Matrix Solvers

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Lecture-12
Equations with Singular Matrices

Welcome in last few lectures we have discussed about few of the direct solution techniques of matrix equations, which are Gauss elimination, Gauss Jordan method, a LU decomposition and tri diagonal matrix algorithm. One interesting thing is that all these matrices, were based on non zero pivots, the diagonal element must not be 0 and the divisions using the diagonal elements.

And these matrix solvers operated only for matrices which we call full rank matrix, or non zero determinant matrix which are unique solution. For the matrices, where there are infinite solutions Gauss elimination, Gauss Jordan, LU or TDMA all these methods will fail.

But there can be cases when there are infinite solution of a matrix and, we need some solver to solve this. So, I said a in for matrices with infinite solutions, the solvers we have discussed earlier we like starting from Gauss elimination to TDMA will not worked.

So, you have to see how we can solve matrices with infinite solutions and, the subsequent lectures few next couple of weeks lectures will be focusing on matrices which will have infinite solutions. How to solve them, how what will be their solutions, we right now we do not have any idea, how will this infinite solution look like expect for very 2 2 2 by 2 or 3 by 3 system, which you can solved by hand and say that the solution will be something like that we do not have any idea how will it be. So, how to solve matrices with infinite solutions, that is what are the method for that.

And also for the cases when there is no solution, what is the best possible solution for that is there is any approximate solution, which nearly satisfies the matrices which have no solution. So, we look in to the cases where there is no new unique solution of matrix equations infinite solution, or no solution.

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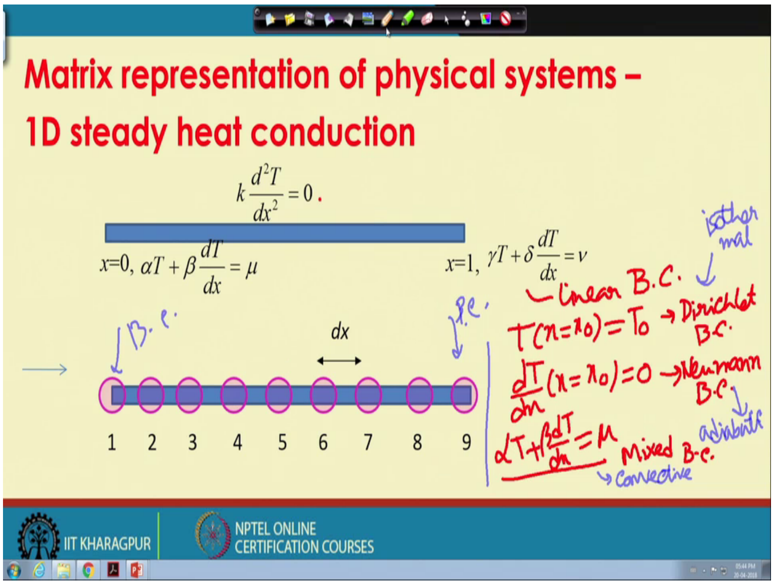

We tried to make get an idea from physical problem equivalence of the matrix all find. So, you go back to our favorite heat conduction equation, which is k del 2 T del x square d 2 T d x square is equal to 0 in a single dimensional space, x is equal to 0 the boundary condition is temperature, say I have constant multiplied with temperature with the constant multiplied temperature gradient is a constant.

Similarly x equals to 1 gamma T plus delta d T dx is equal to nu, why are we using this type of boundary condition. They are called linear boundary conditions, linear bound B and C for boundary condition linear basis at T at x is equal to x 0, at some point if temperature is specified to be T 0, this is known as Dirichlat boundary condition.

And we will see it is very important to have a Dirichlative boundary at least one Dirichlate boundary condition for any equations like, this in order to have unique solution. They can be boundary condition given informed d T dx at x is equal to x 0 is equal to 0, which is called Neumann boundary condition, if we see from B transfer perspective this is for an this is an basically isothermal temperature constant boundary condition and, this is adiabatic boundary condition.

If we mix these two boundary condition temperature plus temperature gradient is 0 which is T B value for a convective boundary, we can write that alpha T plus beta d T d x is equal to mu, this is mixed boundary condition and, we see this type of boundary conditions for convective boundaries. These are the only linear possible linear boundary condition apart from that, we can have something like radiative boundary condition where del delta d T d x is function of T to the powerful, but will not discussed this does not the matrix solvers we will primarily deal with this set of boundary conditions.

So, now when we transform this differential equation to the difference space are from a continuous space we go to a discrete space, we convert this as this to this cases and, we will apply the equivalent boundary condition, for the point for the boundary points 1 and 9. And now once we have done, this we will get a we have seen will get a general tri diagonal form here.

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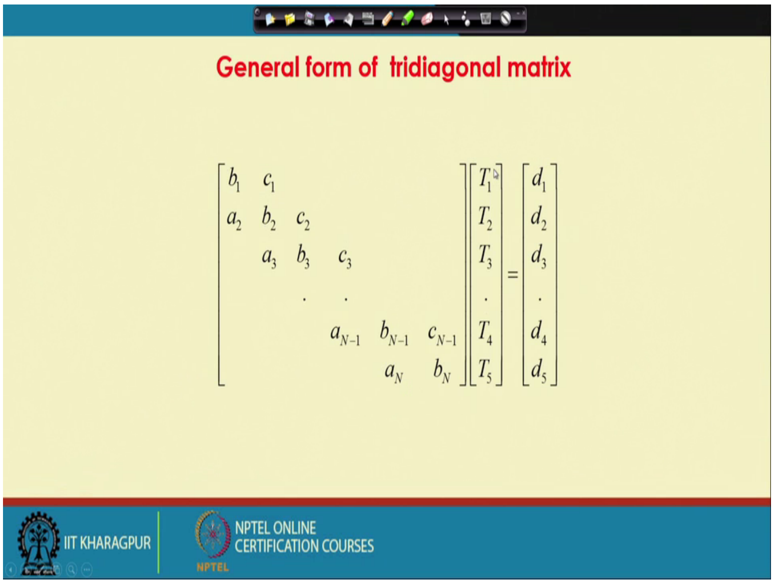

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\begin{bmatrix} b_1 & c_1 & & & & \\ a_2 & b_2 & c_2 & & & \\ & a_3 & b_3 & c_3 & & \\ & & . & . & & \\ & & & a_{N-1} & b_{N-1} & c_{N-1} \\ & & & & a_N & b_N \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ . \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ . \\ d_4 \\ d_5 \end{bmatrix}
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We have to solve this tri diagonal system of equation. Now, if you go back to the physical system of equation and do little quick in with the boundary condition use certain value on the boundary condition.

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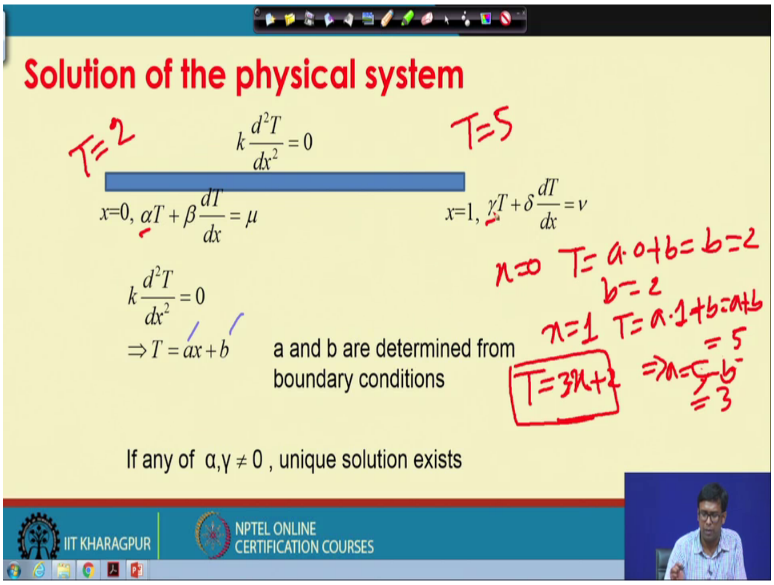

This is the general condition we have assumed we solve this equation will get d d 2 T dx square is equal to 0, which will give T is equal to a x plus b, this a and b these are the at this stage they are arbitrary unknowns. And then we will use determine a and b from boundary conditions, for a example if the given boundary condition is T is equal to 2 at x is equal to 0 and T is equal to 5 at x is equal to 1.

So, what will happen me at x is equal to 0 T is equal to a into 0 plus b is equal to b which is 2 so, b is equal to 2. And that x is equal to 1 T is equal to a into 1 plus b which is a plus b and, this is 5 b is equal to 2. So, a is equal to b minus 5 is equal to sorry a is equal to 5 minus b, 5 minus b is equal to 5 minus 2 is equal to 3.

So, we will get T is equal to 3 a plus 2 sorry, I mean (Refer Time: 06:59) T a is equal to T is equal to 3 x a x plus b which is 3 x plus 2. So, you can using boundary condition you can find out these values. And if alpha or b gamma any of them are non zero, then unique solution exist or for unique solution, you have to have either alpha, or gamma or both should be non zero, then only will have unique solution, otherwise you will not have unique solution in this particular.

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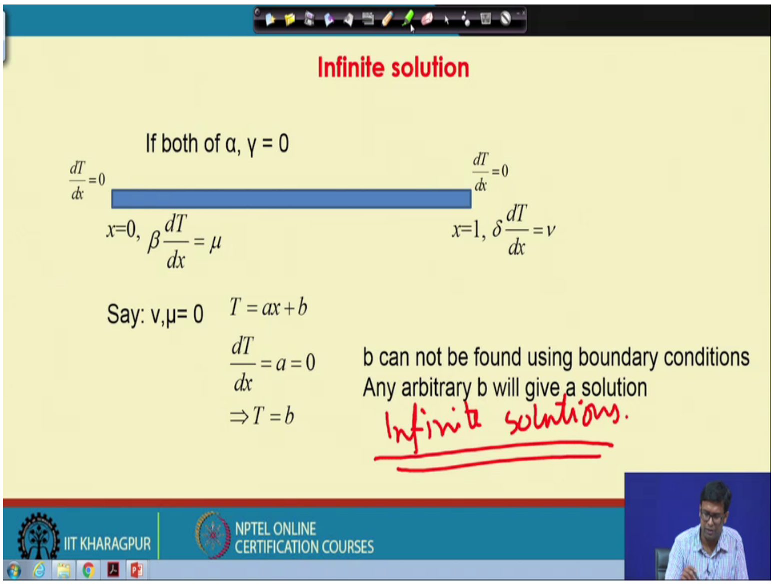

So, what can we what will you have in the case. We assume both a gamma is equal to 0 and, these are the form of the boundary condition here say, will say it both nu and mu is equal to 0 both Neumann boundary conditions, both have adiabatic boundaries. So, will get d T x is equal to 0 d T x is equal to 0 here, T is equal to a x plus b d T dx at x is equal to 0 is equal to a, which is 0 or any where d T dx is equal to 0.

So, at x is equal to 0 we put d T dx is equal to a x is equal to 0 d T dx is equal to 0 so, a is equal to 0, T is equal to b, now b is an arbitrary constant. If you do not have any Dirischlet boundary condition we cannot find b so, b cannot be found without using boundary conditions, any arbitrary b will give a solution. And there can be infinite arbitrary values of b, or we can have infinite solutions.

So, these equations and boundary condition combination, when transform to a matrix equation must also give me infinite solution because, physically there are infinite solutions and, the infinite solution is temperature is of any constant value. So, if we try to do a matrix representation here, you should get infinite solution of that particular matrix.

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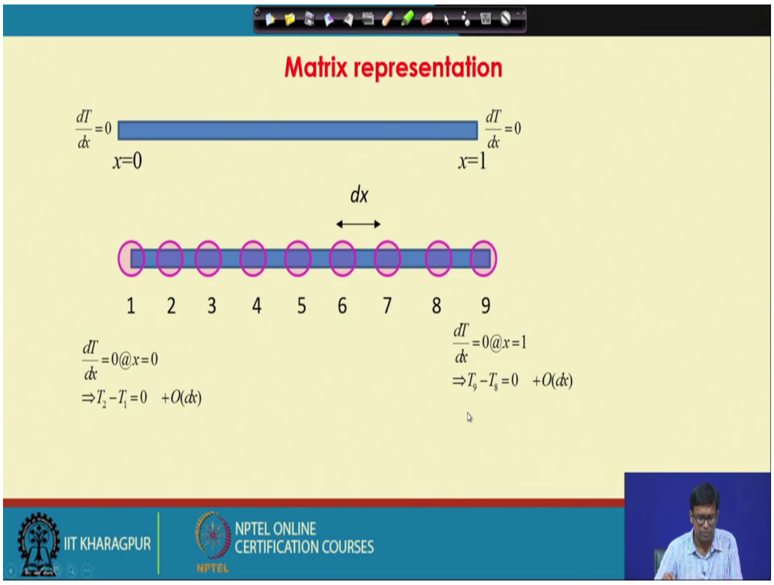

An x is equal to 0 and x is equal to 1, the boundary condition d T dx is equal to 0. So, x is equal to 0 so, d T dx is equal to 0 will be converted again do that from Taylor's. These expression, we looked into couple of classes before the T 2 minus T 1 is equal to 0 and the (Refer Time: 09:24) is not second order. It is a first order similarly for the last time we can write T 9 minus T 8 is equals to 0 eradies of the first order.

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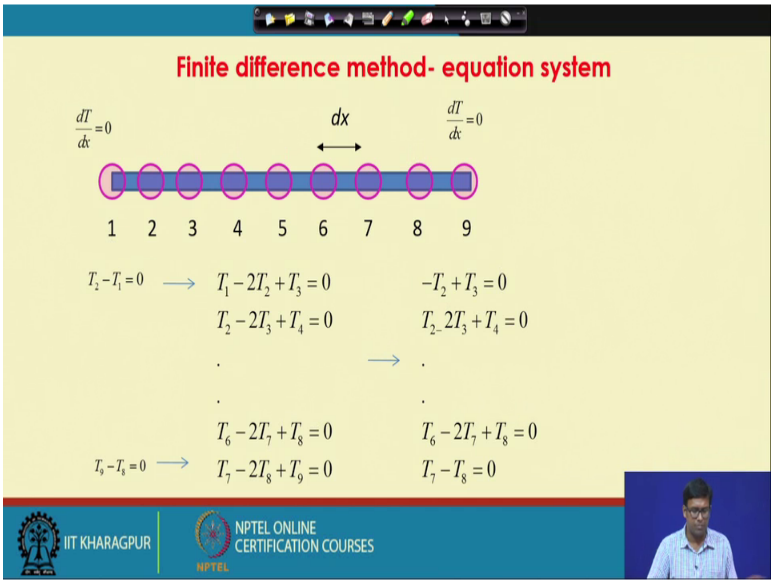

So, you can substitute name into the destitute form of equation that d T dx is equal to 0 gives me T 2 minus T 1 is equal to 0, which should come here and, which should come in to the first equation and d T dx equal to 0 at x is equal to 1 gives me T 9 minus T 8 is equal to 0, which comes in to the second equation and, the transformed equation gives minus T 2 plus T 3, 2 T two things and T 2 minus T 2, T 3 plus T 4 is equal to 0 to T 7 minus T 8 is equal to 0.

Now, if we substitute solved try to solved this equation we will get T is equal to T 2 is equal to T 8 something like that.

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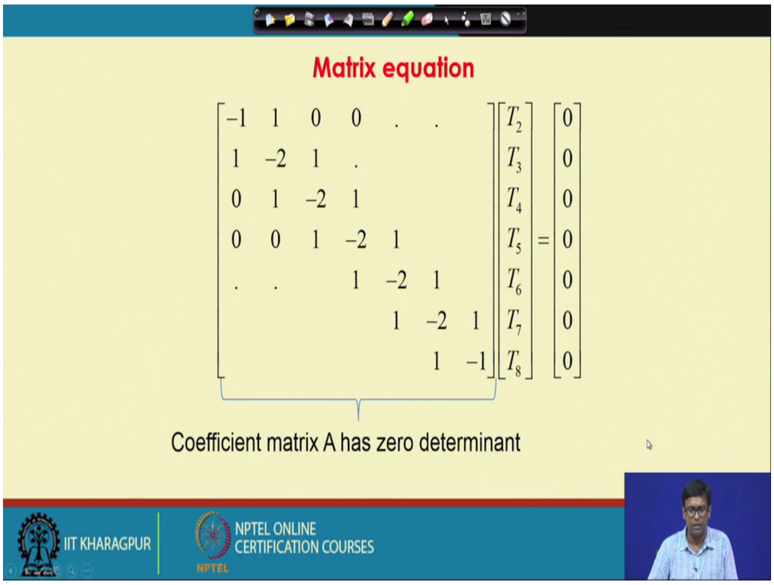

However, so there means that means, that this matrix equation must have unique of infinite solution does not have unique solution, there is the solution which there are solution we have seen from the physical problem, but there are no unique solution.

So, if we tried to solve we Gauss elimination it will fail because, the co efficient matrix is a 0 determinant and after couple of steps will see 0 diagonal coming in ; however, 0 P what coming in however, there are solutions and the Gauss elimination cannot give a solution. So, we have to see how we can get solutions, there can be cases with no solution.

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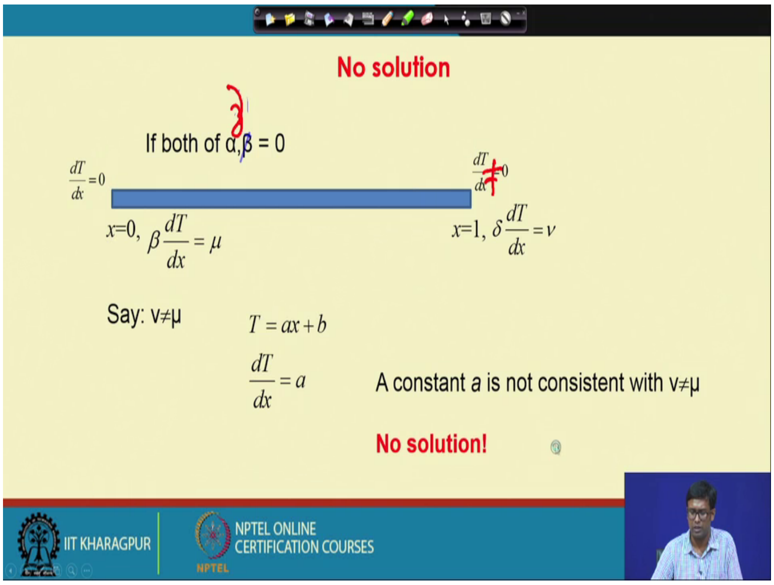

What is that we get similar case I will find sorry I will find a nu is equal to 0 beta, I will find nu is equal to 0 I will find gamma is equal to 0. So, we with the Dirichlet component I will find the gamma is equal to 0 and, we have Neumann type of equations here; however, we will see that nu is equal to mu is not is equal to nu.

So, d T dx is equal to 0 and this d T dx is non zero d T dx is not equal to 0 here, mu is not equal to 0. So, d T dx is non zero here so, we get T is equal to d T dx is equal to 0 here d T dx is non zero, here mu is not is equal to nu. We get T is equal to ax plus b d T dx is equal to a but as mu is not is equal to nu, there is no a which can be consistent with d T dx values at the both the boundaries. So, a consistent a, a constant a can not to be found which is consistent with nu not is equal to mu, boundary conditions can both boundary conditions can not to be satisfied together.

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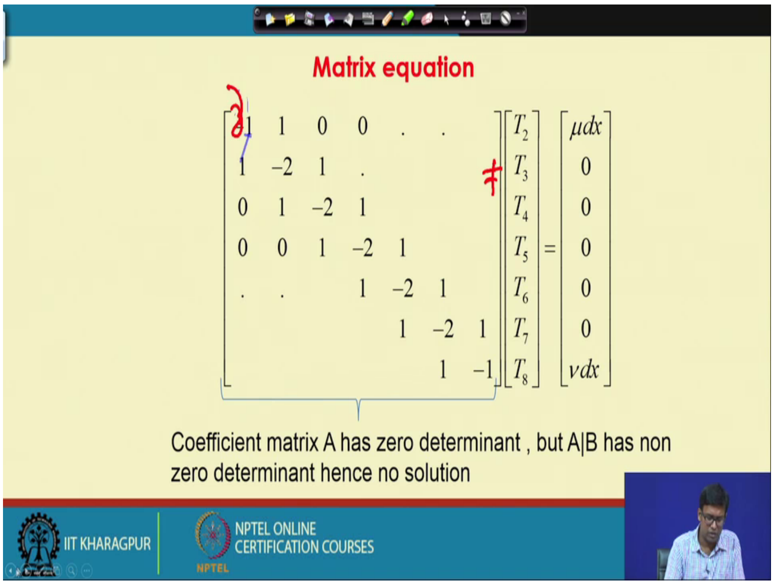

Therefore there is what we see no solution interestingly what you have seen no solution case, or infinite solution or unique solution these cases are dependent on the boundary conditions. If we have Dirichly boundary at least one Dirichly boundary condition, there must be unique solution and, that is why Dirichly boundary condition are called essential boundary conditions, in order to have you Gauss elimination working you have to essentially you have to a Dirichly boundary condition.

Here if you have no Dirichly boundary condition, there will not be unique solution it can be infinite solution if both the side has same del T del x, if that does not match there will be in you a infinite solution. There will be not infinite unique solution there will be no solution.

So, if I look into this no solution case, where nu the d T dx at both the ends are not matching, will see a matrix equation where mu is not is equal to nu and, co efficient matrix a has 0 determinant, but a augmented with b has non zero determinant and therefore, there is no solution.

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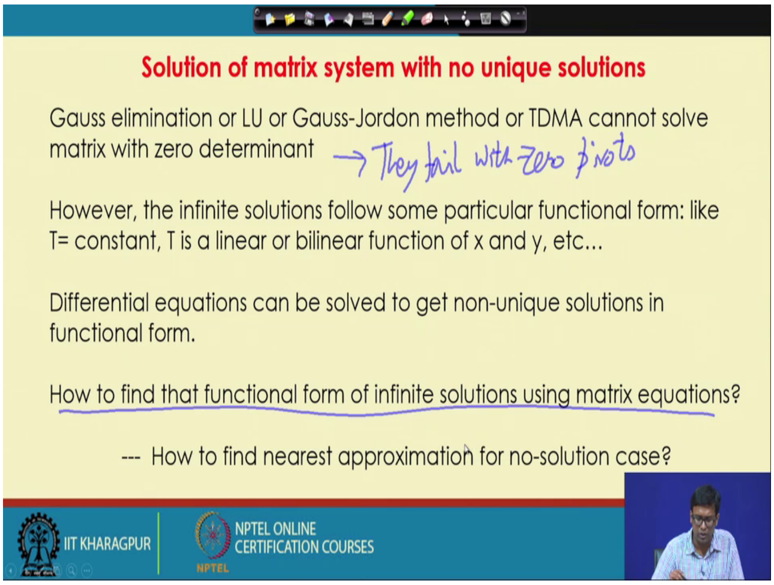

So, how can we solve matrix system with no unique solution Gauss elimination, or LU or Gauss Jordon cannot solve matrix with zero determinant, as they fail with 0 pivots. If we have anywhere pivot element 0, this solution method will fail. However, the infinite solutions follow some particular functional form like T is equal to constant. We can apt T to be a linear function of x or bilinear function of x and y for 2 D and a multi dimensional problem, or for some other problem.

So, though they are for infinite solution forget no solution, because just look into infinite solution case. Gauss elimination type of methods cannot give a solution, there is certain function will form of solution, there are infinite solution which follows certain functional form.

So, it might be of interest look into that functional form. So, it cannot do Gauss elimination, but you have to look into some or other method to get then get that particular functional form, differential equations can be solved to get non unique solution into functional form, for using differential equation and we can get that.

But difference equations you will interested to get it through difference equation and, get it through difference equations and, get it a through a discretized method. How do find finite functional form of infinite solution using matrix equation, that becomes the important question which will try to address in next few classes. And also we will be interested to look into how to find nearest approximation for the no solution case.

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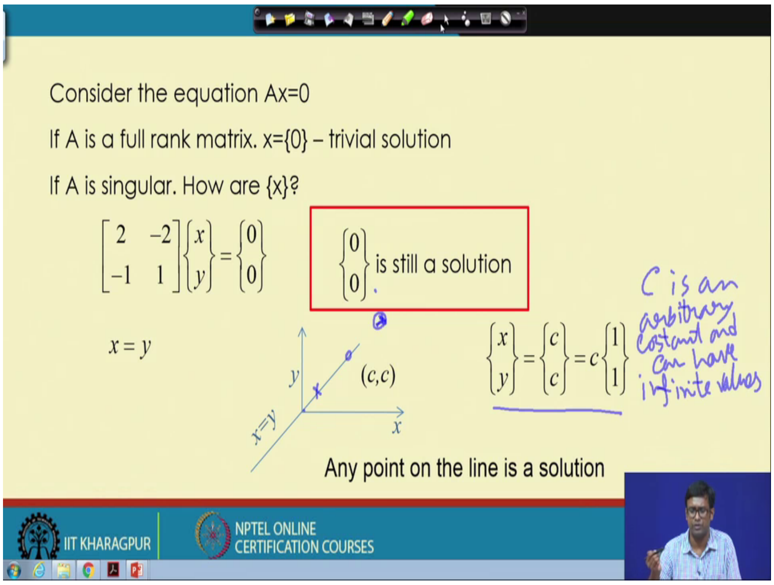

So, let us consider an equation before going into main equation x is equal to Ax is equal to 0, if A is full rank matrix x is equal to 0, which is trivial solution. If A is not of full rank matrix, A is singular how are x.

So, it take a small example minus 2, 2 minus 2 minus 1 1 this is a singular matrix x y is equal to 0, you can solve it quickly the solution is x is equal to y. So, infinite number of solution are possible x, which is x is equal to y and 0 0 is also number of x a also one solution where x is equal to y, in trivial solution is also a part of solution of a singular equation A equation. However, there are infinite solution so, on the solutions are any point on this x y line.

So, you can write any point of this line is a solution, solution 0 0 is still a also a point in this line so, it is also solution and, the solution is x y is equal to c c, c is a arbitrary constant. So, c is an arbitrary constant and, can have infinite values, for all values of c there will be a solution and therefore, it will have infinite solution 0 0 which is trivial solution for full rank matrix Ax is equal 0 is the solution on of that, but there can be infinite solution.

Because it is the solutions lying on the line and the line is collection of infinite number of points. Now, will do this is for a very simple case 2 by 2 matrix is in reality the matrices are not that is simple of much higher order, order is say 512 into 512 order is 56000 into 50000 m is not equal to n because, the infinite solutions are there we non singular matrix is there, the m and n may not be same the number of equations may be different the number of equation.

So, however so, how will be the solution, what we can see about the solution, what we can see is that if this point is the solution. So, if we stretch this point if we multiplied the coordinates by a certain factor that will be another solution. Similarly if this is the solution this is the solution if we add them that the third point will also be a solution. So, if I know one solution we can construct the other solution, if I know two solutions we can construct the third solution and we can thus.

That way if I just know one solution apart from 0 0 0 0 is a trivial solution for all the cases with 0 0 we cannot do anything, but if you know any solution other than 0 0, we can have infinite solutions just using that only. And we can have if you know 2 2 different solutions so, you can have a third solutions. So, in that way we also get number of solutions. So, how can we get difference solutions that is one important question for infinite solution case.

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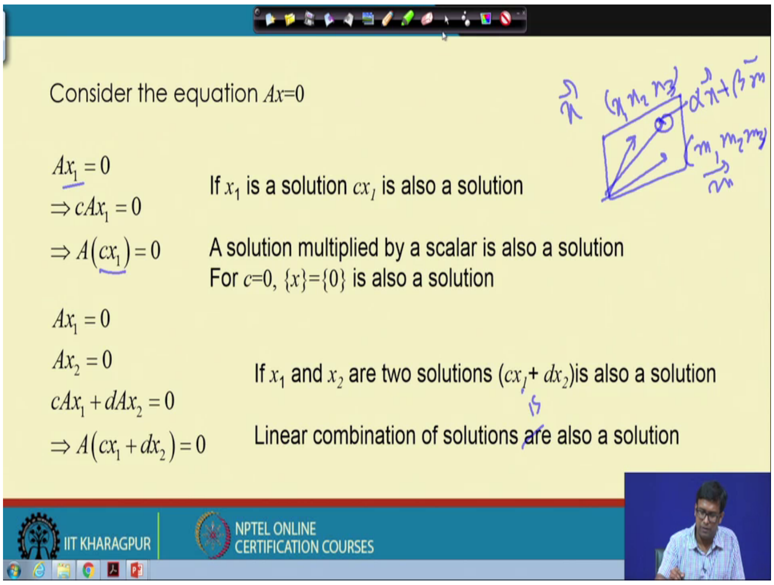

A again consider the equation A x is equal to 0, if A x 1 is equal to 0 multiplied both the part x 1 is a solution. So, this says x 1 is a solution here x 1 is equal to 0, we multiply this by c c A x 1 is equal to 0 A into c x 1 is equal to 0. So, c x 1 becomes another solution if x 1 is a solution c x 1 is equal to 0. And if we considered c is equal to 0 c x 1 will be 0.

So, a solution multiplied by a scalar is also a solution and, for c is equal to 0 x is equal to 0 becomes a solution. If we have a solution A x 1 is equal to 0 and A x 2 is equal to 0, we multiply A x 1 with c and A x 2 by d and will see a into c x 1 plus d x 2 is equal to 0. So, if x 1 and x 2 are two solutions c x 1 and plus d x 2 is also a solution. So, linear combinations of solutions are also a solution is also is also a solution, linear combination of solution is also a solution and, we can have infinite linear combination of a solution to get a solution.

So, solutions of a A x is equal to 0 system where it is there is no trivial solution, should be there are infinite solutions. And once if you know 1 solution you multiplied do with scalar that will also be a solution to the equation system. If you look two solution do linear combination of them that will be the other solution and, that way we can get some idea about how the solutions are. For example, if I am solving it in the 3 D space and I get 2 solutions, which lie in a particular plane like this is one solution x 1, x 2 x 3 which lies on the particular plane and this is one solution say m 1 m 2 m 3 which also lies in that plane.

So, with these 2 lines in that plane I can construct any other line in this plane and, so this is x vector and this is m vector. So, any other line which is alpha x plus beta m, the co ordinates of this particular point will also be a solution to the equation. And that is how if I know few basic solutions, I can construct the entire solution space here.

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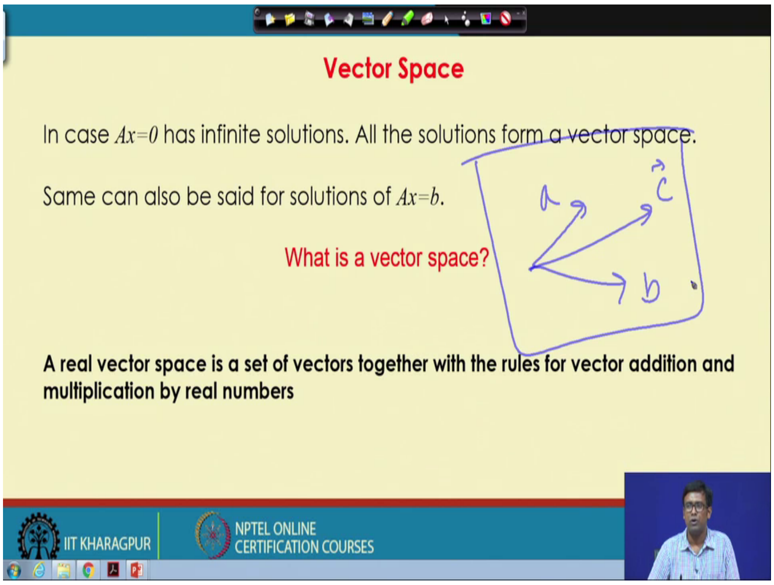

So, we come to a term called vectors space, in case A x is equal to 0 has infinite solutions all the solutions form a vector space. Also same can be said for the solution of A x is equal to b, if A x is equal to b has infinite solutions all the solution also construct the vector space.

And the question becomes what is a vector space. So, what we have seen about the solutions of A x is equal to 0 in the previous cases, that if there is one solution we multiply it by certain constant that will be a solution. If there are two solutions, or three solutions we do linear combinations that will be a solution. And a real vector space is a set of vectors, where this rules of vector addition and multiplication of real number rules.

So, if we can think of a space span by vector. So, a vector because the solutions are 1 columns, column vectors or the solutions are a matrices with only 1 columns and we can call consider them to be vectors. So, if we can think of multidimensional vectors in and this one of the one of this one vector at least which is the solution to 1 x is equal to 0 and, we multiplied with 1 constant and get a solution.

And we can get a we constant we get another solution, you think of another vector for which x is equal to 0 and, then the space constructed by these 2 vectors. For example, we just initially got 2 vectors x 1 and x 2 x 1 is equal to 0 x 2 is equal to 0, the space constructed by x 1 and x 2 by taking they are leaned combination using vector addition and multiplication. All the vectors belonging to that must be a solution of A x is equal to 0 equation.

So, it is a very I am giving a sketchy idea vectors space, but I am actually trying to introduce vectors space, vectors space is the set of vectors, which follows the laws of multiplication by real numbers and addition you, for examples if I have 2 vectors we making a certain angle between them and, we take all their linear combinations to in 3 D space we have just 2 vector so, I have a vector a.

And I have a vector b and I make all their combinations, I will get inter plain in vector line here will be a member of this, say c will be member of this vector space which (Refer Time: 24:30) the entire space. So, and if a and b gives the co ordinates of the point which are solution to one particular equations c will also be a solution to that equation.

And that is the idea we can take away from here, that solution of A x is equal to 0 and A x is equal to b, if there are infinite solution they will construct vectors space. And, we will look into what is the vector space will look in to different properties of vectors space and, from there we will try to see how we can find out solution for the cases where, there are infinite solutions both for A x is equal to 0, which is the homogeneous equation and A x is equal to b which is non homogeneous equation, how will be the solutions look like.

We will also see how we can get best solutions of most near most approximated solutions of Ax is equal to b when there are solutions, solution which will give least error. If you plug it back into the actually equation how to get that these two things we will look into next few classes. And, for that we will have some detail little little details study on vectors space, we will really need not go in to infinite dimensional vectors space imaged detailed, which are actually the continuous functions, where to look in to the finite dimensional vectors space.

And see what are the fundamental vectors spaces, what of the basis and dimensional vectors spaces, what is the linear independence and linear dependence of vectors spaces. And, how to get solution of these two particular equations A x is equal to 0 and A x is equal to b using basic vector space algebra.

Thank you.