Constrained and Unconstrained Optimization

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Lecture – 07
Simplex Method

So, in the last class what we have done? We have started the simplex algorithm. Just the basic concepts we have told. Today let us start with the theory behind the simplex method, that is what kind of theory we are using in simplex method. For this one you consider the LPP. Maximize z equals maximize z equals c x subject to A x equals b and the non negativity condition that is x greater than 0.

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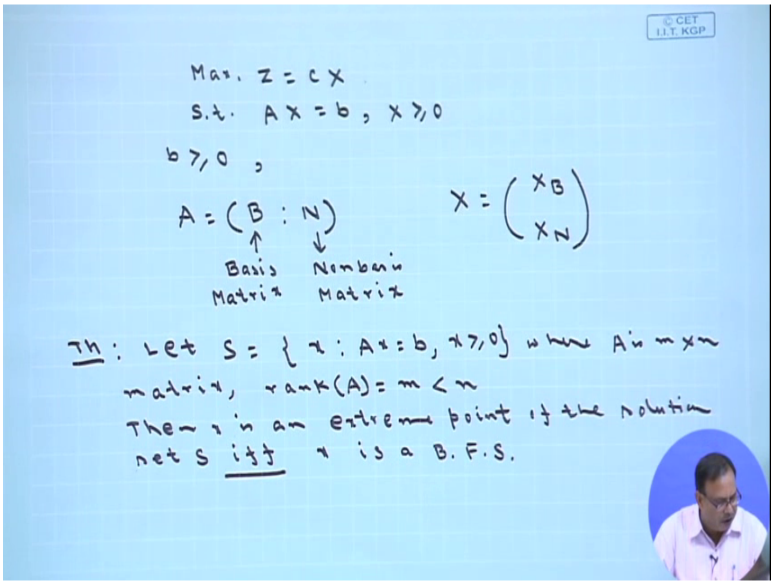

Let us back to assumptions, one assumption is this b is greater than equals 0. And the a has the full rank. These 2 assumptions we are making that is b greater than equals 0, and a has the rank. Then we can write down A as like this. B and N basically this we call as the basis matrix B is the basis matrix. And N is non basis matrix. That is from there linearly independent vectors whatever you are getting, in terms of the unit matrix that one will be the basis matrix and this n will be the non basis matrix.

Similarly your x also can be written as X B and X N, where your X B consist of the N, N X B consists of basic variables and X N consists of non basic variables. So, x is also i am separating X B consists of basic variables which corresponds to the basis matrix B, and

X N consists of non basis variable corresponds to the non basis matrix N. There is a theorem, the theorem states that let s equals x such that A x equals b, x greater than equals 0. Where a is m cross n matrix where a is m cross n matrix, and rank of A is equals to m which is less than n. Then your x is an extreme point. Already we know what is the extreme point, x is an extreme point of the solution set, we can denote it by s if and only if x is a basic feasible solution.

So, effectively we are saying by this theorem that, if s is a set where such that x such that A x equals b x greater than equals 0, where a is your m plus n matrix, and rank of a equals m which is less than n. Then x is an extreme point of the solution set is if and only if x is a basic feasible solution. That is if x is extreme point then it will be a basic feasible solution, on the other way if x is basic feasible solution in that case your x is an extreme point.

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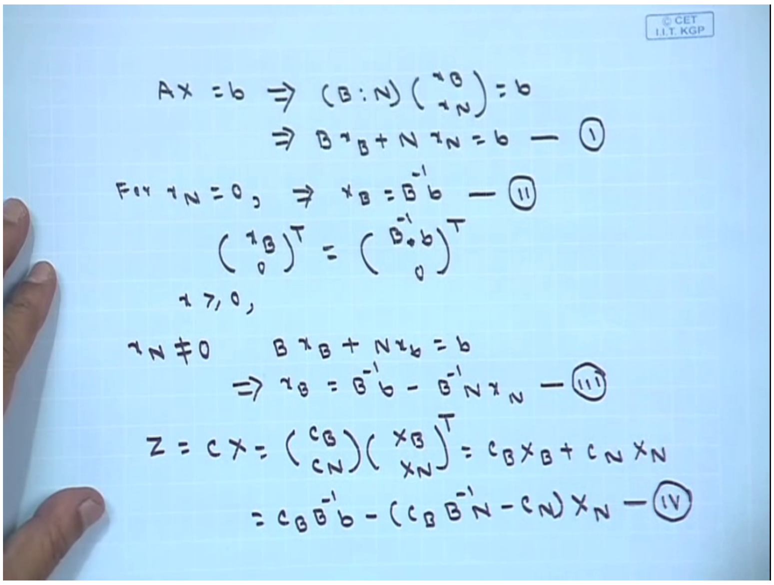

Let us see this one now. Your A x is equals to b which you can write it as from the earlier one a can be denoted by this. X can be denoted by X B X N this is equals b, which implies your B into X B plus N into X N simply we can write it b X B plus n into X N is equals to b. So, this is equation number 1.

Now, whenever X N is equals to 0 for X N equals 0, which implies that your X B equals B inverse b from here you can write down. So, this is equation number 2. Your X B is B inverse b. And since the basic solution you can write it in this form X B 0 transpose which is equals to B inverse b 0 transpose. Basic solution is this one X B 0 transpose is equals to X B is B inverse b. So, B inverse b 0 transpose, and if x is greater than equals 0 which is already mentioned. So, on in this case we have already the basic feasible solution. So, for X N equals 0 we have the basic feasible solution, if x is greater than equals 0 of the form B inverse b 0 transpose. So, this we are getting.

Now, whenever X N is not equals to 0. Say if X N is not equals to 0 in that case you are having B x B plus n X B this is equals b after manipulation you can write down X B equals B inverse b minus B inverse n into X N. So, this is equation number 3. So, for X N not equals to 0 from equation one basically we can write down X B in this form which is we are denoting as equation number 3, your z is equals to now c x and c x means this also we can write it as C B, C N and X B X N transpose in matrix notation we are writing it.

So, that you will get it as C B x B plus C N X N. And if you put the value of X B from 3 here you will obtain. So, it is basically C B x B plus C N X N. From here you put the value of X B from equation 3 and after manipulation you will obtain C B B inverse b minus C B B inverse N minus C N into X N which is equation number 4. So, you are getting this let us take the other pen. So, you are obtaining z equals this one, your z is the objective function which you want to optimize evaluating equation 4.

Now, your X B is B inverse b minus B inverse N X N which i got it in equation 3 in equation 3 you got X B equals B inverse b minus B inverse n X N.

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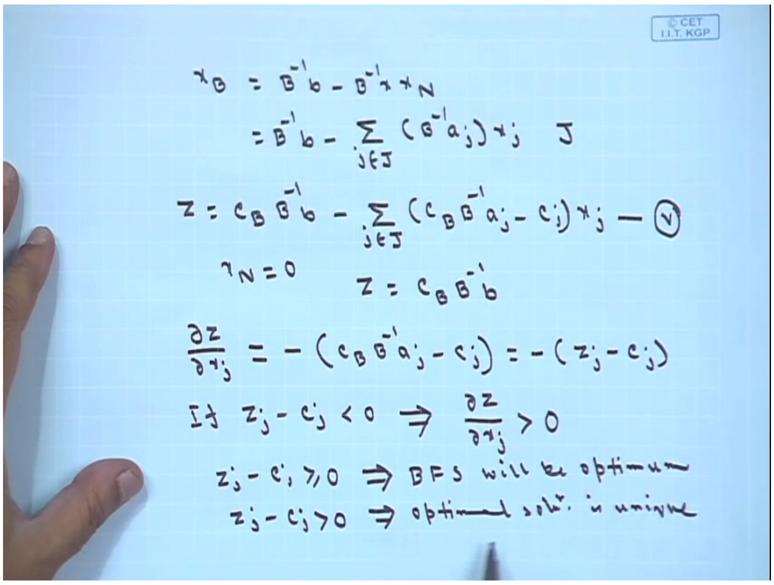

So, that only we are writing and this equals you can write down B inverse b minus summation over j belongs to J. B inverse aj into xj. This B inverse n you are writing in this format B inverse aj into xj, where capital J is the index of non basic variable, please note this one here j capital J is the index of non basic variable. So, using this your z can be written from this equation, if i substitute in this equation 4 z equals this one i will obtain z equals C B B inverse b minus summation, j belongs to capital J C B B inverse aj minus cj into x j. This is our say equation number 5.

So, for basic solution; obviously, we know that your X N should be 0. For basic solution always X N should be 0. So, that you can obtain z equals C B B inverse b. So, once you are getting z equals C B B inverse b. Now from equation 5, i can if i differentiate z with respect to xj with respect to the variable xj del z del xj. This will be equals to minus C B B inverse aj minus cj. And this c inverse C B B inverse aj is nothing but the zj this is nothing but zj. So, this becomes zj minus cj, where zj minus cj represents the reduced cost value, zj minus cj represents the reduced cost value for the jth index.

Now, if zj minus cj if this is less than 0 this implies del z del xj, which is greater than 0. So, we can say from here that if the non basic variable value of the non basic variable xj increases from it is current value then the value of z also will increase, please note this one. That the value of the non basic variable xj if it increases since the first derivative is greater than 0. Therefore, the value of the objective function also will increase; that means, whenever i will manipulate our table. I will check which variables should enter into the basis into the table. For that one we will calculate the value of zj minus cj and the most negative zj minus cj we will choose So that we get the maximum value of the objective function j.

So, this we will use whenever we are doing the algorithm at that time it will be required. So, if you see here one more thing is there, that is the central idea if of the simplex algorithm is very simple that is you are starting with an extreme point. And you are checking what is the value of the objective function. Then you are traversing around the adjacent extreme points one after another, and you are checking what is the value of the objective function. At that and by that way you can obtain the optimum value of the objective function. Since you may be knowing that it is known to us that the value optimum value of the objective function can be obtained only at the extreme point of the feasible region.

So, basically what we are doing in simplex algorithm we are starting with one extreme point, we are checking what is the value of the objective function. If that is not satisfying our criteria that is zj minus cj is greater than equals 0, in that case we are choosing another extreme point and we are depleting this process. This is the basic idea behind the simplex algorithm. Now if your bfs will be optimal if zj minus cj is greater than equals 0. From here we can say that if all zj minus cj for all j if this is greater than equals 0, then the basic feasible solution whatever you have taken that will be optimum.

Whereas if zj minus cj is greater than 0 for all j, then we say that the optimal solution is unique please note this one. Optimal solution is unique. So, there are 2 cases what is if zj minus cj greater than equals 0. In that case the basic feasible solution will be optimum. And if zj minus cj strictly greater than 0 then the optimum solution is unique. So, in the last class if you remember, we were talking about that we will transform, we have the original problem the original problem is written in the canonical form. From the canonical form we are writing the corresponding a kept from canonical form we are writing it into the corresponding standard form. And the standard form is written in the form of a table. Where some basic variables will be there some non basic variables will be there.

Now, if you zj minus cj all zj minus cj are not greater than equals 0; that means, the solution is not optimum. So, then i have to find out what should be the entering vector; that means, from basis one vector will be going out and one vector will come inside. So now, we will check which vectors will be entering and which vectors will go out.

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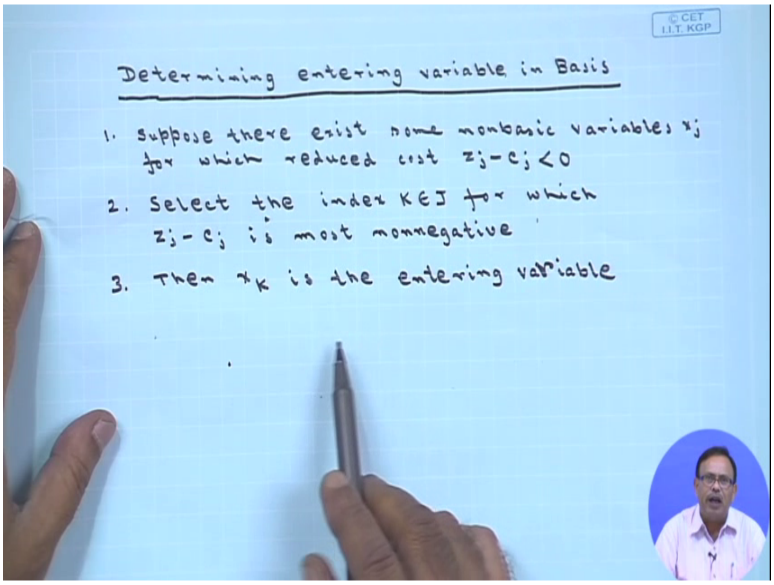

For that one First let us see this thing determining the entering variable in basis suppose there exists some non basic variable xj for which the reduce set zj minus cj is less than 0. So, as we have told if zj minus cj greater than equals 0, we have obtained the optimum solution.

So, if there is some variable xj for which zj minus cj less than 0. Now from this select the index k which belongs to the non basics variable capital J denotes the index of the non basis variables, from there i will select k for which zj minus cj is most non negative. Please note this one zj minus cj is most non negative. So, there may be some values out of that the zj minus cj which gives you the most negative value that we will choose and corresponding to that index k xk will be the entering variable. So, for entering variable what i have to do i have to check the value of zj minus cj, and most negative zj minus cj i will choose and corresponding index and corresponding variable xk will be the entering variable.

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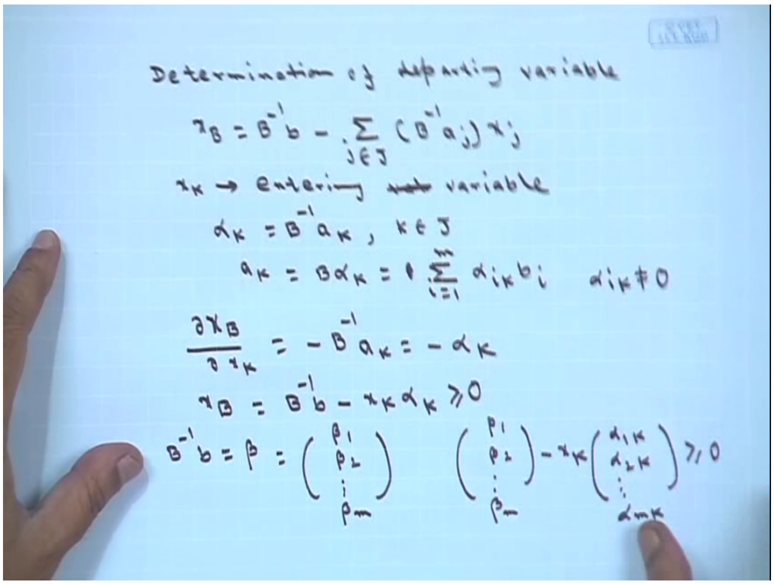

Now, next one is determination of determination of departing variable, that is which variable will go out from basis. This one earlier one we told that which basis which variable will be in basis. So, since one vector is going in basis, one vector or one variable should go out. So, next step is for this one. You know your X B is equals to B inverse b minus summation over j belongs to j B inverse aj into xj. Now suppose xk denotes the entering variable. So, first i am choosing the which one will be the entering variable. So, i am choosing xk as the entering variable. Then i can write down alpha k equals beta B inverse ak where k belongs to the index of non basis variable. And from here you can write down ak equals B into alpha k and alpha k and B into alpha k i can write down, summation i equals 1 to m alpha ik into b i.

So, your ak can be written by this. So, by replacement theorem of linear algebra we can say that ak can be replaced by any b, such that alpha ik is where alpha ik is not equals 0. So, from here we are saying by replacement theorem of linear algebra, ak can be replaced by any bi provided alpha ik is not equals to 0 i will come to this why alpha ik should not be 0. Now this implies that del X B del xk this is equals minus B inverse ak and which is nothing but minus alpha k. So, that X B becomes B inverse b minus xk alpha k, B inverse b minus xk alpha k.

Now, put B inverse b this is equals beta, and which is nothing but beta 1 beta 2 like this way beta m say B inverse b I am assuming as beta and which i am writing as this thing.

In that case you can tell that beta 1 beta 2 like this way beta m minus xk into this, actually this X B always will be greater than equals 0. Xk into alpha 1 k alpha 2 k like this way alpha mk and this is greater than equals 0. So, from here X B greater than equals 0, and if I assume B inverse b equals beta take this column vector then income in terms of column vectors I can write down this thing.

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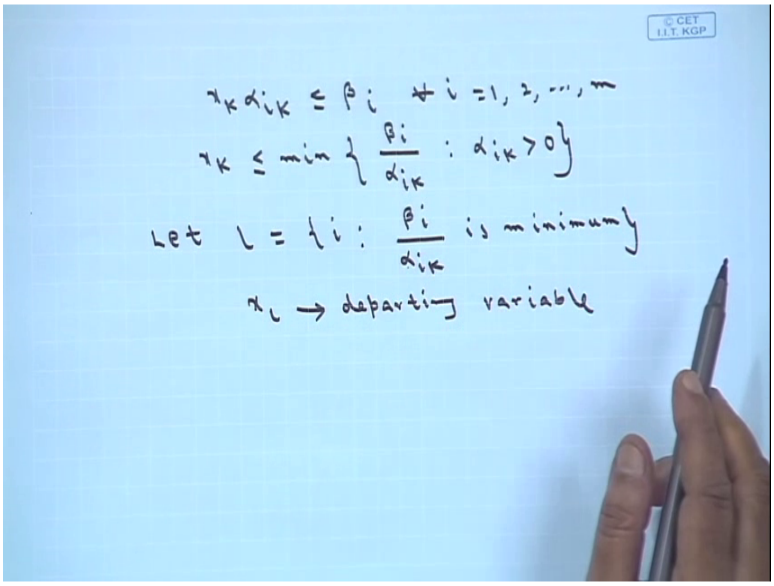

Now, from here we can write down xk alpha ik. Xk alpha ik is less than equals beta i for all i equals 1 2 like this way m. So, from here for individual things we can tell that a xk alpha ik should be less than equals beta i for all i equals 1 to m. Or I can say xk is less than equals minimum of beta i by alpha ik. Where such that alpha ik is greater than 0 minimum of this. So, xk now we are saying it should be minimum of anyone of this where i varies from one to m. If you assume let l equals i such that beta i by alpha ik is minimum.

So, we are assuming l equals i such that beta i by alpha ik is minimum. Then we say that this index l; that means, xl will be the departing variable. Or in other sense if i have to say I will say that we are finding the ratio of beta i by alpha ik, whenever we are solving the problem we will see to it. So, when beta i by alpha ik is minimum for which value corresponding to that variable will be the departing variable. So, i think that now it is little bit clear, whenever I am going for the entering variable, how to use the entering variable and then how to select the departing variable.

So, let us take now one example how to find out the solution of this type of equations.

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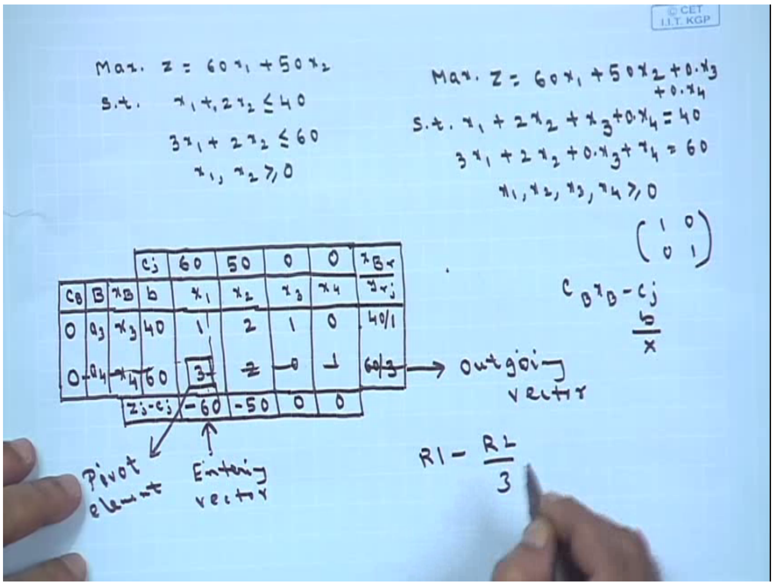

Let us take a problem maximize z equal 60 x 1 plus 50 x 2, subject to x 1 plus 2 x 2 less than equals 40, 3 x 1 plus 2 x 2 less than equals 60 and x 1 x 2 greater than 0. So, since I have 2 less than equals variables first thing is I have to convert it into the equality type the constant. And this equality type I can use by I can do it by using the slack variable, this we have done earlier. We have told it earlier that whenever you have less than equals type equation by adding the slack variable I can make it an equality type.

So, your problem in standard form i can write it as maximize z equals 60 x 1 plus 50 x 2, subject to x 1 plus 2 x 2 plus x 3 because this is your slack variable and this is equals to 40 we have told the other one second one will be 3 x 1 plus 2 x 2 3 x 1 plus 2 x 2 plus I can write it 0 into x 3 plus x 4 equals to 60. So, here also i can make 0 into x 4, where x 1 x 2 x 3 x 4 all are greater than equals 0. X 1 x 2 x 3 x 4 all are greater than equals 0. So, therefore, what is happening here since Ii have 2 less than equals a type of inequality?

So, I am adding to slack variables x 3 and x 4 and i am adding here. So, what is the cost associated with this? Cost associated with this will be for the variable slack variable always it will be 0. So, in the cost function for the slack variables the coefficient will be always 0 please note this one. That for slack variable the coefficient in the cost function it will be 0. Now see this one C B, B, X B like this way it is written. If you note here this particular portion this side if you see, this is forming one linearly independent set of vectors, because you need vector you know it always will be that one.

So, what are the variables associated here, x 3 and x 4. Therefore, in your basis basically the variables will be x 3 and x 4. So, this I am writing as x 3 and x 4. So, using these wherever you are obtaining the unit matrix. Corresponding variables will go into basis those will be basis basic variables, and the other 2 variables that is x 1 and x 2 will be the non basic variables. If you remember we have told that in the table the first thing is that the first row represents the coefficient of the cost. Like for x 1 coefficient in the cost function z is 60. So, you are writing 60 for x 2 it is 50 we are writing 50 for x 3 it is 0 for x 4 it is 0.

This b corresponds to these 2 values 40 and 60. So, i am writing this as 40 and 60 and these coefficients are one 2 for x 3 it is 1; it is 0. The next one is 3 coefficient of x 2 is 2, then it is 0 and then it is 1. Your B case we write the variables as a 3 and a 4. For C B what is C B? C B is the value coefficient of x 3 and x 4 in the cost function. Coefficient of x 3 and x 4 in the cost function that is this 0 and 0. So, these are the 0's.

Now, you have to calculate the value of zj minus cj. Zj minus cj is C B x B minus cj C B X B minus cj. That is this into this the 0 into 1 plus 0 into 3 minus 60. So, it will be minus 60 since we have 0 basically. So, it is C B is this part into X B value whatever is there minus cj. So, for the second variable x 2 it will be 0 into 2 plus 0 into 2 minus 50. So, that the value will become minus 50. The next one is 0 next one will also be 0. So now, you see we have told if all zj minus cj greater than equals 0, then only we obtain the optimum value, but for this case that is not true.

So, some are non negative which is the most non negative here it is minus 60 is the most non negative. So, therefore, this will be the entering vector. This will be the entering vector. So, basically from this basis one variable will go out and x 1 will enter. Next is i have to calculate which variable will depart. For that one I have to calculate the ratio. What is the ratio? Ratio is this b divided by x nothing else. Whatever that alpha we have told here it is b divided by x. What is b? Here it is b for this entering variable this x. So, it will be 40 by 1 and this case it will be 60 by 3.

So, therefore, this value 60 by 3 is minimum. So, your outgoing vector will be this thing, outgoing vector is this one. So, in the next table what will happen? X 1 will enter, and x

4 will go out let us see quickly what happens to the next table. In the next table, in that case what will happen here?

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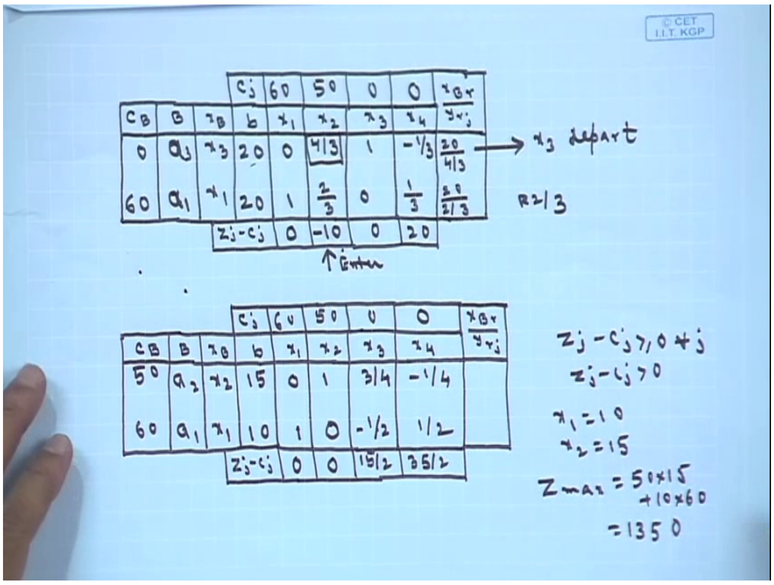

In this case for X B it will be x 3 and x 4 will be replaced by this. These values remains same that is 60, 50, 0 and 0 your b value is there sorry, this is X B is x 3 and x 1 this is a 1 and a 3 corresponding to x 3 and x 1 the cost coefficients corresponding to x 3 it is 0, corresponding to x 1 cost coefficient is 60.

So, I am writing 60 over here. Now what i have to do if you see this one you are having this is the departing variable this is the entering variable. Wherever they meet that point is known as the pivot element. This point we call it as the pivot element. So, what i have to do in the next table. I will make this pivot element as one, and in the all other elements of this column first column i have to make 0. So, I have to manipulate the matrix in such a way that this element have to make one, and all other elements of this table will be 0.

So, to make this particular element as 0, I have to divide the entire row by 3. So, that I will obtain here in this case i will obtain here 20, it was one, then it is 2 by 3 0 and 1 by 3 2 by 3. 0 and 1 by 3 simply by dividing this. Now how to make this element as 0. This is row 1 minus row 2 by 3, if i make row 1 minus row 2 by 3. So, in the first case what i have done i have made it this i obtained by making a 2 divided by 3 to obtain this element as 1. Whereas, to make this element as 0, row 1 minus row 2 divided by 3. So, I have to start from here itself. So, that it will be 20 row 2 minus this thing it was 40 and 60.

So, these minus 60 by 3 60 by 3. So, it will be 20. So, 40 minus 20. So, that you will obtain 20 in the next element if you see 1 minus 3 by 3 it will be 0. So, you are writing 0. Like this way we can put the other values 4 by 3 1 and minus 1 by 3. Now calculate again zj minus cj in zj minus cj this into this plus this into this minus this. So, 0 into 0 plus 60 into 1 minus 60 it is 0, next one will be minus 10 you can calculate it. This is 0 this is 20. So, this zj minus cj are not greater than equals 0. So, you have one entering vector. So, this will enter this vector x 2 now we will enter into basis.

Now, you have to calculate the negative the which vector will go out, for that one b divided by this x 2 this I have to use. So, here you will get 20 divided by 4 by 3, and in this case we will obtain 20 divided by 2 by 3. The minimum value you will obtain here. So, therefore, your x 3 will depart. So, in the next table your x 3 will depart So that here now x 3 will be replaced by x 2 x 1 will be there. All these values are fixed 0 0. So, corresponding to this it is a 2 and a 1. These values are corresponding to x 2. C B is 50 corresponding to x 1 this is 60. If you see here for this the pivot element is 4 by 3 this one; that means, now i have to make this element as one and all other elements of this as 0. Using the earlier approach whatever we have told.

So, I am not explaining that one now I am directly writing this one i have to basically divide multiply by 3 by 4 to get it one. So, in that case other values will be this thing 3 by 4 and minus 1 by 4. And whenever i have to make this x 2 as 0 in this case. So, you will obtain the values as b will be 10 this one is x 1 is 1. Then it is 0 then minus half and plus half. Now if you again calculate this into these 50 into 0 60 into 1 minus 60 will be 0, this one is 0 this will be 15 by 2 and this is 35 by 2.

Now, you see for this case zj minus cj greater than equals 0 for all j. Zj minus cj is greater than equals 0 for all j, and in this case zj minus cj is greater than 0 for non basic variables, x 3 and x 4. So, we obtain the optimum value. What will be the optimum value? The optimum value you can obtain from here that is x 1 will take the b value that is x 1 equals 10. X 2 this is equals 15 and the value of z z max you can obtain by calculating, these values that is this into this plus this into this. 50 into 15 plus 10 into i am just writing 50 into 15 plus 10 into 60. So, it will be 1 3 5 0. So, this is the solution of this problem. So, in the next class we will explain one more example to illustrate this algorithm.