Matrix Solvers

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Lecture – 01
Introduction to Matrix Algebra – I

Hello welcome to the class of Matrix Solvers, this is the very first class of this course, number of engineering and physical systems as well as systems involving social sciences like economics deal with large number of equations. Equations which are generated from systems which deal with a very huge number of variables, and these equations are often represented as matrix equations.

And in many times we need very first efficient solution of the matrix equations and also there are situations, when you do not need solutions of the systems of the equations rather we need to see how the system behaves just by looking in to the system of equations.

So, in this class will try to cover large group of solvers which can address the system of equations, and will also see the cases when this equation systems are solvable, what are the other issues with the equation systems and how the equation system tells about coupling between the behavior of different parameters. So, today's class is on introduction to matrix algebra this is very elementary level class will have discussion on the definitions, notations, and few of the elementary matrix operations.

The main intension behind this class is that getting the getting acquainted with the right conventions, so that throughout the entire syllabus or all the classes we be all of us been the same page on the definitions and notations use for different matrix operations. But before going in to that quickly let us see what this course will cover.

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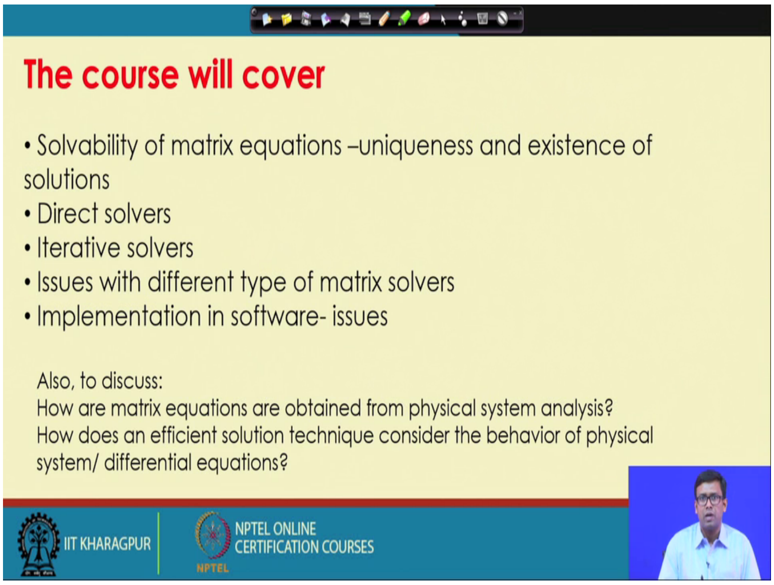

So, this course will primarily discussion these issues, first is solvability of matrix equations in which cases the matrix equation has an unique solution and there are multiple cases where the solution will not have unique solutions. One case is that the solution will not have any solution thus equation systems in not have any solutions or will call that to be a inconsistence system of equation.

And there can be another case when infinites solutions exists for given equation system. And usually this cases arise when number of variables or number of independent variables is not equal to the number of equations given to us.

And then will look into different types of solvers, one is the direct solver and another is iterative solvers. Direct solvers that as the name suggests directly solves the matrix equation gives us what is the solution in terms of values of different variable. What is iterative solvers start with a gaze value and then does not iterative technique using the matrix itself to update the gaze value. So, that is finally, converges to a right solution.

The main important thing with iterative solvers is that what is the rate of convergence of this solvers, and they are pros and cons behind each solvers all not only behind choosing direct solver or iterative solver also in choosing different types of solvers within the general class iterative solvers. So, will see what are the issues with different type of matrix solvers and will also see how to implement the solvers in software.

Because for a large equation system in today's world we many a time we solved million by million 100 million by 100 million even billion by billion matrices and when you are solving this large matrix systems. We use efficient softwares which are implemented over modern computer architecture and one goal of this class is also look into implementation of this matrix solvers in different softwares and what are issues with that.

Also will discuss how these matrix equations are obtained from different physical system analysis. And knowing that these equation, which we are solving is coming from analysis our physical system, how the behavior of the physical system can be utilized while devising an efficient matrix solvers, or the or usually this solvers in number of cases there will be were this solvers are coming as a projection of differential equations.

So, when computers cannot solve differential equations, computers do not understand what is derivatives. So, when we try to solve it differential equations using computer you convert them to difference equations and these are multi degree of freedom different difference equations which come as matrix equations. So, if you can use the very basic nature of those differential equations to solve the projected matrix solve equation.

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So, these things will also discuss at different parts of this class. The main to textbook will follow this classes one is linear algebra and it is application by professor Gilbert Strang, this is one of the very extremely sighted and refer textbook in linear algebra and matrix calculations and this book will primarily use to see the behavior of different matrices and above different matrix operations and to also to looking to the cases when a matrix equation has unique solution or has infinites solutions.

And the and also to looking into the direct solvers for the iterative solvers will mostly use the second book which is professor Yousef Saads iterative methods for sparse linear system. It is again addition was published by Siam first addition is already available online. In professor Saads web page and this equation will deal with number of iterative solvers their issues the algorithms will probably directly used from Yousef Saads books on this solvers.

So, now we move to our main discussion and today's discussion is on elementary matrix algebra the first question comes what is a matrix. And we can see matrix is a regular array of numbers or symbols or expressions and so as it is a as it is a array of numbers and symbols and expressions it has should have to component it is a rectangular arrays.

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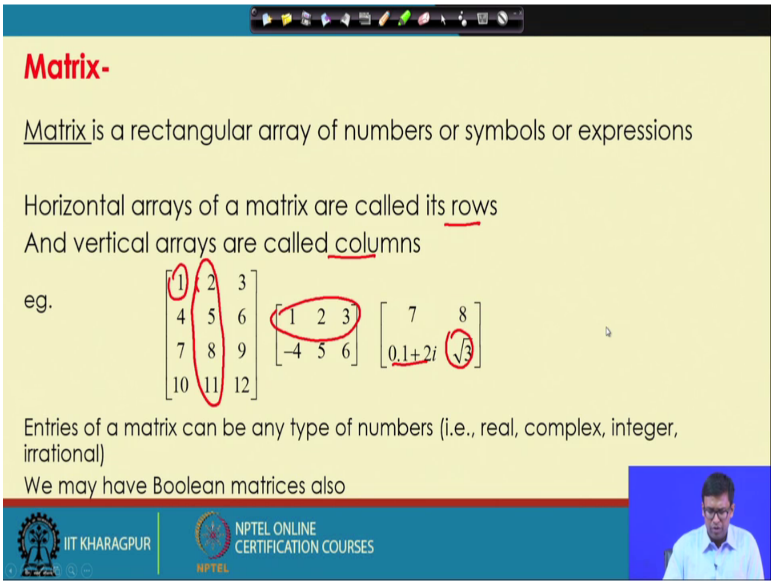

So, it should have a something in a horizontal direction as well as something in a vertical direction. The horizontal array is within a matrix are called it is rows and the vertical arrays are called it is columns.

So, if we look into few example matrices there are a 3 matrices shown here and in 3 of the matrix or each element of the matrix can be integer, can be relational number, can be any real number, can be complex number, also there are cases what these entries are logical entries or we can have Boolean matrices.

Now, if we looking to each matrix it is as I say it is a rectangular array of numbers and we can see both vertical arrays which we call columns and will also see horizontal arrays which we call rows. Now, rows and columns are also matrices even a single number can also be thought of a matrix. Now, as I said entries of a matrix can be any type of numbers, it can be real complex integer imaginary irrational and we may have Boolean matrices also.

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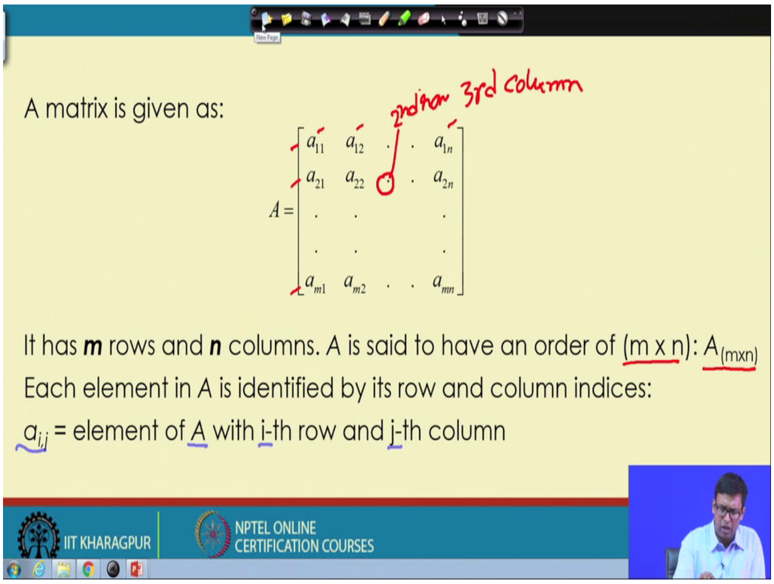

So, matrix is usually given as a rectangular array where the increase can be anything which is shown as a matrix which has number of entries inside it as directed along rows and columns it has it is it has total 1, 2, 3, 4 m number of n number of columns and 1, 2, 3, 4 n number of rows. So, this if you looking to this matrix there are total m rows 1, 2, 3 m rows and 1 2 3 n columns.

And inside this matrix any number can be identified by it is position according to row and column. For example, this number goes to the third row sorry second row this number goes to second row and 1, 2, 3 third column. So, each entry inside the matrix will have a unique identification based on it is row and column.

And we can see these matrix has m n m rows and n columns and the matrix a is said to have an order m into n and usually written when we need to show it is order also a and in subscripts m into n and each element in a is identified by it is row and column index.

So, ai j is an element small ai j is an element of the matrix capital A small ai j is an element of the matrix capital A with i th row and j th column ok. This is how an element inside the matrix is defined and the order of the matrix is also defined.

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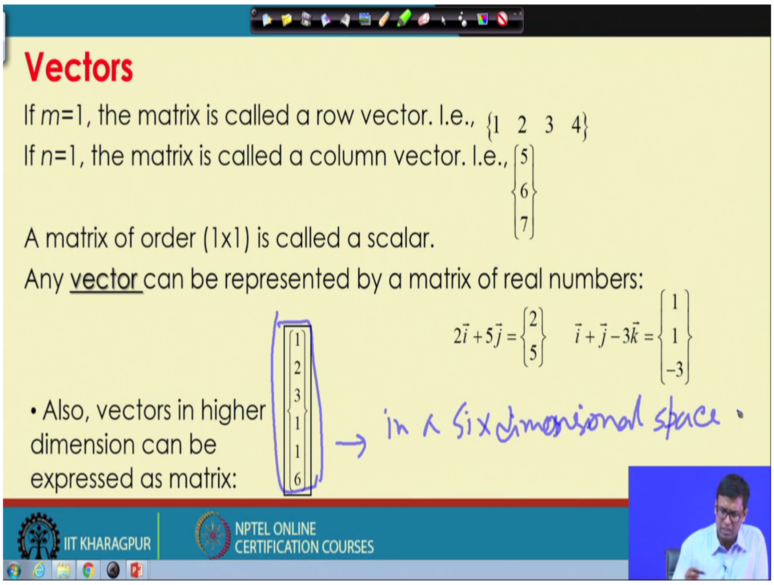

Now, 1 dimensional matrix is called a vector; that means, row is also a vector if m is equal to 1, the row will become a vector for like 1, 2, 3, 4 arranged in a 1 horizontal that is the vector, and if n is equal to 1 that is also vertical array which is called column vectors. So, there are 2 types of vectors usually row vectors and column vector and the matrix which has only 1 increase 1 into 1 that is also in rectangular arrangement if you see that is called a scalar.

Any vector can be representated by a matrix of real numbers; that means, when we talk about a vector for example, say we talk about 2 i plus 5 j which is a vector. So, in x direction there is 2 units in y direction there are 5 units and we can write it as 2 and 5 these 2 numbers similarly 3 dimensional vector can be witness column of rows size 3 or the column with or column matrix with order 3 into 1. There are 3 rows and 1 column.

So, instead of 2 D and 3 D because we are dealing with very large system of equations in several time; So, we think of higher dimensional matrices and higher dimensional spaces is not a physical space, but we imagine the physical space to have more and more dimensions after the third dimension which is of course, we cannot easily visualize by it we can think of it. So, for example, I have x, y, z and I think of another coordinate zeta that gives me 4 dimensional space similarly higher dimensional vector can also be representated as a column matrix which will have 1, 2, 3, 4, 5, 6, 6 entries.

So, this is a in a 6 dimensional space a vector in a 6 dimensional space and several times we have to work with higher dimensional spaces in matrix algebra. So, we start looking into some of the important matrices basically we look in to the definitions of this thing. So, when the notation is used will figure out that actually this type of matrix is been has been discussed.

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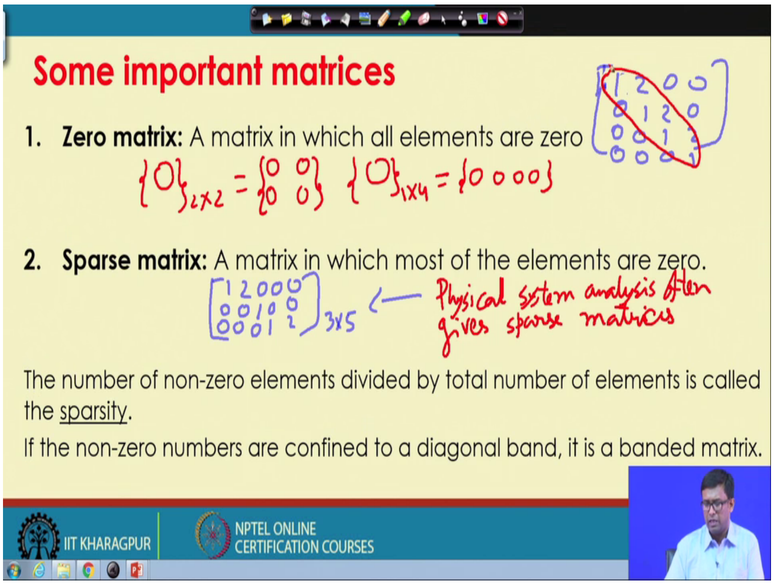

So, 1 is a 0 matrix. So, matrix which has all elements 0 and it can be matrix of any order like for example, if we write by O 2 into 2 that will be a 0 matrix which has 0 0 0 0, which is of 2 into 2. Similarly, if we write O 1 into 4 that will be this is this 1 means it has 1 rows, so it is a row vector with 4 columns, so again there are 4 0’s.

In any space vector 0 vector or a 0 specially, 0 vector in any space will denote the origin where everything all the coordinates are 0.

So, next is not a 0 vector 0 matrix are the matrix where most of the elements are most of the elements are 0. Sorry most of the elements are nonzero where there is a there I am sorry there is a mistake here. So, you should make it vector where most of the elements are 0. A sparse matrix is a matrix in which most of the elements are 0 for example, if you think of a matrix say 1, 2, 0, 0, 0 0, 0, 1, 0, 0, 0, 0, 1, 2 like this like there are 15 increased 3 into 5 of this matrix has an order which is 3 into 5 around this 15 entries only 5 are nonzero.

So, these are called sparse matrix. So, and most of the elements are 0 and this type of matrices are often arise well we are solving physical systems. So, physical systems analysis often give sparse matrices and will see sparse matrices in as a class of matrix coming from problems in computational mechanics specially in as example.

And there are specific matrix solving algorithms which are design for sparse matrices because of that there is a huge applicability and as this is a little qualitative definition that most of the elements are 0. So, quantitative definition also comes on sparsity that is a total number of nonzero elements divided by total number of elements for example, in this particular application total number of elements is 5 into 3 15 and nonzero numbers are 5.

So, sparsity will be 5 by 15, sparsity should be very low in order to have a sparse matrix and if the nonzero numbers are configured to a diagonal band then we call it to be a diagonal a banded matrix for example, if you have a matrix like this 1, 2 1, 2, 0, 0, 0, 1, 2, 0, 0, 0, 1, 2, 0, 0, 0, 1. So, the all the nonzero numbers are distributed along a band near the diagonal and this is call the banded matrix.

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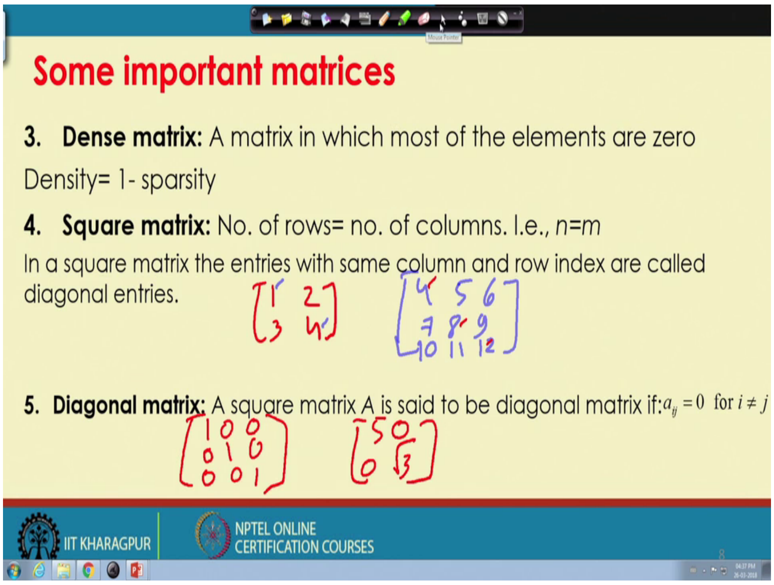

There is a dense matrix where opposite to sparse matrix most of the elements are 0 and density will be 1 minus sparsity. So, number of nonzero number of 0 a values divided by total number of number of 0 elements divided by total number of elements, square matrix is when number of rows is equal to number of columns.

And which is in a square matrix the entries with same row and column index are called the diagonal entries. So, if you see an example for example, in this matrix 1, 2, 3, 4 the entries with i is equal to 1 and j is equal to 1, i is equal to 2 and j is equal to 2 are the diagonal entries. Similarly if you take a 3 into 3 matrix select 4, 5, 6, 7, 8, 9. 10, 11, 12 the diagonal entries are identified as 4, 8, 12.

So, which has same row and column index and if square matrix has all the off diagonal non diagonal entries 0, we call it to be the diagonal matrix. So, only the diagonal values on nonzero or maybe nonzero there are can be some zeros also, but all the off diagonal values should be 0. So, a matrix like this 1, 0, 0, 0 1, 0 0, 0, 1 is a diagonal matrix or a matrix like 5, 0 0, root 3. This is also a diagonal matrix, were only the diagonals are nonzero.

So, question comes whether the diagonal matrix is a sparse matrix or not and of course, because most of the elements are 0 here, only very few elements are nonzero and if we see the definition of the sparse matrix that if you see the definition of the sparse matrix. Where the most where most of the elements are 0?

So, if we have a we have anything greater than 2 2 by 2 ordered matrix will have the diagonals only nonzero off diagonals are much more than the diagonal terms and therefore, it should be sparse matrix. Rectangular form matrices, so we are discussing only about square matrices their rectangular matrices.

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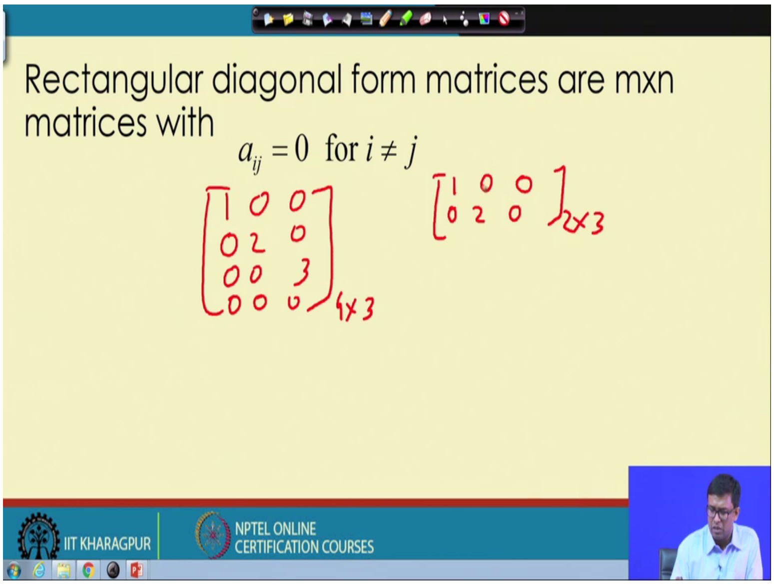

In which case in cases they follow the definition that the terms with different row and column indexes are the 0 are called matrices with rectangular diagonal form and an example can be a rectangular matrix 1, 0, 0, 0, 2, 0, 0, 0, 3, and 0 0, 0. So, any this matrix has number of rows more than number of column. So any row which is beyond than the maximum column id should have all 0s for this form.

Similarly, if we have a say 2 into 3 matrix 1, 0, 0, 0, 2, 0 this is also of diagonal form and any column which is more than maximum row number should have all 0 in case here. So, these are these are few of this definition will see few more some of the important more, more important matrices 1 is a diagonal matrix must be a square matrix if diagonal entries are 1 it is called an identity matrix.

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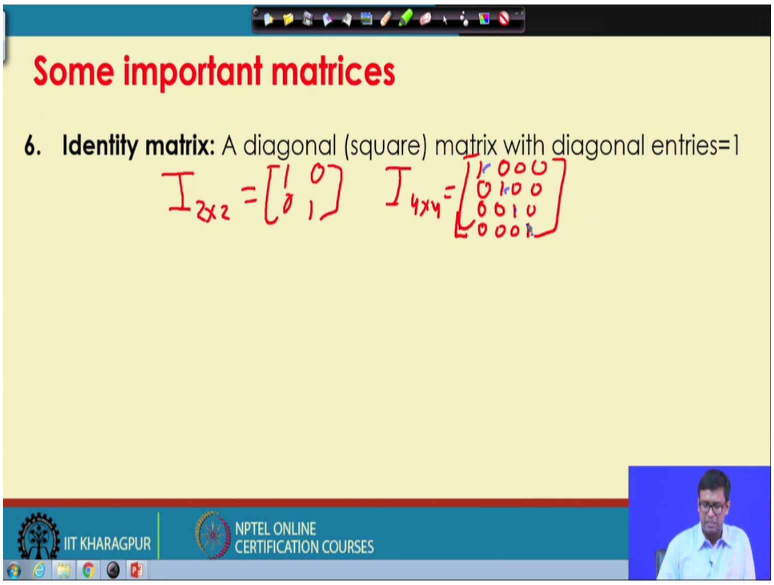

So, an identity matrices are generally denoted by I for example, I 2 into 2 is 1 0 0 1, I 4 into 4 is equal to 1, 0, 0, 0, 0, 1, 0, 0, 0, 0 1, 0, 0, 0, 0, 1, and now there is a restriction on what type of entries will matrices have these will only have integers because 1 is an integer in their diagonals.

So, we cannot have complex matrices or imaginary entries in a identity matrix there are there are some other definitions upper triangular matrix let me clear. This upper triangular matrix is a matrix for which any term which has a row index more than column in column index is 0 and it is looks like a triangle and it is it is also for a square matrix.

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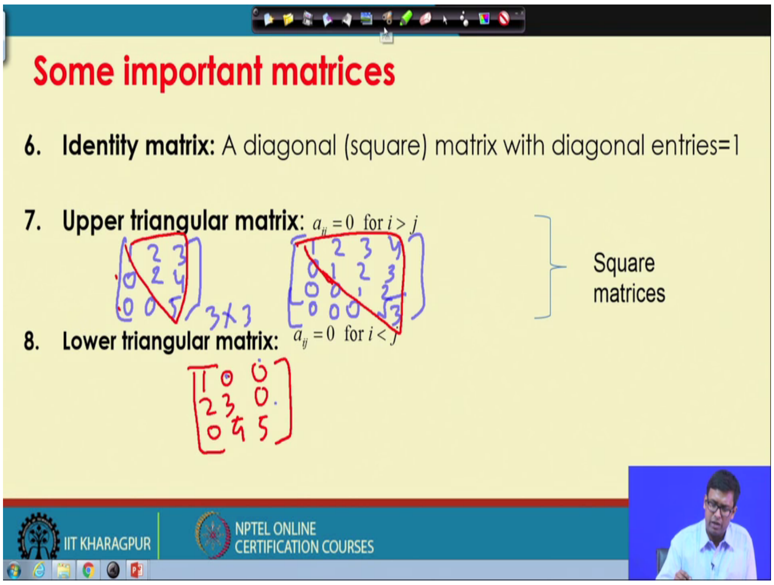

So, if we have for example, if we have a 3 by 3 matrix, 1, 2, 3, this is first row and first if first row and column is 1, 2, 3. So, this is nonzero this is second row the column number is 1, so this is 0 maybe 2 4 and this is 0, 0, 5. This is what is called an upper triangular matrix is it can be something like this also 1, 2, 3, 4 in this is first row so first column onwards everything is nonzero, but second row first column is 0, 1, 2, 3, 0, 0 1, 2, 0, 0, 1 maybe root of 3 it can have any entry in that sense.

And the lower triangular matrices opposite on so we can see that the matrix actually the numbers given in the matrix sorry this is the wrong side I am sorry this is also 0. The number the nonzero numbers given in the matrix forms a triangular shape and a lower triangular matrix is just opposite it has where the column index is more than row index it will be 0. So, 1, 0, 0, 2, 3, 0, 0, 4 interestingly the it is mandatory.

That this terms the terms with column ID greater than row ID or incase of lower triangular matrix like this terms or in case of upper triangular matrix row ID greater than column id has to be 0, but other any of the other terms and many of the other terms can also be 0 sparse matrix can have lower triangular and upper triangular form that sense.

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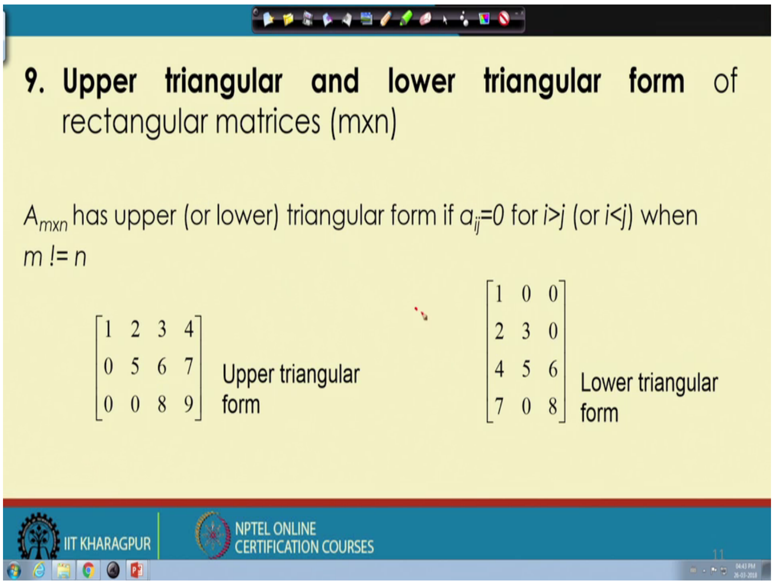

We go to upper triangular and lower triangular form of rectangular matrices a matrix which is not a square matrix if it follows the definition of upper triangular or lower triangular that is row id greater than column ID it is 0 then it is called upper triangular column ID greater than row id all that terms are 0 it is called lower triangular. So, there are there can be few examples of that.

So, this is a rectangular matrix which is of upper triangular form this is a rectangular matrix which is of lower triangular form, it is they are not called the upper triangular and lower triangular matrix they are called rectangular matrices of form of upper triangular form or lower triangular form. In many cases we see matrices square matrices which are not of triangular form, but of nearly triangular form, sorry or nearly triangular form.

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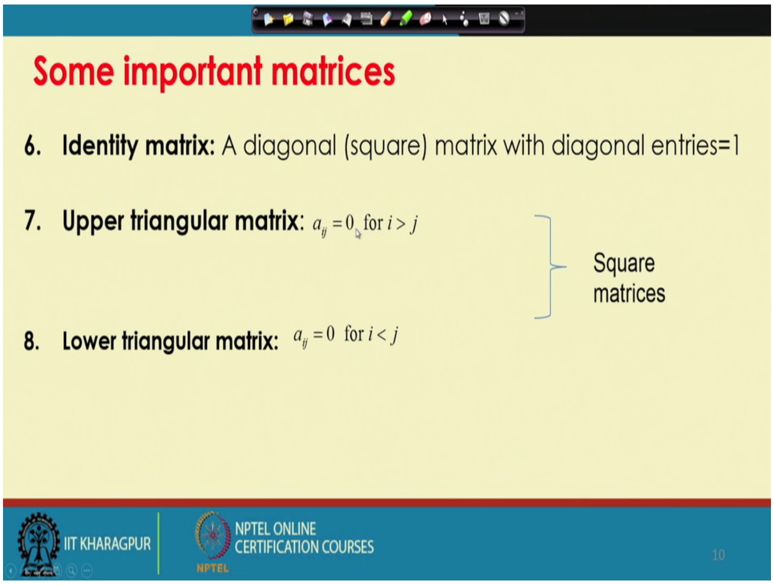

That means, they follow this role very nearly, but they exactly do not follow this to maybe one more band and one more term near after the diagonal or before the diagonal is still nonzero and we call them to be Hessenberg matrices; it is upper or lower Hessenberg matrix is that a i j is equal to 0 for i greater than j plus 1.

That means, we can allow still one more row to be term with one more row index to be nonzero which is after the column index and similarly we can have lower Hessenberg form.

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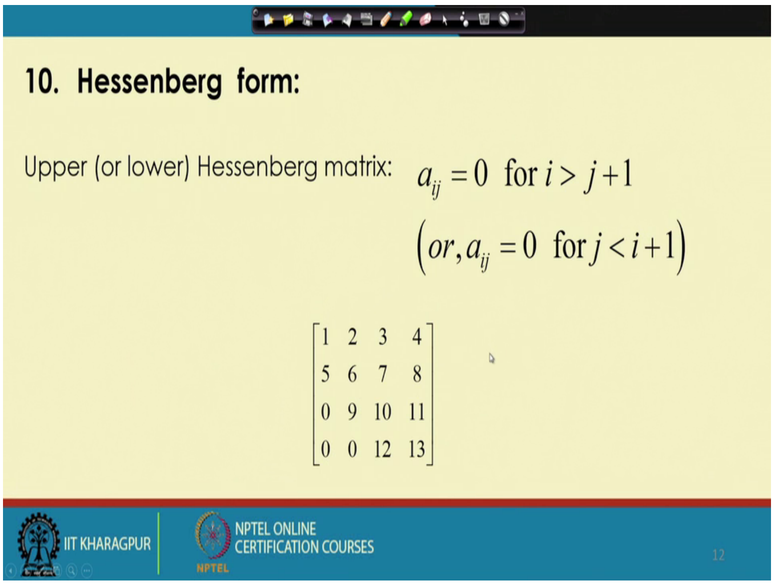

In this Hessenberg matrices will (Refer Time: 26:14) when will discuss about one of the very recent developments in matrix solvers which is called (Refer Time: 26:23) of space solvers. Interestingly matrix solvers has been developed over last more than 2 decades, less more than 2 centuries, earliest matrix solvers which we will discuss is was obtained in first century writing of Chinese mathematician. However, those solvers are not discarded here they not obsolete we use those solvers heavily in different type of calculations even in very modern computer programs also any way.

So, one of the recent developments are in last decade which are called Kryle space methods and they used not last decade at the end of last century 1990s and they uses Hessenberg matrices, which is nearly triangular matrix. The basic matrix operations come to it what is equality 2 matrices are called equal when they have equal order and each term with the with the particular row column indexes same with the same similar corresponding row column index in the other matrix. And then we called if a i j is equal to b i j and a and b have same order then we call for all i j we call a is equal to b.

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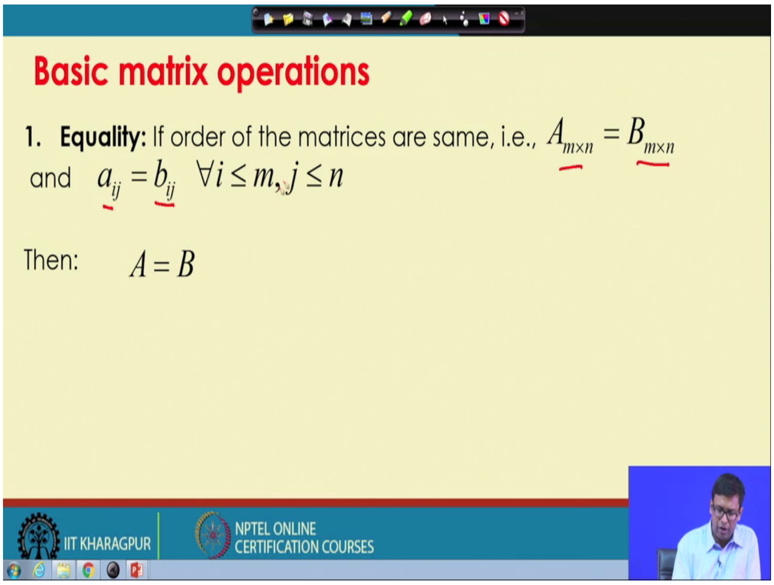

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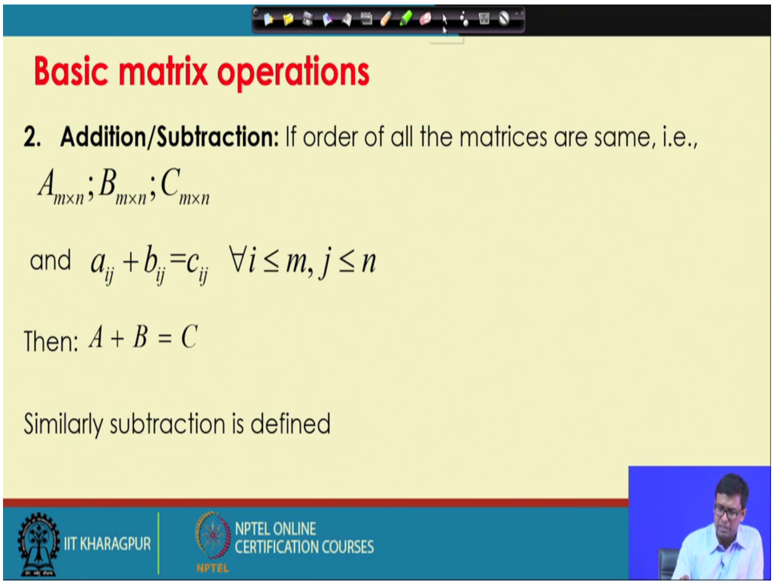

Two matrices can be added and subtracted if they have same order and each term of 1 matrix each i j index term of 1 matrix is added with the similar i j inject index term of the other matrix to give the resultant matrix.

And similarly subtraction and then we called A plus B is equal to C, similarly subtraction is also defined. If I multiply a matrix by a scalar all the terms inside the matrix are multiplied by that is scalar and very important operation is matrix multiplication.

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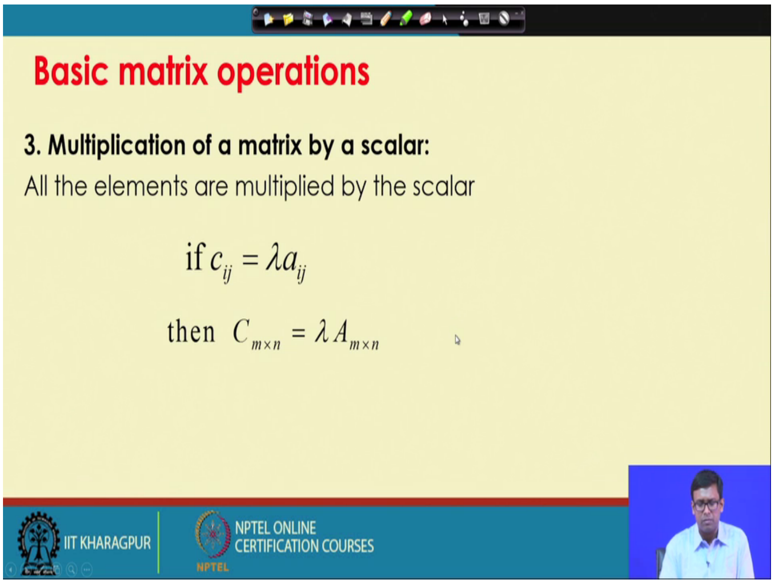

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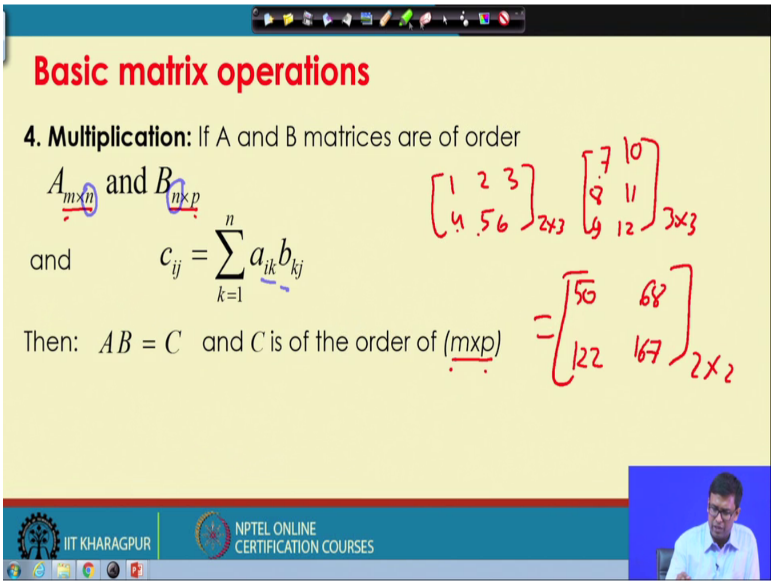

When you multiply two matrices then and get a result in matrix the matrix multiplications defined as there is only defined when length order of one matrix and order of other matrix will have something common. That is the number of rows of number of columns of the first matrix is equal to number of rows of the other matrix and that why, because the resultant matrix will have one term which is summation of each terms of one row of the first matrix multiplied with each term of one particular column of that matrix.

And then we called AB is equal to C and interestingly the order of C will come from the order of A and B the number of rows of C will be number of rows of A number of columns of C will be number of columns of B. And we can see a quick example for example, I have 1, 2, 3, 4, 5, 6, which is 2 into 3 multiplied with 7, 8, 9, 10, 11, 12. which is 3 into 2 I, will get something which is 2 into 2

And each term here this is 1 into 7 plus 2 into 8 plus 3 into 9 which will give me 50, this is 1 into 10 plus 2 into 11 plus 3 into 12 which will give me 68. This is 1 into 4 into 7 plus 5 into 8 plus 6 into 9 which will give me 122 and this is 4 into 10 plus 5 into 11 plus 6 into 12 is give me 167. So, this is how matrix multiplication is defined and we use matrix multiplication in there are method of application in matrix multiplication in linear algebra courses.

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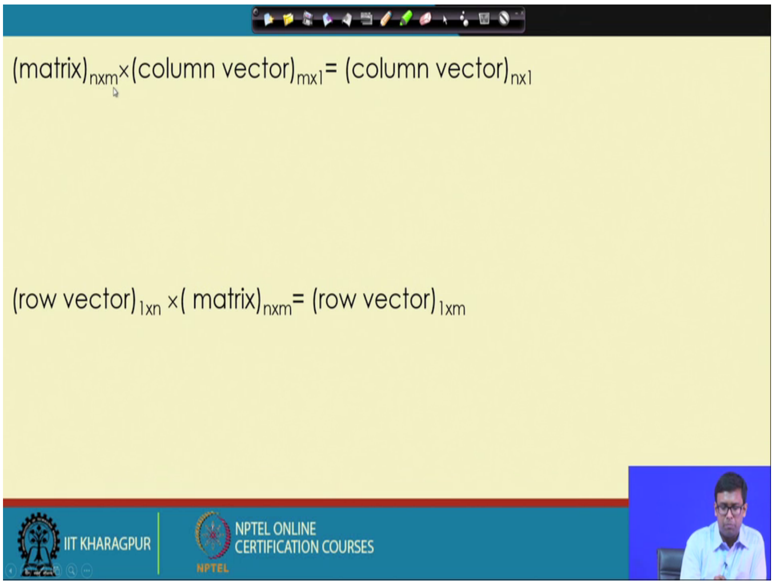

Now, a matrix multiplied by a column vector just if you look into their indices gives us a column vector small is simply or and row vector multiplied by a matrix gives us a row vector.

So, the order of multiplication is important as the sorry as a number of columns of the pre multiplier has to match with the number of columns of the post multiplier, sorry. A matrix will be post multiplied by a column where are matrix has to be pre multiplied by a row.

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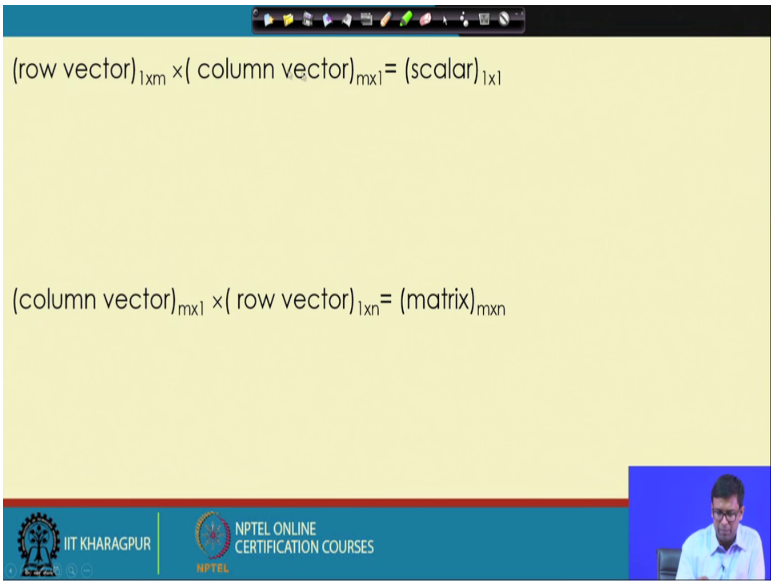

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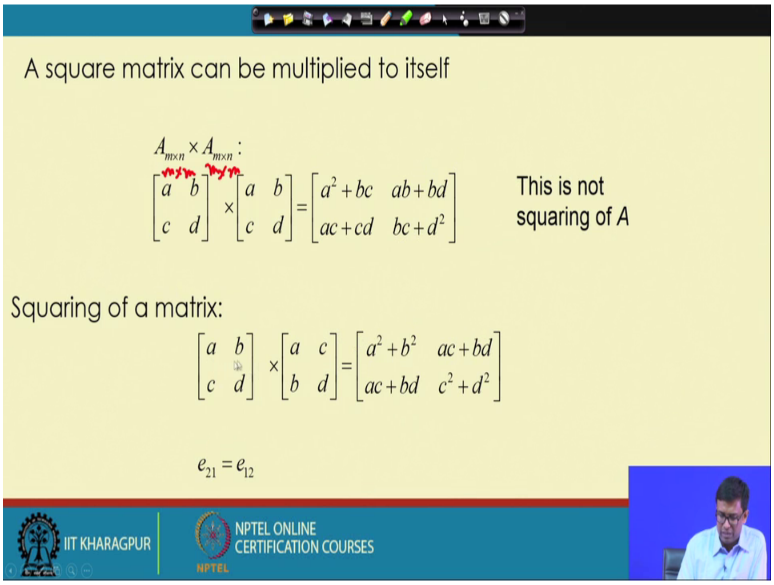

And sorry and the row vector multiplied by column vector gives a scalar, but the opposite a column vector post multiplied by row vector will give a matrix of m into n order. A square matrix can be multiplied to itself, a row vector cannot be multiplied to a row vector because the order in terms of pre multipliers column and post multipliers row do not match.

Similarly a column vector cannot be multiplied to a column vector itself, square matrix can be multiply to itself because it has same order this is sorry this should be m into n, this would be m into as this is a square matrix. So, as this is a square matrix this will be m into n the orders are same as the number of row is equal to number of column.

It can be multiplied itself; however, when you multiply it we see that this is not squaring of each terms rather it is something different. Probably a better squaring of the terms can be obtained if I multiply a square matrix with rotating the square matrix in terms of row and column. For example, a, b into c, d is multiplied with the matrix a, b is the row is changed as a column here and we get product as a square plus b square c square plus d square.

Interestingly in the product matrix we see that the off diagonal terms are equal and this is an interesting class of matrix when we see similar off diagonal terms and we call them to be symmetric matrices. And also when row and columns are changed of a matrix rows be has become columns and columns have become rows these matrix like this the first matrix and second matrix they are called their transposes. So, there are these are some of the matrix operations, in next class will see some more matrix operations and definition of some more important matrices

Thank you.