Constrained and Unconstrained Optimization
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Lecture - 06
Solution of LLP: Simplex Method

Now in this particular class, we will start the simplex method that is solution of linear programming problem by simplex method.

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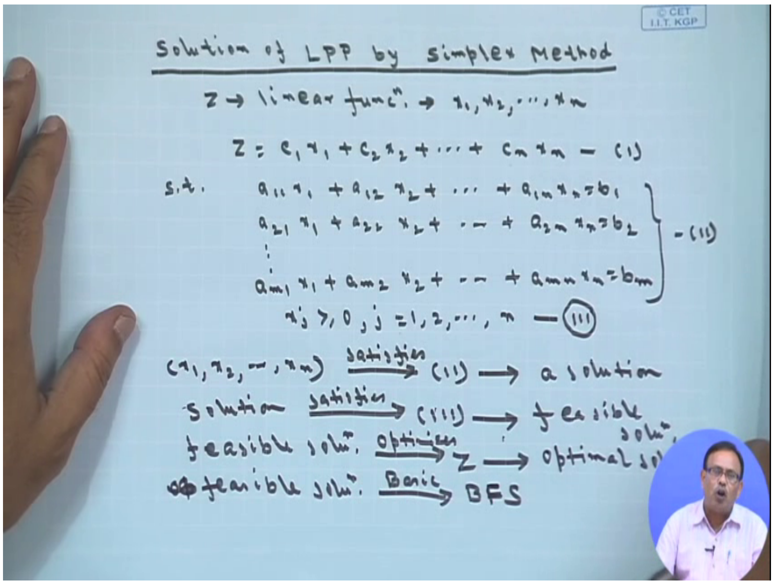

Let us see we are having z which is a linear function because as you know LLP the objective functions as well as the constraints are linear. It is a linear function of n variables say x 1 x 2 and x n. We are writing z equals c 1 x 1 plus c 2 x 2 like this way, plus c 2 x n cn xn say this is one subject to I am just again repeating, I have written earlier a 1 1 x 1, plus a 1 2 x 2 plus a 1 n xn this is equals B 1, a 2 1, x 1 plus a 2 2 x 2 plus a 2 n xn. This is equals B 2 like this way a m 1 x 1 plus a m 2 x 2 plus a m n xn equals B m.

this we are denoting as number 2 and x j greater than equals 0, where j takes the values from 1 2 n. So, this is number 3. So, the general LLP can be defined as maximum optimization that is maximization or minimization of a function z equal c 1 c 2 c 1 x 1 plus c 2 x 2, plus c n x n defined dr by one subject to satisfying the constraint 2 and 3 whenever you are trying to satisfy the constraint 3 basically denotes that it is a non negativity constraint of the decision variable.

So, any n tuple solutions x 1 x 2 xn which satisfies the constraint 3, we call it as the non solution. Sorry, this is a non negativity now you take n tuples x 1 x 2 xn. A set of n tuples x 1 x 2 x n these satisfies the constraint 2, then you will obtain a solution of the LLP. Please note this one is, if I obtain a set of tuples n tuples x 1 x 2 xn which satisfies the constraint or condition 2 then it is a solution of the LLP. Now any solution which also satisfies, if a solution satisfies the non negativity condition 3, then we call it as a feasible solution.

So, you are getting solution, then you are having the feasible solution. Now whenever you are having a feasible solution if this feasible solution optimizes the objective function z then we call it as the optimal solution. So, please note this one that whenever a feasible solution optimizes the objective function z then it is an optimal solution. And if the solution which you are getting that is if I have a feasible solution, and if this solution is basic then we call it as bfs or basic feasible solution.

So, please note this one I am reemphasize these things, because afterwards very frequently we will use these terms. You have a problem that is you have to either maximize or minimize a function z given by equation. One subject to satisfying a set of constraints given by 2 and 3, 3 denotes the non negativity constraint of the decision variables. Now a set of n tuples x 1 x 2 xn which satisfies the condition 2 then we call it as a solution. Whenever the solution satisfies the non negativity constraint 3 we call it as a feasible solution if the feasible solution is basic then we call it as a basic feasible solution.

And if the feasible solution optimizes the objective function z then we call it as a optimal solution. Now let us see what is the solution of LPP by simplex method. What are the steps the what are the steps used in this particular method? The steps are first compute the trial basic feasible solution now as I have told you at the feasible region you have to find out the basic feasible solution, we have defined earlier what is basic feasible solution. What is basic feasible basic solution and from the basic solution, how to find out the basic feasible solution. Here we will use the concept of linearly independent vectors whenever we are going through the examples we will do discuss it.

So, first is the compute trial bfs number 2 is test the initial basic feasible solution for optimality, that is whether the basic feasible solution is optimal or not how to test whether it is optimal or not that is we will see after some time. Then number 3 improve the initial basic feasible solution by a set of rules, that is you have the initial basic feasible solution if the solution is not optimal. In that case I have to improve it by certain rules or techniques. And I have to repeat the step 2 and step 3 until I derive to the optimal solution or I terminate that there is no solution.

So, these are the basic 4 steps please note this one. That I have to first find out the trial or initial basic feasible solution I have to test whether this initial basic feasible solution produces the optimum value of the objective function or not. If it is not producing the optimal value of the objective function in that case I have to improve it by certain rules and then I have to repeat the step 2 and 3 until optimal solution is derived.

now whenever I have a problem in simplest method, the problem we represent in tabular from each terms of each term at of rows and columns. Each row is represented by one constraint we will show it each column is represented by one variable. And top row corresponds to the of objective function. So, basically we are giving the input in terms of a table, where each row is represented first row of the table is corresponding to the objective function and subsequent rows corresponds to the constraints. And each column is represented by one variable. Let us see how it is being represented.

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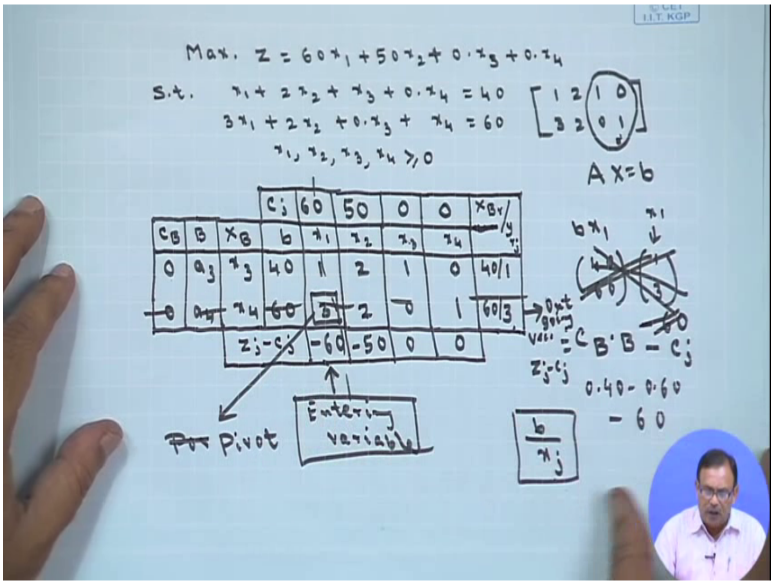

Suppose you have a function I have to maximize z equal 60 x 1 plus 50 x 2 plus 0 into x 3 plus 0 into x 4. I will come to this why I am writing this one subject to this constraint the constraints always should be equality.

So, we represent it in a tabular form like this, but we will discuss what is cb B xb x 1 x 2 x 3 x 4 cj, your cj is here in this particular case your cj is the coefficient of the cost functions that is coefficient of cost function is in x 1 is 60. So, here I will write down 60 coefficient of cost function with respect to variable x 2 is 50. So, I am writing 50 here then x 3 with respect to x 3 the coefficient is 0 here, it is 0 I am writing 0 here with respect to x 4 the coefficient here it is 0. So, I am writing over here.

Now, from here what is which portion of this if you see this is a matrix which we can write down 1 2 1 0 3 2 0 and 1. So, if you see these 2 always will form a linearly independent vector. So, I have to chi see where I am getting the unit matrix. And corresponding to this unit matrix, whatever decision variables are associated those will be the here basic variables. So, basic variables are those corresponds to those that unit matrix which I am declaring in this constraint set.

So, for x B I will write down here x 3 and x 4. So, x B what it corresponds to, x B corresponds to the basic variables which is arising from the unit matrix in the constraint. So, I my aim is to form one unit matrix in the hole constraint set a and from there I am obtaining the basic values. Of these variables what is B as you know your equation is of the form ax equals B that is B is this value. So, I will write down 40 and 60 here as we have told first row corresponds to the objective function that is coefficients of the decision variables in the objective function. And subsequent rows correspond to the coefficients in the constraints of the decision variables.

So, your first constraint is x 1 plus 2 x 2 plus x 3 into 0 x 4. So, coefficient of x 1 is one x 2 is 2 x 3 is 1 and x 4 is0. So, you are writing this 3. And similarly your x 4 next row will represent the second constraint, here it is coefficients of x 1 x 2 x 3 and x 4 are 3 2 0 and 1. Your cb value is nothing, but the value corresponds to the objective function here corresponding to x 3 and x 4. Or in other sense the coefficients of x 3 and x 4 in the objective function is the value of cb. B value we may write we may not write this is actually a 3 a 4 to denote the variables.

now what is your z value, sorry your zz minus c j this represents the reduced cost that is actual cost zj is the total cost minus cj is the cost of this one. So, in this case what will be this value B into x 1 I am just writing for this one, B into x 1 plus B into or in other sense if I have to write down 40 60 into 1 3 that is this corresponds to x 1 minus cj that is c 1 here it is c 1 is 60. So, this one. So, or in other sense it is 40 minus 180 minus r sorry zj this plus these into this plus these minus 60.

So, we will obtain the value as minus 60. So, like this way sorry I am I have written wrongly here, totally I have written wrongly this is cb into this B value minus cj your z j minus cj is this one, cb into B minus cj that is 0 into 40 0 into 40 minus 0 into 60 minus 60. So, that the value here it becomes minus 60. Similarly, for this case 0 into this value 2 plus 0 into this value 2 minus 50.

So, it will be minus 50 since both are 0 here. So, it will be 0 and this will be 0. So, like this way here, I have to check where it is most negative. Why it is most negative that I will come to the point whenever we are going through the going to the theory of simplex method. So, maximum value is obtained over here. So, whatever wherever maximum value is coming we say that this will be the entering variable. Please note this one this will be the entering variable wherever we are getting the maximum value of zj minus cj.

So, we will get this one now to calculate where is the which one with the outgoing. So, basically if you see you have the decision variables x 3 and x 4 and you are getting for the reduced cost negative values. So, I have to continue or this feasible solution it is not providing me the optimum value. So, to get the optimum value there will be one entering vector and there will be one outgoing vector entering vector will be that one corresponding to that column where zj cj value has most negative value. And similarly now you calculate xbr minus yj what xb r minus yj. This is basically B by xj value this one this is basically B by xj value xj may be one 2 3 4.

So, here this is the entering variable. So, variable is x 1. So, that is it will be 40 by 1. And it will be 60 by 3. So, the minimum value of these 2 is obtained here minimum value is obtained for this one. So, the outgoing this we say as the outgoing vector, please note that this is outgoing vector. So, you have one entering vector depending upon where the value of zj minus cj is most negative. Then you have to calculate the ratio xbr by yr j xbr by yr j is nothing, but bb by xj the represent corresponding to the entering variable. For this case it is 40 by 1. It is 60 by 3 here it will be 40 by 1 and 60 by 3.

So, outgoing variable is this vector will be x 4 and incoming vector will be this one. So, these 2 needs means wherever if I draw a line column corresponding to this is this outgoing is this. So, they meet at a point 3, say here this we call as the pivot element. This we call as the pivot element. And this value will see what to do. So, effectively what is happening for this particular case here. If you see now wherever I am transforming my LLP in some form like this from there. I am putting it in a tabular form I am calculating certain values of zj minus cj from here.

Then I am checking what will be the entering vector what will be the outgoing vector and basically I will repeat this process, until my reduced cost zj minus cj is non negative that is greater than equals 0. This I will discuss what is the determining entering vector and outgoing vector also.

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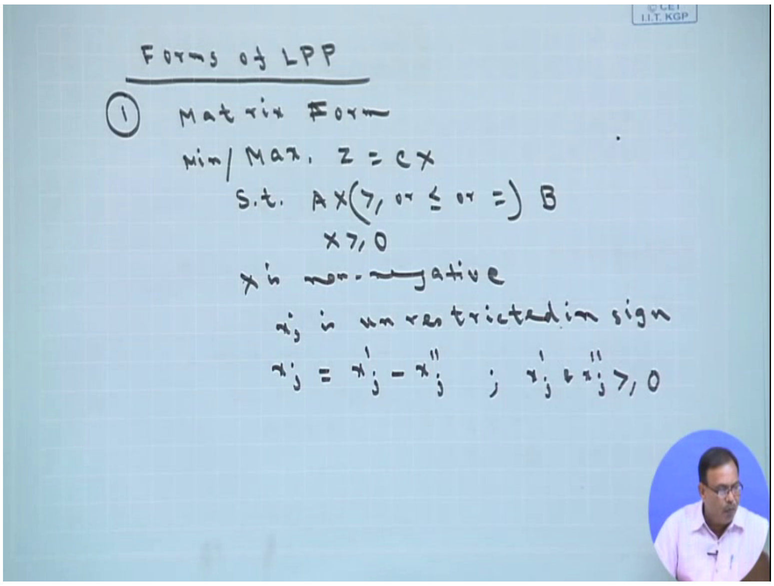

Now, different forms are available forms of LLP number one. We call it as matrix form in matrix form we will take the maximization problem because always if I have a minimization by doing minus I can change it to maximization problem.

So, maximize z equals cx subject to a x greater than equals or less than equals or equals B, I am writing the standard form and x greater than equals 0, where I am right not writing what is a B and c a is m cross n matrix B is also column matrix 1 cross m and c is a row matrix 1 cross n. And x is the in a decision variable that is the column matrix one 2 n this we have told earlier. So, this is your one form that is matrix form, from this matrix form we come to the second form which we call as canonical form.

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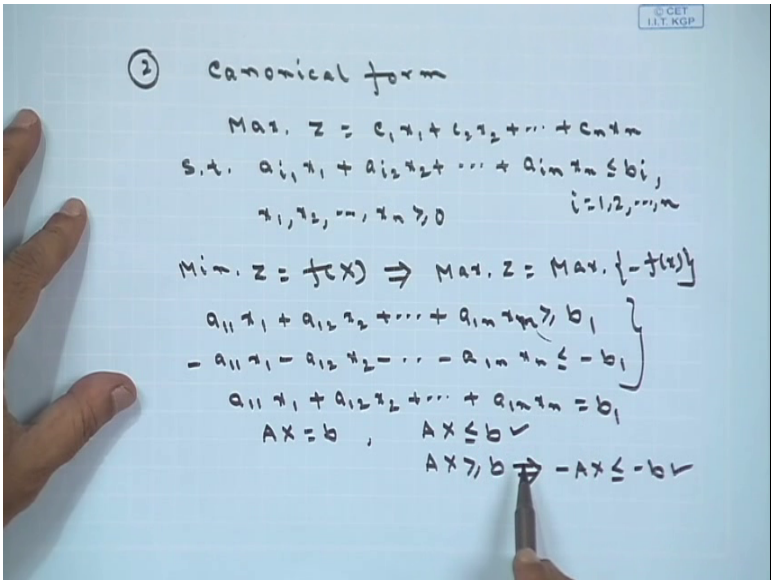

So, whenever I am writing matrix form then please write here minimum or maximum then it will be all right because your initial problem may be minimization problem or it may be maximization problem. A subject to ax greater than equals or less than equals or equals B and x is greater than equals 0. In canonical form always we write the LLP in this form maximize z equals c 1 x 1 plus c 2 x 2 plus c n xn subject to ai 1 x 1 plus a i 2 x 2 like this way plus a i n x l less than equals B i, I take the values 1 2 like this way, n and; obviously, x 1 x 2 and xn all the decision variables are greater than equals 0.

So, please note one thing; obviously, here that we have changed now the form, whether it is maximization problem or minimization problem I have to transfer it as a maximization problem. All the inequalities has to be only less than equals.

So, whenever we will solve we will always follow this notation problem will be maximization and the constraints will be the minimization. So, suppose I have a function minimize z equals some function of fx. So, this problem I can transform it as maximize z equals maximization of minus fx, that is if I make the negative of this then automatically the minimization problem will be transformed into minimization problem will be transformed into maximization. So, if I my problem is minimization problem. In that case I will transform my problem into minimization by making the negative of the objective function.

now I may have something like this a 1 one x 1 plus a 1 2 x 2 like this way plus a 1 n xn which is greater than equals say B 1. So, in this case again this is greater than equals, but I have to change it into less than equals. So, what I will do as we know the normal algebra, if I multiply both side by negative sign this greater than equals will be converted into less than equals.

So, I will write down these like this minus a 1 x 2 minus like this way minus a 1 xn is less than equals minus v 1. So, this is another type and your constraint may be something like this, a 1 1 x 1 plus a 1 2 x 2 plus a 1 and xn. This is equals B 1 that is equality type, but we have told always we will take it as the less than equals. So, this equality I can write down or in other sense I am writing in matrix notation, if I am having x equals B I can write down ax less than equals B. I can write down a x greater than equals B equality can be written like this, ax less than equals B and ax greater than equals B again this is not acceptable here.

Because the constraint always has to be less than equals only. So, this I will write down minus ax less than equals minus B. So, now, both this constraint and this constraint r less than equality the less than equals. So, therefore, if I have equality first I will convert it into 2 inequalities of less than equals type, and greater than equals type later the greater than equals type will be converted into the less than equals type, one more thing can come here that is we always say x greater than equals 0 or x is non negative. Right, but if it is given that xj is unrestricted in sign that is we cannot talk about the sign, it is not non negative it may be positive it may be pos negative anything.

But in our canonical form what we have told always all the decision variables must be greater than equals 0. So, if this is the case we can write down xj equals I can represent it by 2 more decision variables. And I can write it x j equals xj dash minus xj double dash where both xj dash and xj double dash both are greater than equals 0. Where both xj dash and x j double dash greater than equals 0, this is required whenever we are we will work with dual simplex method.

So, therefore, if xj is unrestricted in sign then, we write down xj equals xj star minus xj double star. And by this way we write down this one. This is the second form that is canonical form.

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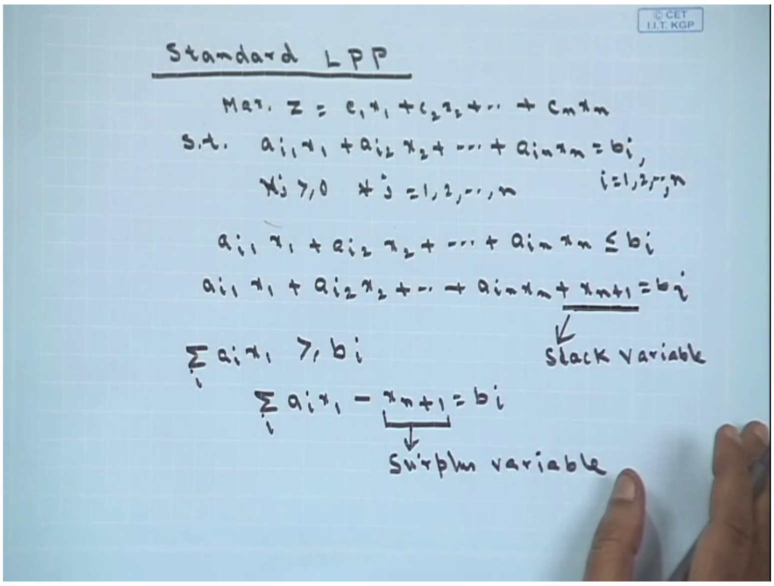

And the third form is which we call as standard LLP. That is, I have the original problem from there I am transforming it into the canonical form. And from the canonical form I have to transform it into the standard form my standard form will be maximized z equals c 1 x 1 plus c 2 x 2 like this way, plus c n xn subject to ai 1 x 1 plus ai 2 x 2 like this way plus a I n xn this is equals bi, i equals 1 to n and xj greater than equals xj greater than equals 0 for all j equals 1 to n. So, please note that in canonical form we made the inequality as less than equal sign, but whenever I will transform it. I have to make it equality sign in graphical method, if you remember we have told that if I have inequality sign make it equality sign and then the draw the line in 2 dimensional space. So, here also whether I have less than equal sign or greater than equal sign by some means I have to convert it into the equality sign.

So, and all the constraints are equations except this non negativity constraint. Now suppose I have this one, ai 1 x 1 plus ai 2 x 2 plus a I n xn this is less than equals B i, whenever the equality is less than equals the it is less than equals type of inequality to make it a equality I had to add some variable or I have to add some value. So, I can write something like this, ai 1 x 1 plus ai 2 x 2 plus a I n xn plus say some slack variable xn plus 1 this is equals B i.

So, please note that I have added one more value since it was less than equals type. And this x n plus 1 we call it as slack variable. So, basically what happens we add slack variable in 1, inequality sign where inequality is less than equals type or in other sense slack variable is used to make the less than equals type of inequality into the equation type of equation. Similarly, if I have the summation over ai into x 1 over i, this is greater than equals B i say in that case to make it equality, I have to subtract one variable. So, this I can write down summation over i a 1 x 1 minus x n plus 1 this is equals B 1.

Because if I subtract then this will be equals to B i. So, this variable we are subtracting to make it an equality and this variable is known as basically the surplus variable. So, if you see we are using the slack variable we are using the surplus variable to make one inequality type of equation into equality type of equation. So, these are the 3 type of forms the matrix forms from the matrix form we transform it into the standard form sorry canonical form from canonical form, we transform it into the standard form then we transform the standard form into the tabular form and we get the using the initial basic feasible solution and try to improve it and obtain the basic optimum solution. So, in the next class we will see the theory of the simplex algorithm; that means, how one vector is entering which vector will out go and all these things.