimension, but the number of components in that vector must come from the fact that what is the dimension of the real coordinate space which is model of this subspaces.

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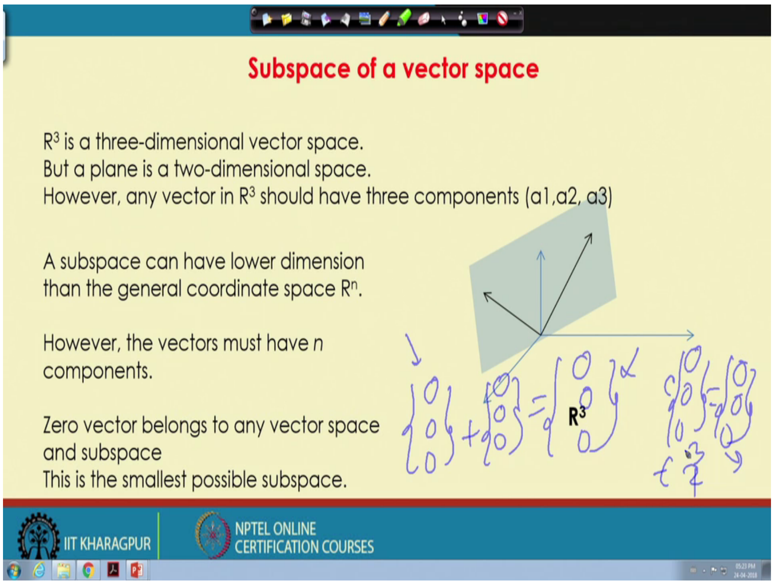

And zero vector becomes the smallest possible subspace. Zero vector is itself a subspace y if I have a in 3 d I have 0 0 0, I add it with another and this is this is this belongs to a subspace.

The only member of the subspace is a zero vector. I add with 1, I get so zero vector only which is member of the subspace or I now, I multiply a zero vector with a zero vector, I get the zero vector only. So, 0 is a zero-dimensional subspace or the smallest possible subspace and I will again, iterate it because this is very important that what is the dimension and what is the order or what are the number of components.

Zero vector has 0 dimension, but the number of components comes from the fact that they it is a member of R 3 which real coordinate space it belong to that says what it should be its number of components ok.

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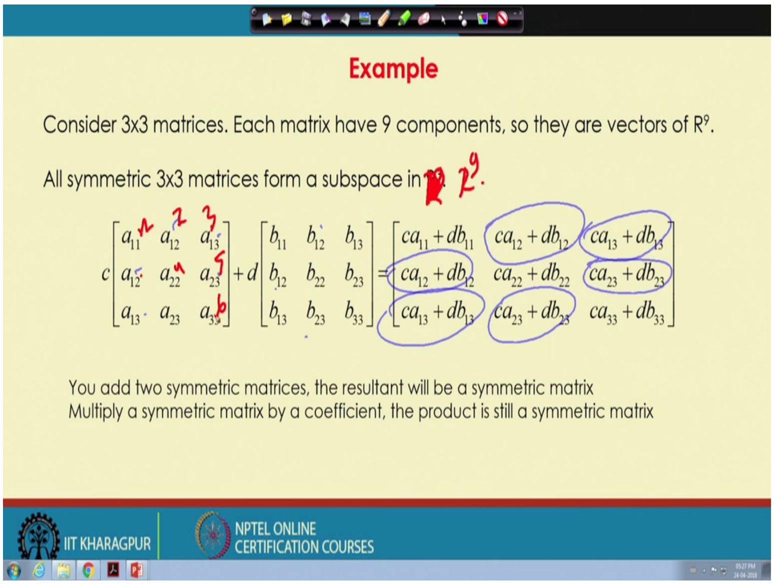

So, we consider look into few examples of the subspaces. We consider a 3 by 3 matrix. Each matrix has nine components. So, they are vectors of R 9 right. 3 by 3 matrix, they have 9 components and if I try to write a 3 by 3 matrix we have to write 9 components. Therefore, there must be number of real coordinate space with dimension 9 because I can express a nine-dimensional vector using a 3 by 3 matrix, all the components of 3 by 3 matrix.

So, this is R. So, all symmetric 3 by 3 matrices they form a subspace in R 3. Any 3 by 3 matrix is a member of R R 3 R nine sorry this form a subspace of R 9. Let me block it, they are subspace of R 9. Any matrix is a member is a vector which is a member of R 9. Any symmetric matrix, now we are putting a restriction over it. Any symmetric matrix is a member of a subspace of R 9.

So, it is not all the matrices all the members of R 9; only the symmetric matrices; that means, only where 1 1 1 2 is equal to 2 1 or the second component is equal to fourth component, (Refer Time: 17:30) vectors are members are form a subspace.

How will it be? If we see, if we linearly combine 2 symmetric matrices the linear combination is also a symmetric matrix like this, this, this, this. This is symmetric matrix a and b, they are 2 symmetric matrix and the linear combination the 2 1 term is same as 1 2 term; the 1 3 term is same as 3 1 term; that 2 3 term is same as 3 2 term. So, the linear combination is also a symmetric matrix.

So, symmetric matrices form a subspace in R 9. You can add 2 symmetric matrices the resultant will be a symmetric matrix; if multiply a symmetric matrix by a coefficient and the product is still a symmetric matrix. Therefore, all symmetric matrices form a subspace.

Now, the question can be we can probably address it later. The question can be that what is the dimension of that particular subspace right. And of course, maybe we will see here that in order to express a symmetric matrix we sorry, in order to express a symmetric matrix we need 1 2 3 4 sorry not 4; 1 2 3 4 5 6 terms. So, it should be a six-dimensional space though it is a member of R 9. Anyway we will see few more examples of a subspace.

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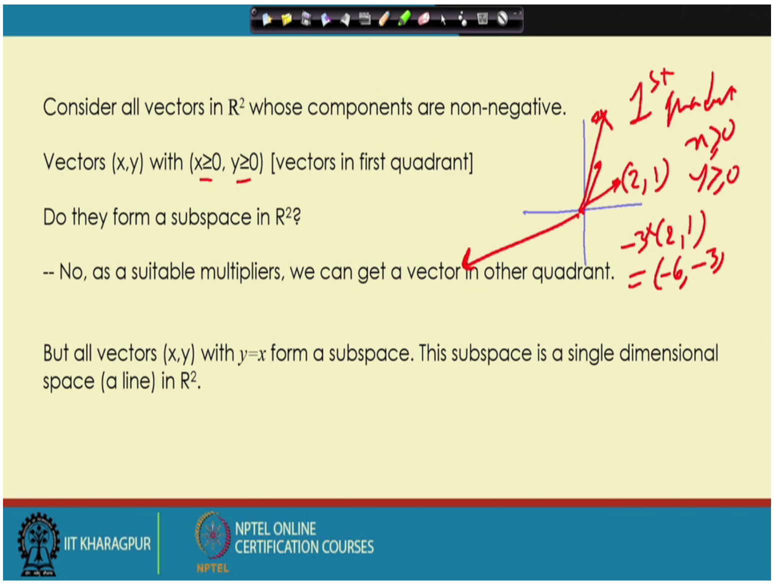

Consider all vectors in R 2 whose components are non-negative. Vectors x is greater than equal to 0; y is greater than equal to 0. So, any vector in the first quadrant; so, we consider all the vectors that will belong to the first quadrant; x is greater than 0; y is greater than equal to 0, we include the origin also by writing greater than equal to 0.

So, all this vectors; Do they form a subspace in R 2? If we thi multiply say we take a vector 2 1 and then, multiply it with minus 3; minus 3 into 2 1. So, that that will be minus 6 into minus 3 which is something here, which is not lying in the first quadrant.

So, if I take the first quadrant, we test that whether all vectors in first quadrant and forming a subspace in R 2. No, if we can multiply a vector with a negative number and it will come out of the first quadrant. So, this cannot be a subspace. A suitable multiply or we can get a vector in another quadrant. So, we can linearly combine 2 vectors and get multiply 1 with negative number and can get a vector in the third quadrant fourth quadrant and so on.

But all vectors in x y; So, with y is equal to x form a subspace and again why is it so?

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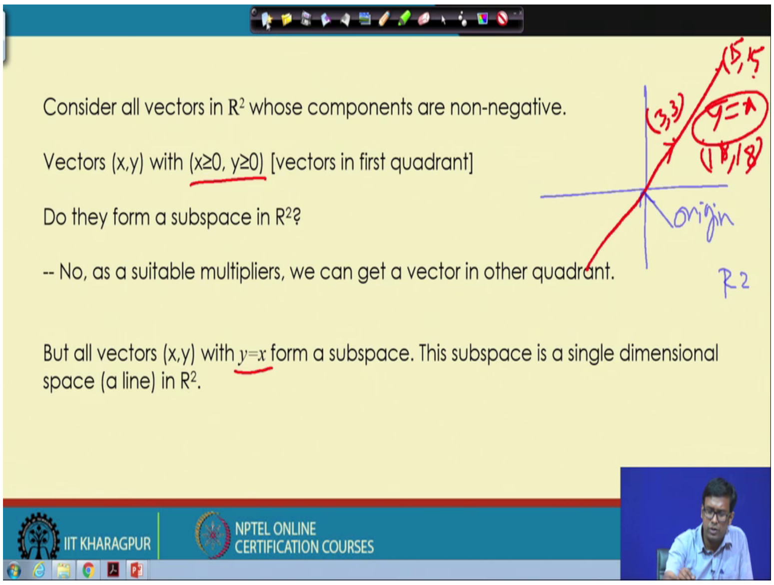

So if we think of the R 2 plane and this is the line, I told earlier that line passing through (Refer Time: 21:14) origin is a subspace. So, any vector this is y is equal to x; any vector 3 3 can be multiplied with any number and that will be say 15 15 still a vector with y is equal to x or if I add 3 3 and 15 15, I will get 18 18 which is again as the same y is equal to x criteria. So, this criteria is satisfied when we do linear combination or multiplication or addition of vectors.

The criteria satisfies here, but the criteria x greater than 0; y greater than 0 does not satisfy oh when we do a linear combination of the vectors in the first 2 example. Therefore, the this becomes a subspace.

The second one y is equal to x becomes a subspace and also it is important to see that this is passing through origin. Now, the third question if I think of all vectors pointing to a parabola; all vector with y is equal to x square do they form a subspace? So, we can probably try to see.

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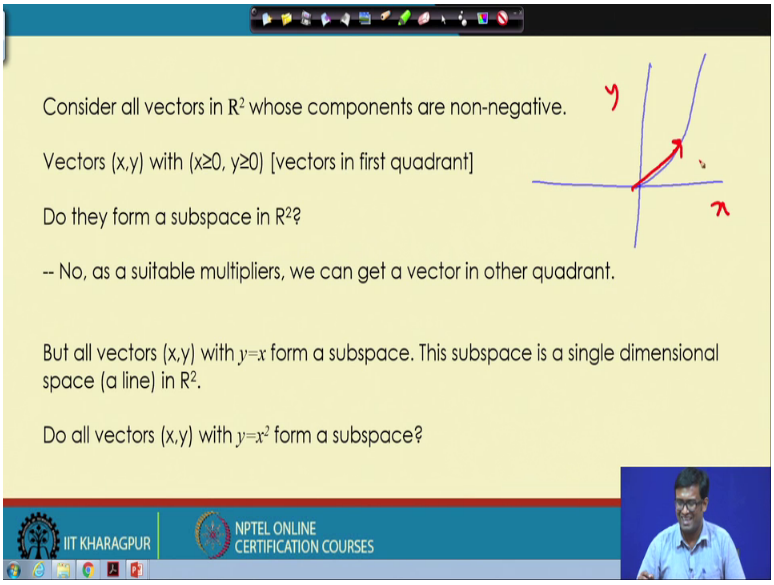

So, this is y is equal to x square. Now, I will think about any vector pointing here. So, this is x and this is y. So, this is this does not become y is equal to x square sorry. So, I will draw it again; y is equal to. So, y will be more than x y is equal to x square.

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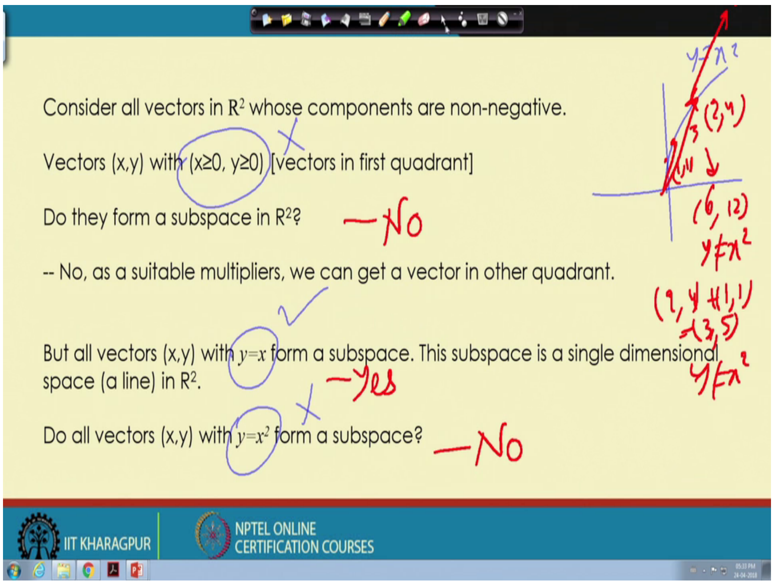

So, let us assume it to be this. So, this is 2 4. A vector here which is pointing to the location 2 4; the vector is 2 y plus 4. Now, if I multiply this with say 3; we get 6. The resultant vector will be 3, thrice increase this 6 12 and we can say that this is coming out of the parabola also; y is not is equal to x square.

So, or we can add 2 4 with 1 1. 1 1 must be in the parabola, 1 1 somewhere here. The resultant 2 4 plus 1 1, the resultant is equal to 3 5 that is also not in the parabola y is not is equal to x square. So, the answer is no. If we multiply a vector with another constant or if we add 2 vectors, it does not satisfy y is equal to x square. Therefore, vector x y with y is equal to x square will not be a subspace also.

So, this is important that they have to go through origin that is 2. Along with going through the origin, they have to also satisfy the criteria that if we add 2 vectors that will follow the same criteria through which we are defining the subspace or if we multiply the vectors they will follow the same criteria in through which we are defining the subspace. And this criteria is not satisfied by resultant in the first and third case; where, it is satisfied in the second case and that is why this is an yes case and these two are the no cases.

So, I will keep on exploring it and now one important thing is that what are the types of subspace we can have? The subspaces will have a dimension less than the actual dimension of umm dimension of the actual real coordinate space of the real coordinate space or maybe it is equal to the dimension also. You see what can be the different dimensions or different subspaces in that way.

For example, we can see that a line can be a subspace right; line passing through the origin can give subspace in R 2. A plane can be a subspace in R 3. Also line can be a subspace in R 3. So, what are the different types of subspaces?

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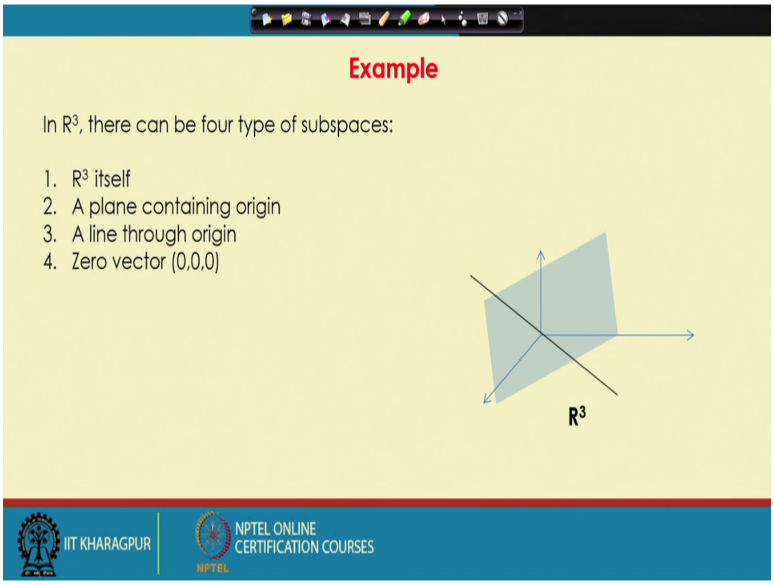

In R 3, there can be 4 types of subspaces. R 3 itself is a subspace; anything belonging to R 3, we add it we will get R 3 vector belonging to R 3 or if we multiply it, will get a vector in R 3 only. So, R 3 itself is a subspace. R 3 itself is a subspace any vector here, if we add or multiply we will get here.

Then the next subspace can be a plane containing the origin and we have seen that that any two vectors in this plane we can add this two vectors, we will get another vector in that particular plane only. We can multiply this will get a vector in that plane only.

A line passing through the origin; So, any vector along this particular line, if we add this with another vector along this particular line; this will be the resultant will be a vector on this line only or if we multiply any vector in this line with some constant, this will be only in this line.

And interestingly the origin is as which is to be noted origin is a member or zero vector 0 0 0 in R 3. 3 zeros a vec zero vector with 3 components is the member of all the subspaces and the fourth one, we have discussed it earlier also is the zero vector itself; this is also a subspace of R 3.

So, now I tried to demonstrate the definition of vector space and subspace which will be very important when we will do our next discussion which is on in a motivation behind all this thing is solving x is equal to b which is will start looking into the a matrix. What are the subspaces inside a matrix, we can find out?

We have seen that matrix can be itself a ma a mean to n th order vector, but there are other vectors inside a matrix and we can look it from other prospective also. If we remember the first few lectures, we have tried to write a matrix in terms of combination of matrix equation in terms of combination of column vectors.

So, the does this do this column vectors form a subspace also. We will try to look into it and then, we will see that what type of subspace or vector space is formed by the solution of the equation x is equal to 0 or x is equal to b. The solutions, what type of space do they form? This we will explore in the next few classes.

Thank you.