Matrix Solvers
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Lecture – 13
Introduction to Vector Space

Welcome, in last session we have discussed that solution of A x is equal to b in the cases where there is no unique solution lie in a vector space. And we stopped our discussion there and today will see what a vector space is? And how later we will see how this solutions lie on vector space? How are the solutions span over a particular vector space? And how can we obtain the solutions in cases where there are non unique solution, where there are especially infinite solution and our typically Gauss elimination type of solvers fail for this type of equations.

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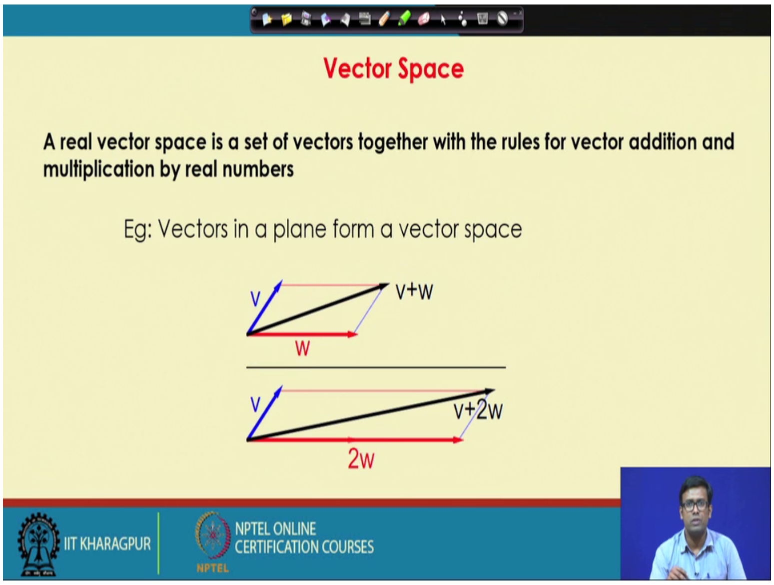

So, what is a vector space? A real vector space is a set of vectors together with the rules of vector addition and multiplication by real numbers; that means that there will be a number of vectors and we can add those factors, we can multiply a vector with a coefficient. And then also we can add the vectors multiplied by a coefficient with another vector and the resultant vectors will also lie on that particular space.

So if we see how is it? For example, if we see vectors in a plane, they form a vector space in order to explain that we can say that there are 2 vectors v and w and we say that they lie in a plane. So, if we do an addition v plus w that will also lie on a plane and so, this plane is now a vector space, where all the vectors are lying; also if I multiply w by 2 and we get 2 w. This vector 2 w when added with v; this is 2 w itself lying in the same plane and v plus w is also lying on the plane. So, all vectors in a which are go planar it can be said it can be said that they belong to one particular vector space; we will see this in more detail.

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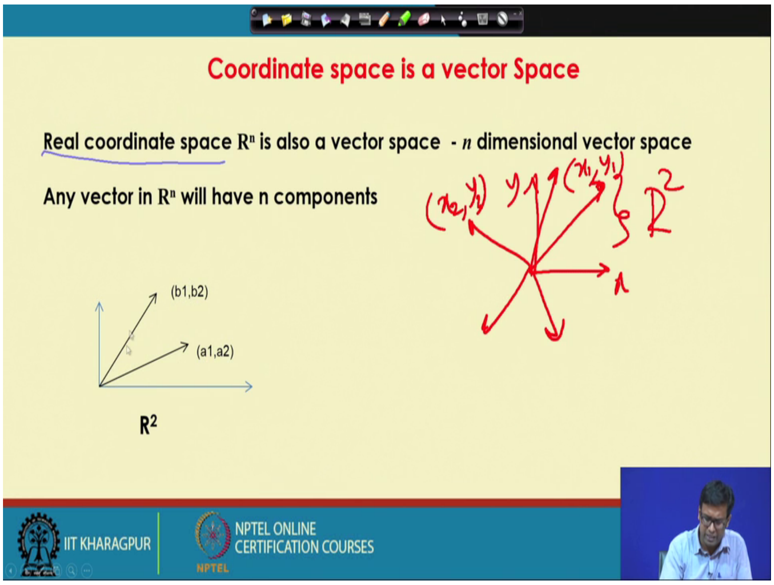

Coordinate space is a vector space and this is something interesting and something more real life like. Coordinate space is space span by the coordinates x y or xyz in 3 dimensional or the former was in 2 dimensional case. So, if we think a vector which can be expressed in an which lies in an x y plane. So, it has a x component and has an y component.

So, this is a vector like that and all the vectors lying in x y plane; they belong to a vector space which is the coordinate space and a coordinate space or the real coordinate space. Because there is no imaginary coordinate here all the distances measured along a axis in the coordinate space is a real distance. So, it is given as R with the superscript n and this is called an n dimensional space when there are n components or n of the vector or n dimensions expressed by the coordinate plane.

So, if we see a coordinate plane by x y only this coordinate place space gives me a vector space which is R 2 because this is a 2 dimensional space. Similarly we can think of a 3 dimensional space any vector in R n will have n components. So, any vector or before coming into it any we can see any vector here. For example, this vector is given by x 1 y 1 or x 1; I plus y 1 gets something like that or a vector here can be given as x 2 and y 2.

So, any vector in a 2 D space will have 2 components; so, any vector in n th order coordinate space will have n component then coordinate space is a vector space. So, if I look into a 3 D vector sorry this is a if I look into a 2 D vector in R 2; this is a 1 a 2, b 1 b 2 these are the vectors and all these vectors will give us a coordinate a vector space which is the coordinate space.

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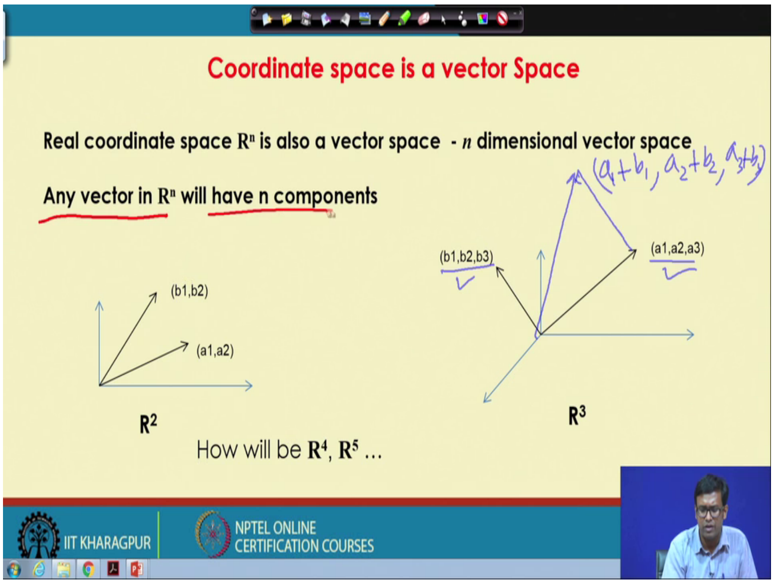

And if we look into a 3 D vector or 3 D coordinate space; this is there will be 3 axis because it is a 3 dimensional plane. And any vector will be in that particular 3 dimensional plane or in that coordinate space which is R 3; 3 dimensional coordinate space and any of this vector will have 3 components, I will require 3 components to express any of this vector.

Now, why this is a vector space? Because we can add these 2 vectors and get a resultant vector which will also lie on the same coordinate space or same vector space. A vector with 3 components can only be added with a vector with 3 components and the resultant will be another vector with 3 components; so they lie in the same vector space. Also if I multiply a vector; it is just magnifying the vector in the same coordinate space and it lies in the same coordinate space, it is same vector space.

So, coordinate spaces or the physical geometries with given axis x axis, y axis or R axis, theta axis or theta phi axis z axis or x y z axis this spaces 2 D spaces or 3 D spaces what we can call (Refer Time: 06:02) it physically or vector spaces. Because any vector in a 2 D space can be added to an another vector in 2 D space and I will get another vector in 2 D space only.

So, what is important in defining a vector space; we can also get that that if I have a vector with 3 components, it can be added with another vector with 3 components and this addition if we add them. So, the addition will be a 1 plus b 1; a 2 plus b 2, a 3 plus b 3 that should also have 3 components. It sounds very trivial and kind of an that we can very really accept it that 3 dimensional things can only be added to 3 dimensional things.

However, this is the basic a definition of a vector space that if we add 2 vectors with they should have same components and the resultant same; they should have same if we add 2 vectors, they should have same number of components and the resultant will also have similar number of components and the space in which the first 2 vectors are lying; that resultant vector should also lie there; in 2 D and 3 D these are apparent.

The problem or the more difficulty in perception will probably down when we go for third dimensional spaces. And we also have to see why we are we doing vector space how is it related with matrix algebra? We will come into it later. So, the question becomes 2 D and 3 D we can understand, even 1 D we can probably also understand that its if we move only along x distance; we may get distance only along with its distance that also gives me the coordinate space which we are now calling as a vector space.

That we can add vectors in that space, we can multiply a vector by another constant and then get another vector in that space only. But how will be R 4 or R 5 or even higher dimensional spaces which we really cannot perceive using our geometric visualization capacity; we cannot perceive what is the 4 dimensional space 5 dimensional space or even R dimensional spaces.

So, how will they be? But we can probably get an idea what say is that any vector in n dimensional coordinate space which will have n components. And what is what lies in coordinate space anything till infinity; can lie in coordinate space. I can have a line infinitely long and same 2 D plane 2 dimensional coordinate space or R 2, it is tends something which x and y both tends to infinity.

Similarly, I can have anything in R 3; any 3 be really 3 dimensional geometry any 3 dimensional vector or length can be translated into a vector any length in 3 dimensional geometry can be translated into a vector which is belonging to R 3. So, now, question comes that what will be R 4 and R 5? And we will start to looking to higher dimensional vector space.

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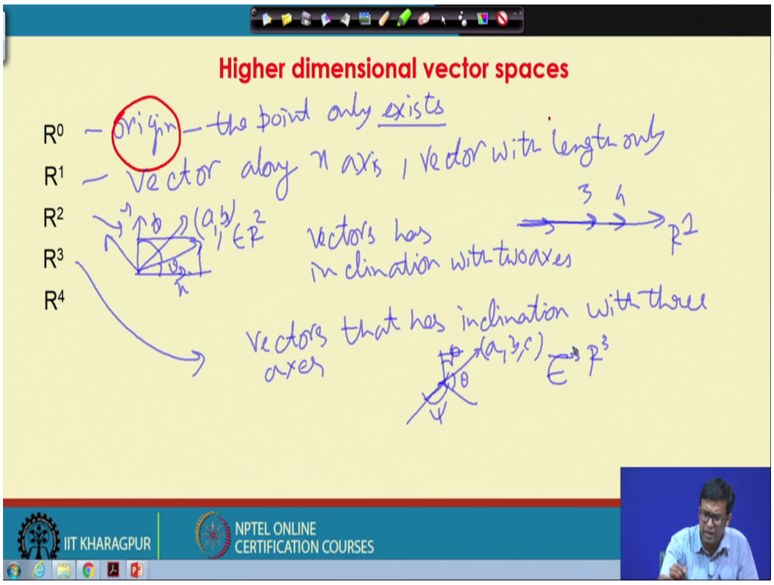

We will start with what is zeroth dimensional vector space? What is 0 dimension? 0 dimension means we set a point has a 0 dimension what is the point? It has no length, it has no orientation; it only has its presence.

So, zeroth dimensional vector space or R 0 can be the origin only it is a point; the point only exists, it is mere existence of a point which can be thought of a vector space of 0 dimension. Why it should be a vector space? Because a point if we add another if we multiply with something, it will still remain a point it has no dimension its length is 0; it cannot be increased.

Also if we add a point with a point on that same location I draw point, I just add another point on this is this computer screen is not working, but I can have a point here I can add another point over this and it will be the same point. So, it is also a vector space I can multiply and I can do linear combinations here.

Now, this is zeroth dimensional space which is origin or merely a just a point what will be R 1? It is single dimensional vector space we call you can think of a vector along x axis or vector with link only. It has no inclination, no orientation it I consider there is only x axis and any vector along this is a is R 1; why? Because if I try to express this vector I only have to write only one component say 3; it is 3 meter or 3 centimeter or 4.

This should be this only one component should be able to express the entire vector because I have only measuring along one particular axis. Now if we see R 2 and very in the real life like we do lot 2 D approximations in many cases and this is x axis and this is y axis. And any of this vectors belong to all this vectors are belong to R 2; so, vectors which are member of vector space R 2 or cordial coordinate space R 2 has inclination with two axis.

It has an inclination along x axis sum theta it has sum inclination phi along x axis of course, theta plus phi is 90 degree. And we can get a link along x axis, we can also get an length along y axis we can express this vector as a 1, b 1 where a 1 is along x axis, b 1 y axis. So, the components required to express this vector is 2 when it is belongs to R 2.

Now, let us see what is in R 3 and that is again can write it down very clearly vectors that has inclination with 3 axis. So, for example, if I have a vector like this; this has some inclination here, some inclination here, some inclination here. And it has some length along each of the coordinate axis; we can express is that a b c; where this is basically a i; bj plus c k. And therefore, anything belonging to R 3 should have which belongs to R 3, this belongs to R 3 should have 3 components.

Now anything belonging to R 2 has 2 component, anything belonging to R 3 has 3 components, anything belonging to R 4 should have 4 components. And how will it be? How to perceive a vector which is a member of a 4 dimensional real coordinate space or what can be a 4 dimensional real coordinate space? We can think of something; this is say it is this we cannot visually perceive it, but in we can try to think how to or we can try to perceive it in a different way correlating with our visual perceptions.

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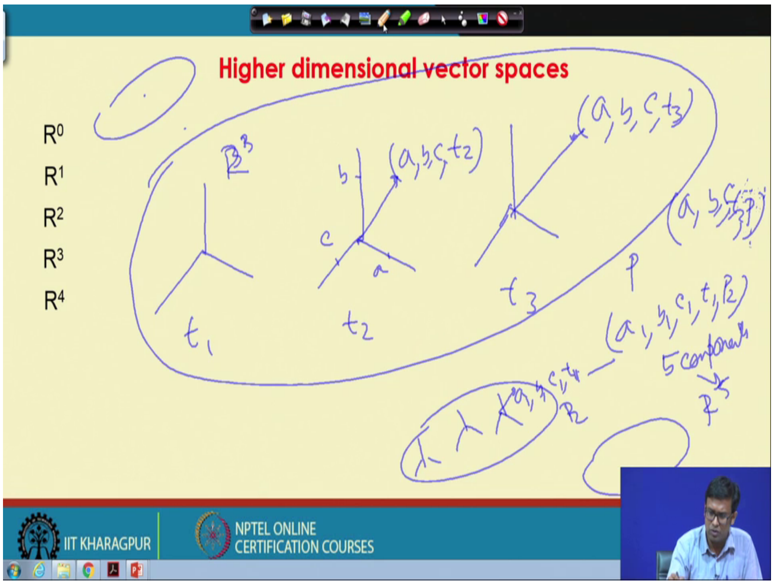

So, let us think of something this is an R 3 space; R 3 space and any vector can belong this. And now this R 3 spaces changing in time and the best example of 4 dimension is time all know even from our science friction studies.

So, these are this space will change in time, but there is another R 3 space here that will further change in time; so with time continuously this space is changing. Now at one particular time; I see a particular vector with orientation a here with certain inclination and intercept a along x axis, b along say b along y axis and c along z axis which is a b c. Now this is t 1, this is t 2, this is t 3 and it does not exists there it only exists in t 2 only.

So, will say that it has this 4 coordinates this a b c is existent only when t is equal to t 2 at from t is equal to 0 of t is equal to t 2 we can see that. So, this is a member of a 4 dimensional vector space; if we see anything which is a t 3; a b c t 3 which is some vector with same orientation along a x y z axis that is the different vector; however, both of them are the member of the same vector space.

So, now we can get 4 dimensional vector space where there are 4 components to express a vector; what is its length along 3 coordinate axis. And probably it is the time lapsed when till at which we are observing it, what can be a fifth dimensional vector space? So, for example, this is there are we consider that number of planets are waiting around a particular star.

And this is one particular planet where at one particular time we see this, there are some other planets also, but we are not seeing them in those planets, we are not seeing this particular vector in any planet we will see it only at one particular planet P. So, in the first observation this vector is not there, the second observation probably not there similarly when we come to Pth observation; this vector is there.

So, we now when we say that this particular vector will have a, b, c, t 2 P; 5 component sorry t 2; t we need 5 components to express these vector; if I get similarly there are with time the coordinate vector frames are changing here. At one particular coordinate frame we see a 1, b 1, c 1 at time t say t 1 we get something and this is the observation P 2. So, this will have a the vector will be given by a component a 1, b 1, c 1, t 1, P 2 there will be 5 components and this will be in R 5.

Now, I can add this vector with this vector and say that there is a resultant vector; which probably goes to a different observation a different time and the distances along x y z axis are different. So, so on we can think of another set of another star actually; so, this is planet of one particular star. Now we can say consider a collection of star or the galaxy and one star of the galaxy gives one particular observation of a b c t and P and this studies says.

So, a b c t P s there will be 6 components needed to express that particular observation or particular vector which we are observing there. Therefore, it becomes a member of R 6; so, we can keep on increasing the components of vector and the real coordinate space will increase in its dimension or the vice versa. We can think of real coordinate space with one more dimension, one more coordinate to observe and the vector in that coordinate space will have one more component.

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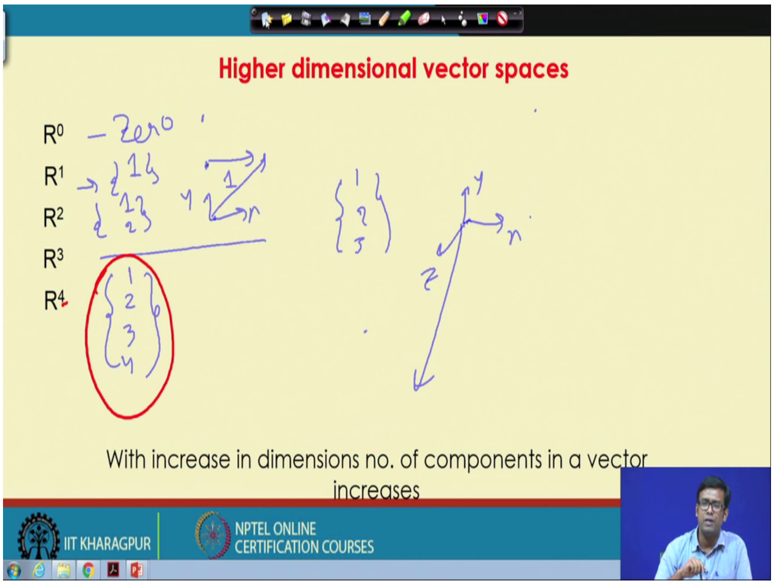

So, what we can write with increase in dimensions of the vector space or of the real coordinate space; the number of components in the vector increases. And that is how we start from our 0, where there is no component, we basically know it is the origin, it is the 0 vector and we can easily call it to be a 0 vector, we do not have to even; we cannot have to write any component any component here any number here.

But after that we can say if we can write one number and it is a member of R 1, we can write 2 numbers it is a member of R 2. So, this means a length from origin this is the origin from origin a length 1; this is means from origin a length 1 along x axis and 2 along y axis; so, a vector like this. Here it we can write 1 2 3; so, it will mean. So, this is x and y could be write that x y and z.

So, 1 along x, 2 along y and 3 along z; so, something like this a vector like; similarly in R 4 we have to write one more component extra. So, as we are increasing the dimension of the vector space the column vector. So, this is this if we see from matrix point of view this what we are writing as a vector is nothing, but a column vector, we are writing it along a column with only one column and the number of rows is basically the number of component which relates with the dimension of the coordinate space. So, with increase in the dimension the number of components in a vector space will increase.

And now the next question, how long can we increase the dimension of the vector space? We can make a one more observation and increase the number of vector or ah; increase the dimension of the vector space or number of components. And that way we can probably increase not probably we can in reality increase it up to infinity. Or we can think of a vector which has infinite components; that means, at different along different axis; the vector will have different values and this axis are infinite how can it be?

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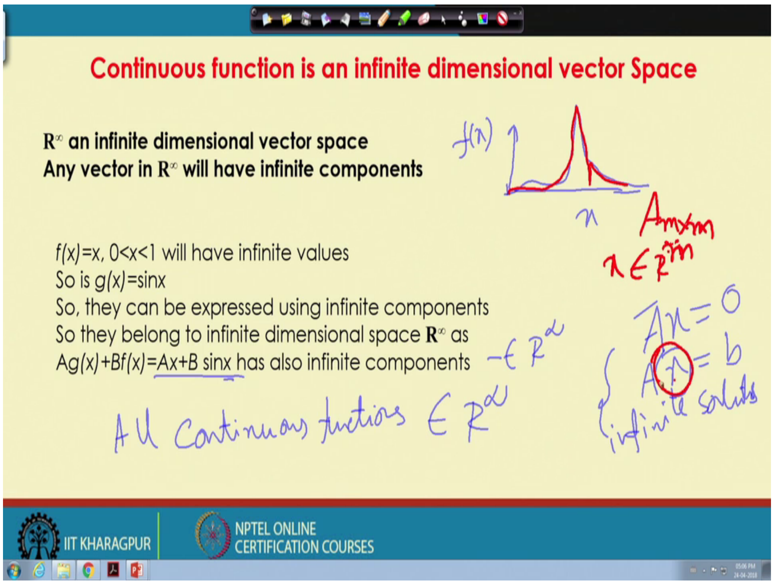

And now we think of a infinite dimensional vector space which is the continuous function.

Why a continuous function is an infinite dimensional vector space? R infinity we define it to be an infinite dimensional vector space and any vector in R; R infinity will have infinite components. So, if we can think of the function f x x is equal to x lies between 0 to 1; how many points are within 0 to 1 range? Infinite points; it is a continuous function.

And each point it will have a different values or specific values. Therefore, it will have infinite values; if I think of a function sorry if I think of a function x and this is f of x say if this is f x is equal to x, but any f x will give me if I think of some f x which is like this.

Now, if at there are infinite points; so, if at any point I change the value; I will get a different function may be it is following the same pattern, but here the value is changed. So, it is a different function; so, you can get a particular vector here which is given by f x has infinite degree of freedom or at infinite points is values are specified. So, we get a particular f x; now f x is equal to x this is the function which has the f x has the value same as x, but at infinite points the values is same as x at any point if it is different than x then it is not is equal to x.

Now, is it a member of a vector space? So, we can express it as a vector because if I write this infinite values the; if I if I right to write all the values of f x I have to really write a vector with infinite components. Is it a member of a vector space? So, we can think of another vector g x is equal to sin x also at there are infinite x points they are they are g axis infinite values.

Now, if I add the both of them can be used using infinite components; if I add they belong to infinite dimension vector space A g x plus B f x which is A x plus B sinx this is also function and has an infinite components. If I I have infinite points describing x for each point, this particular function also has a distinct value. So, this also becomes a member of R infinity and therefore, a continuous function continuous functions R vectors with infinite components and their member of infinite dimensional vector space.

So, a continuous function all continuous functions belongs to R infinity. And now we will start another discussion what is what is sub space of a vector space? Vector space what we are seeing are coordinate spaces are vector spaces and they can have infinite members there. And they only say that what is the dimension of the vector? Based on the dimension of the vector, we get what rather what is the dimension of the coordinate space.

Based on the coordinate space dimension of physical space dimension; we will say what are the number of components in a vector. And if the number of components matches with the dimension of the vector space, then we will say that this vector particular vector belongs to nth dimensional coordinate space or R n. Now, we are interested in some specific properties of vector space why are we interested in that?

Because we started discussing with the solutions of the equations A x sorry A x is equal to 0 or A x is equal to b when there are infinite solutions. Now if these infinite solutions belong to certain vector space and we will be interested to see how is the vector space? But that cannot be R infinity; if it belongs to R infinity. Then if x has say 10 components or R infinity that should not be R n, if x has 10 components and I say x is equal to b has a solution which belongs to R 10 it definitely belongs to R 10.

But then it can be any vector in R 10 it does not tell me anything about the nature of the solution. If I can say that A x the A x; A x is equal to b as a multiple solution and all this solutions of course, they are member if R 10, but they follow a particular criteria, they belong to particular sub space inside R 10 that will be of interest to me. Then I can identify the nature of the solutions in that particular equation like I have A x is equal to b and how will be x? That becomes my question.

So, if A is A mean 2 n x should belong to there will be total m rows and n columns ah. So, x should belongs to R m; x should belong to R n rather let us do it like this if A is equal to m into n, A has an order m into n; x must be belong to R m. But R m is collection of all vectors in that particular vector space. So, where x is belonging, what is it particular nature of x when I am writing x is equal to b that interest me more.

So, what we should start looking into? Is what is a subspace? Vectors coordinate real coordinate space or a larger vector space or universal vector space has all vectors of that particular dimension in that in that R n, but what is the smaller space or sub space to which the solution of A x is equal to b lies? And we will go to the next definition called what is sub space of a vector space?.

And sub space has the name suggest it is a; it is a smaller sub set of that vector space; sub space of a vector space is a non empty sub set that satisfies the requirement for the vector space, linear combination stay in the sub space. So, it is it is also vector space, but it is not the real coordinate space, it is the smaller part of the real coordinate space. And if we can do linear combination like add 2 vectors; that will also lie on that smaller part of the vector space or if I multiply a vector space with another that will also lie there.

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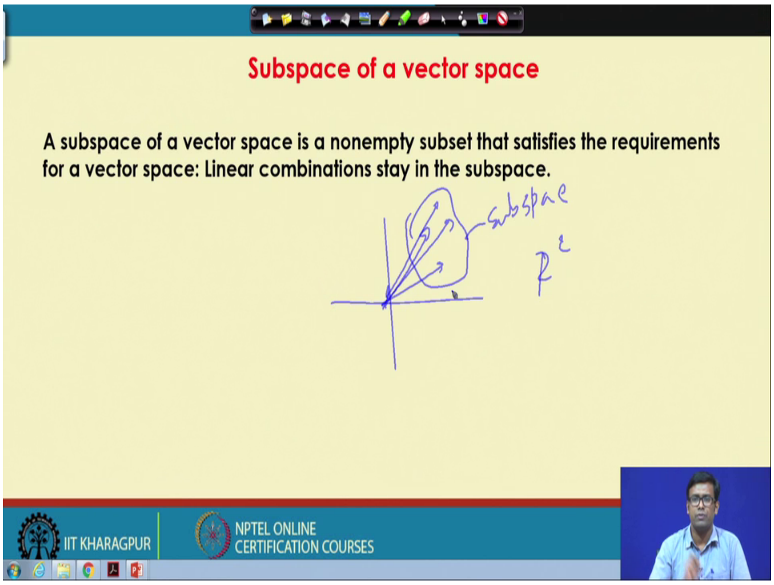

For example, we can think of a think of this is R 2 and we can think that this is a sub space here. So, if I take a vector and multiply it with some constant; this will also lie in that particular sub space how will it happen? That we will have to see or if I take a vector here and add it with another vector; the resultant should also stay in that sub space. And then we call that sub space of R 2 and we will see eventually that all this solution A x is equal b in case of infinite solutions lie in a particular sub space, we will see it in the subsequent classes.

Thank you.