Constrained and Unconstrained Optimization
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Lecture – 09
Algorithm of BIG-M Method

In this particular class we are going to discuss on the big M method. In the last class we started what is big M method, specially in an LPP in the constraints whenever you are having the greater than equals type inequality. In that case we use the surplus variable we subtract one surplus variable from that constraint to make it equality. But since the surplus variable is negative. Therefore, whenever I will make the variables x 1 x 2 xn equals 0. The surplus variables if it is xn plus 1 to xn plus i. Then there will be xn plus 1 equals minus b one xn plus 2 equals minus b 2 like this way xn plus equals minus bm.

So, from there we cannot form the basis for that reason we are using one artificial variable we are adding one artificial variable of that format.

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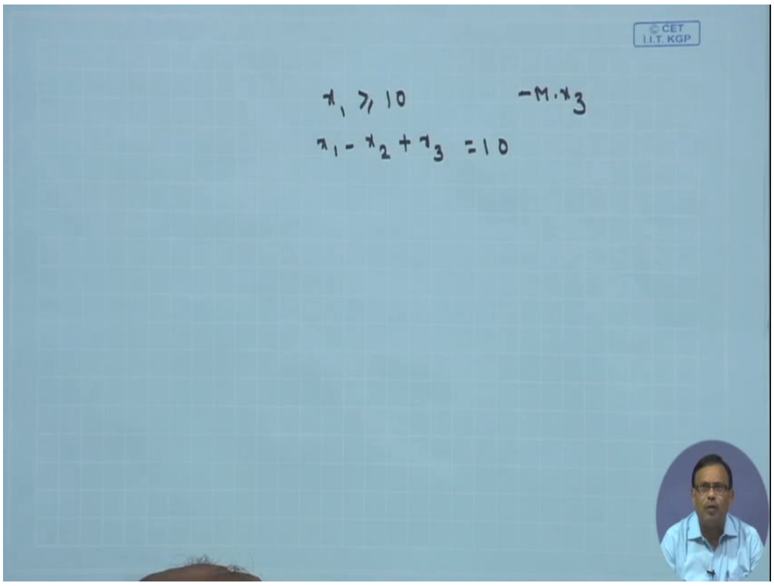

That is whenever you are having x 1 greater than equals say 10. We are writing it as x 1 minus x 2 which is surplus variable, plus x 3 this is equals the 10. And this variable plus x 3 we are telling it as an artificial variable, where in the cost coefficient we are adding a high penalty to that So that we are in the cost function we are adding here minus m into x 3 in the cost function, which I told in the last class itself. Because the original problem and this equivalent problem are not same. Therefore, you have to remove the artificial variable and for that reason we have to add the a very high penalty which is m we assume that whenever we subtract one quantity from m always m will be greater than that particular quantity.

But this is manually whenever we try to solve a problem this is all right, but whenever we want to computationally do it using computer. In that case this approach may not be appropriate. Because sometimes it may be difficult to judge what should be the value of capital M. Because we cannot predict what kind of coefficient, or constant values will be there for a particular problem. Anywhere we will discuss that part afterwards. So now, let us see what is the procedure for the big M approach.

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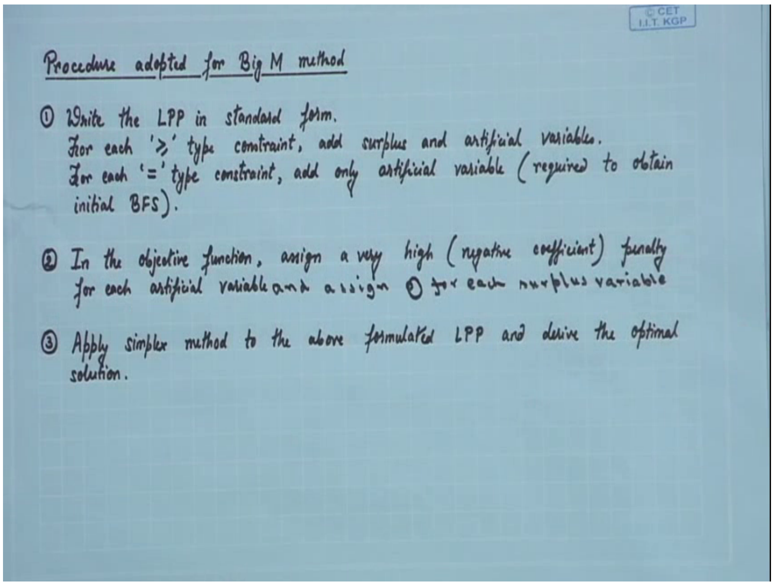

See here the procedure adopted for big M method. First earlier also we have discussed that write the LPP in the standard form. Then for each greater than equals type constraint add surplus and artificial variables.

So, as I discussed for each greater than equals type constraint, you have to add one surplus and one artificial variable and the reason we have discussed earlier. For each equal type constraint if you have equals type constant, in that case I have to add only the artificial variable. This is required to obtain the initial basic feasible solution. Because in the equals type constant if we do not add these artificial variable then initial basic feasible solution will not occur. This we will discuss whenever we are solving the examples. So, this is the first step. In the second step in the objective function we are assigning a very high negative coefficient or penalty or very high penalty which we call as negative coefficient for each artificial variable and assign 0 for each slack and surplus variable. Earlier also you have seen in the objective function we assign 0 coefficient for each slack variable.

Now, for the big M method we will assign 0 to for 0 coefficient for each surplus, and slack variable whereas, we will assign a very high penalty that is a very high negative coefficient for each artificial variable. And once we have done it then apply the simplex method to the above formulated LPP defined in step 2 and we have to derive the optimal solution. And this we are doing whatever we have done earlier using the normal simplex method. So, only thing extra here we are adding the surplus variable and artificial variables for each greater than equals type constraints. If I have less than equals type constraint we will add the slack variable.

Whereas if I have equals type constraint then I may have to add artificial variable and that is to obtain the initial basic feasible solution. So, these are the basic steps which will be followed for the big M method. Next is so, whenever you are doing the LPP.

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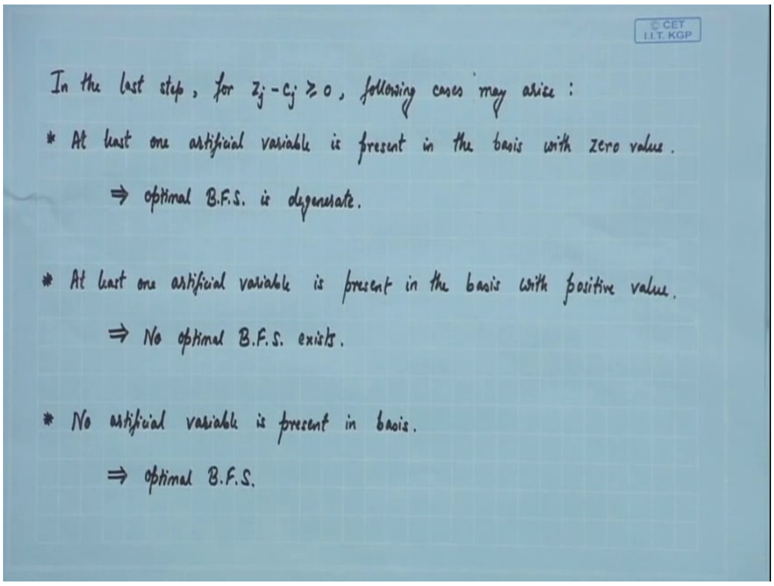

Whenever you are doing solving the LPP, in the last step what do you see we try to find out whether zj minus cj greater than equals 0 or not. Whenever zj minus cj is greater than equals 0 then we say that we can obtain the optimal solution. Now when zj minus cj greater than equals 0, some solutions some cases may arise. One is the first point at least one artificial variable is present in the basis with 0 values.

In the basis means on the left hand side of the table under c B whatever vary with c B and xb under xb, whatever variables we are writing those are the basic variables. So, if all zj minus cj greater than equals 0 for all j. And if one artificial variable is present in the basis with 0 value; that means, the optimal basic feasible solution is degenerate. Now what is the meaning of degenerate I will discuss the degeneracy case afterwards. Degenerate means there may have the redundant constants, that is some constant is there in your problem which is not required which is covered by some other constraints. Or in some other cases in the basics basic variable value is 0.

The problem with degeneracy is that, if you try to improve the solution and if the problem is degenerate. Unless you are using certain techniques, you cannot improve the solution that is you cannot obtain the optimal solution and the result will remain same. So, that will be the problem for degeneracy, and this again as I told we will discuss it later whenever we are solving different problems. The next step is at least one artificial variable is present in the basis with positive value; that means, in the basis you have one artificial variable, but whose value is positive and all zj minus cj is greater than equals 0; that means, you could not or we could not remove all the artificial variables from the basis. And in that case no optimal basic feasible solution exist, again we will see the reason why we are saying this thing.

Number 3 is no artificial variable is present in the basis and zj minus cj is greater than equals 0; that means, we have obtained the optimal basic feasible solution. So, please note that zj minus cj greater than equals 0 for all j greater than equals 0, does not imply that you will obtain the optimal solution. As I told you in the first case if any artificial variable is present in the basic whose value is 0, then basic optimal basic feasible solution will be degenerate. Whereas, if you have one artificial variable it with positive value in the basis in that case no basic feasible solution exist, and if and only if no artificial variable is present in basis then only we can obtain the optimal basic feasible solution.

So now let us see how we can solve the problems, let us take one problem over here.

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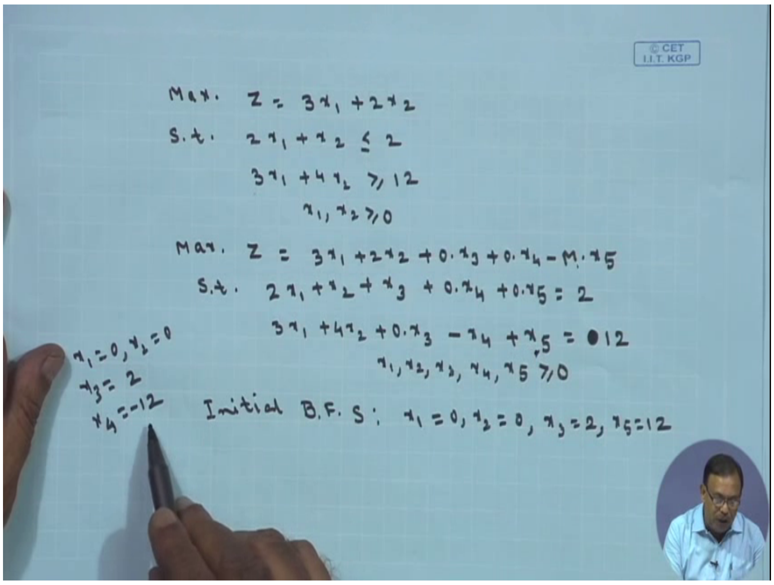

Maximize z equals maximize z equals 3 x 1 plus 2 x 2, subject to 2 x 1 plus x 2 less than equals 2 and 3 x 1 plus 4 x 2, greater than equals 12 and x 1 x 2 greater than equals 0. So, if you see the problem you will find that we have one constant of less than equals type and one constant of greater than equals type over here. So, from here you can tell that I have to use one slack variable for the first constant. Whereas, since the second one in inequality is greater than equals type I have to subtract one surplus variable whereas, I have to add one artificial variable to this problem or to make all both the constants as equality constant.

So, introducing the slack and surplus and a artificial variables your problem reduces to maximize these I will write down z equals this one, subject to 2 x 1 plus x 2 plus x 3. I will write the right side later whereas, 3 x 1 plus 4 x 2 plus 0 into x 3 and I have to add one slack variable plus 1 artificial variable this will be 0. So, that the coefficient of the slack variable and the coefficient of artificial variable in the first constraint is equals to 2, sorry this will not be 2 0, but it will be 12.

So, please note that in this particular problem since the first inequality was less than equals type that is 2 x 1 plus x 2 less than equals 2. So, we have added one slack variable x 3 to make this less than equals type into equality type. Whereas, the second one was the greater than equals type that is 3 x 1 plus 4 x 2 greater than equals 12, this we have subtracted one surplus variable x 4 and to get the initial basic solution, we made we added one artificial variable to this. So, that we are writing this one.

Now, in the objective function what happens if you note the procedure whatever we have told earlier. This is the original function. Coefficient of slack and surplus variables will be 0 whereas, coefficient of the artificial variable that is x 5 here will be a very high non negative number which we will subtract that is negative penalty we are adding. So, that we will write down it as 0 into x 3 plus 0 into x 4 minus m into x 5. So, and of course, x 1 x 2 x 3 x 4 and x 5 all must be greater than equals 0 in this case.

So, you had the original problem this you are writing into the standard form, by introducing slack variables, surplus variable and artificial variable depending upon the nature of the inequality sign in the constant. Here I do not have any equals type of constant that we will take another example afterwards. So now we will use the normal simplex method whatever we have discussed earlier. So, first I have to tell what is the initial BFS that is initial basic feasible solution will be the original variables will be equals to 0, that is x 1 equals 0 and x 2 equals 0.

Now, if I make x 1 0 and x 2 0, this we have used because we have told that the my decision variables are greater than equals 0. X 1 x 2 greater than equals 0. So, we are taking x 1 0 and x 2 0. If we substitute this on the first constraint you will obtain x 3 equals 2, and in the second constant if you put you will find x 5 equals 12. So, basically your initial basic feasible solution is this one by substituting original decision variables equals 0, and then put these values in the constraints So that you will obtain the values of the other variables.

Now, I hope it is clear to you why we have used this particular this artificial variable here. Because if we do not use artificial variable here then the value of x 4 should have been minus 12, but which violates our condition that all my variables are non negative. That is the reason we have added the artificial variable x 5 here. And we have made a high penalty to these So that afterwards we can remove this artificial variable from our problem. So, this is the basic problem or standard format which we are putting. Now this I will write in the tabular form which again we discussed earlier.

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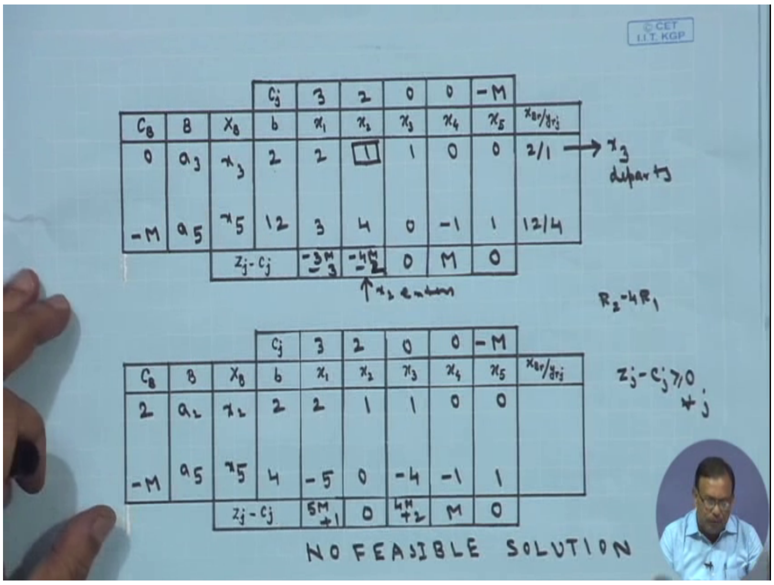

So, for this one if you see if I put it here, your in this particular problem then in the basis there will be 2 vectors that is x 3 and x 5 because all are the 0s in the basis all vectors cannot be 0. So, we will take only the vectors whose values are non 0. So, in the basis the variables are x 3 and x 5. If you see here from this particular problem your cj is the values of the coefficient of the corresponding variables, that is here it is in this case it will be 3 2 0 0 minus 5. Corresponding to x 1 x 2 x 3 x 4 x 5. So, it will be 3 2 0 0 and minus m for x 5, your variables are x 3 and x 5.

So, corresponding values I can write down a 3 and a 5 x 3 coefficient cost coefficient is 0, x 5 it is minus m. So, I am writing here 0 and minus m, now b value is 2 and 12. So, we are writing the b value as 2 and 12 over here. So, I am writing 2. Now this row corresponds to the first constant that is 2 1 1 0 and 0. So, it will be 2 1 1 0 and 0. Second one represents the second constants that is it will be 3 4 0 minus 1 and plus 1. So, I think the this is the initial table which you are obtaining from the standard format of the problem, which I discussed earlier also.

Now, what I have to do I have to find out the value of zj minus cj. That is the reduced cost, and your reduced cost for this particular problem will be how much? This is 3 m minus 3 that is minus 3 m and 3 minus 3. Next one would be minus 4 m minus 2 I think it is clear, that is 0 into this 1 2 plus minus m into 3 that is minus 3 m minus 3. Next one 0 into 1 plus minus m into 4 minus 2. So, minus 4 m minus 2 similarly the next one would be 0 next would be here it is m all others are 0. So, it is m and the next one is this one that is minus m and this is plus m. So, this will be 0.

So, here I obtained the reduced cost value zj minus cj corresponding to each variable and I am finding that the variable x 2 corresponding to variable x 2 which has the most negative value that is minus 4 m minus 2. Because I can make m as large as possible. So, this is your entering variable. So, x 2 will be your entering variable, once you are getting the entering variable in that case now you have to find out the value of x B r by yrj, by finding the ratio of b by x. Your b is this column x is corresponding to x 2. So, here it is b by x means it will be 2 divided by one for this case whereas, for this case it will be 12 by 4.

Now, which will be the departing variable? That we have told the minimum of this ratios. So, minimum of these ratios is this one that is your x 3 is the departing variable in this case. So, your x 3 here is departs and your x 2 enters here. So, therefore, your corresponding to this column and these row they are intersecting at this point. So, your pivot element is 1. So, this is the first step since all zj minus cj not greater than equals 0. So, some variable will enter and some variable will depart.

So, whenever the value of the zjcj which is most negative corresponding to that column the variable will enter into the basis, then I finding the ratio that is b by x format and which I have written as x B r by yrj. This is the 2 by one gives the minimum value of these 2. So, corresponding variable is x 3. So, x 3 will depart from this. So, in the next table x 3 will be replaced by x 2. So, here in the xb value, it will be x 2 and x 5. So, here it will be a 2 and a 5. So, here you have 3 2 0 0 minus m there is no change for a 2 it is 2. So, it will be 2 and minus m.

So, what you have to do? For the other columns as we have told the pivot element has to be 1, and then the corresponding elements of this column I have to make 0. Since this is already one I do not have to do any operation on this row straight away this will remain as it is that is 2 2 1 1 0 and 0. Whereas, for this it is 4. So, I can multiply by this, this second row minus first row into 4 in 5 may then this one will be 0. So, if I perform this operation, that is row 2 minus 4 into row one on this second row. Then this element will be 0.

So, performing these operations you will obtain the values as 4 minus 5, and x 1 will be 5 0 minus 4 minus 1 and 1. So, once I am obtaining this table this you can check it afterwards also. So, after this once I am doing this one I have to find out that zj minus cj value. Zj minus cj already you know how to calculate this c B into xb minus cj. So, this into these minus this into this minus this that is here it will be 5 m plus 4 minus. So, it will be 5 m plus 1, next one will be 0 2 minus 2 0, next one will be 4 m plus 2, next one will be m and next one is 0.

So, if you find here all zj minus cj is greater than equals 0, for all j this is true. So, I should obtain the optimal solution. But please note one thing that artificial variable is present in the basis and with a penalty that is with a non negative penalty. Your artificial variable could not be removed over here. So, if you see the earlier problem what we have told here that if at least one artificial variable is present in the basis with positive value. In that case we say that there is no optimal solution. Since here x 5 is the artificial variable and it has it is present in the basis therefore, and it is value is non negative therefore, this particular problem has no feasible solution, no feasible solution.

So, it is clear to us that whenever we are having a problem, if you see from the original problem just I am brushing up, that you have the less than equals type problem. You have the greater than equals type problem, and this 2 type of inequalities whenever you are having, then you are adding the if you are adding the slack variable to make slack variable x 3 to make the less than equality sign into equality. And after that you are adding the artificial variable, sorry surplus variable you are subtracting and you are adding artificial variable for greater than equals type inequality to make it equality.

Now, the reason for adding this artificial variable is that we are depleting again. If this x 5 was not present here, in that case whenever we try to find out the initial basic feasible solution. We takes one equals 0 and x 2 equals 0 and if we put on these equations your x 3 will be 2, but your x 4 becomes minus 12 from here. If this x 5 artificial variable is not there. So, which is violating our non negativity constraint and for this reason we have to add these artificial variable, and we are making it equals. Now this problem by introducing artificial variable and our original problem they may not be same.

I have to remove this artificial variable and to remove the artificial variable, I have to add some high penalty to this artificial variable in the objective function. So, if you look at the objective function you see this is the original objective function 3 x 1 plus 2 x 2. And the coefficient for each slack and surplus variables will be 0, just like we have given here coefficient of x 3 0 and coefficient of x 4 0, and we have added the coefficient of the artificial variable x 5 as m which is a very large value. Then we are writing it in the tabular form like this. And after that we are using the normal simplex algorithm whatever we have done earlier.

So, in the second step we are finding in the second iteration whenever you are doing it you are finding that zj minus cj is greater than equals 0. All zj minus cj greater than equals 0, but one artificial variable is present in the basis. We could not remove this artificial variable. So, therefore, the cost will be very negative instead of maximization. The cost will be to negative, or in other sense we can say that from our procedure whatever we have discussed earlier that this particular problem has no feasible solution. So, in the next class, we will solve one 2 more problems to clarify this particular method.