Constrained and Unconstrained Optimization
Prof. Debjani Chakraborty
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture – 59
Multi-Objective decision making

Today's topic is multi objective decision making problems. This is a kind of optimization model where we are dealing several objectives together and the objectives are conflicting in nature. In reality we have seen that in any decision making situation, though we are considering that there is only one objective in the mathematical bottle, but in reality we could see that this is a very much un realistic. Because all the time we are having the situation where we need to take several decisions together and decisions are conflicting in nature. And we have to take the decision under certain constraints, that is why in optimization technique we are dealing with multi objective decision making problems.

Now, today I will just introduce what is the concept of it. The subject is very vast and within a class that is very difficult to cover many things together, but I hope with this introduction if anybody is interested further they can go that is why I am dealing with the situation where we are having several objectives together.

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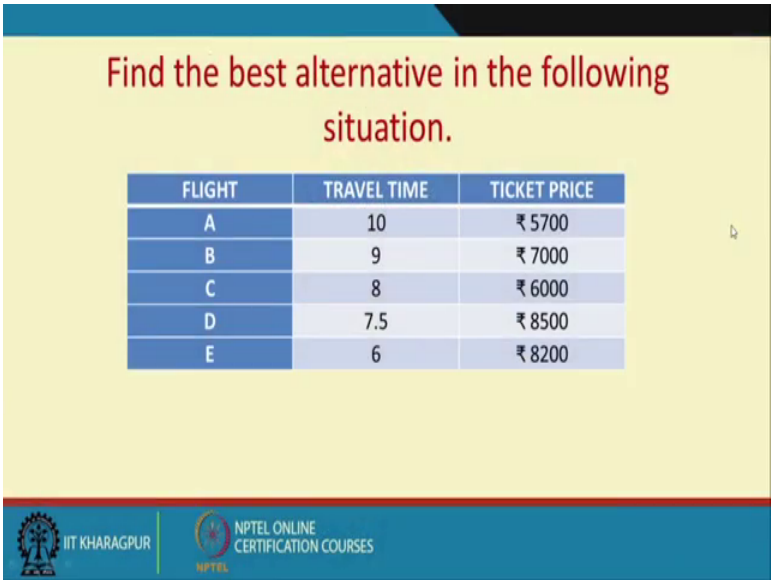

FLIGHT	TRAVEL TIME	TICKET PRICE
A	10	₹ 5700
B	9	₹ 7000
C	8	₹ 6000
D	7.5	₹ 8500
E	6	₹ 8200

For example, we are having a situation we have to take a decision there at 5 options for us there a flight. From one place we are moving to another place the source and destination that is fixed, but we are having several options in front of us. And for individual flight options the travel time and the ticket price both are given.

Now, in this situation we have to take a decision, that which flight should be selected where the travel time is minimum and the ticket price is minimum. That is obvious we do every day without knowing multi objective even, we are taking 2 decisions together which are conflicting you see I say again and again the objectives are conflicting in nature because you see we do not have any option where we will get the travel time minimum and the ticket price minimum. Because the data has been give given in such a way that you see where the travel time is high then the ticket price is low that. So, in this situation taking a single decision, if I ask you what is the optimal solution in this decision making situation very difficult to find out the optimal solution because there is no single optimal in this case. That is why we say this kind of problem as a multi objective decision making problem. One objective is minimization of travel time and the second objective is minimization of ticket price. If we just minimize the travel time, then ticket price is going up and if the ticket price we are minimizing then travel time is going up.

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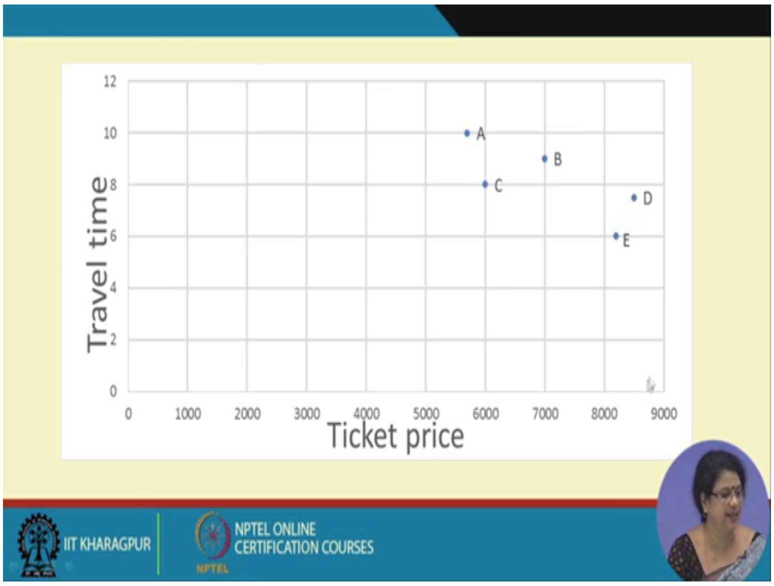

That is why in this situation we have to take a decision which one should be selected. If I just draw it in a 2 dimensional space, where x is x axis as a ticket price and the y axis as a travel time then we could see that.

If we just put all the flight options that is the travel time 10 hours with the ticket price 5700, if we just put then you see we could get this is this as the option A this is B C D E. Now if you consider geometrically even in 2 dimensional space. If I ask you which point is best point, when we are minimizing both, very difficult. Because you see if we say A is minimum with respect to x, but it is maximum with respect to y, C is minimum C is somewhat minimum with respect to x, but still E has a good option. Then C with respect to y that is why in this situation which one should be selected that is the difficult part let us try to compare you see we are having the data in front of us.

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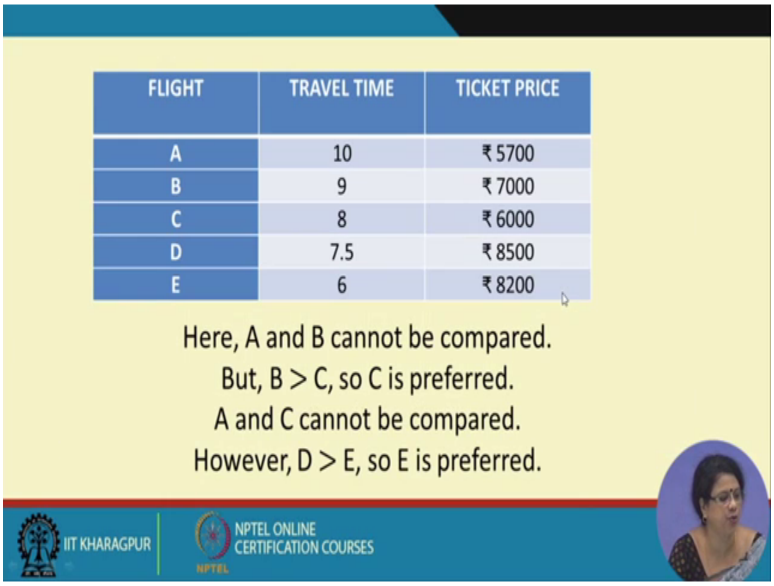

Again let us compare A and B within A and B which one to be selected A and B cannot be compared. Because minimum travel time ticket price minimum, we would not get from here if we include the option C, then we could see within B and C certainly C is the good option than B. But once we are coming the comparison with A and C, it cannot be compared again because travel time minimum and the ticket price minimum either a nor C nothing is accepted to us that is why we could conclude that B and C can be compared, but C is the best option now let us compare D and E, if we compare D and E then E is the best choice than D all right, but if you just include C within that or B or a within that you see we cannot compare that is why from here.

We can summarize the situation in this way that within D and E is E is more preferable within B and C, C is more preferable, but A C and E this 3 options are not really comparable, this is the conclusion only we can make that is why if I ask you to put your choice in a rank and in order then what you will put you will put probably any one of these 3 A C and E, I will select and in the next my choice is will be either D or B. This is the case.

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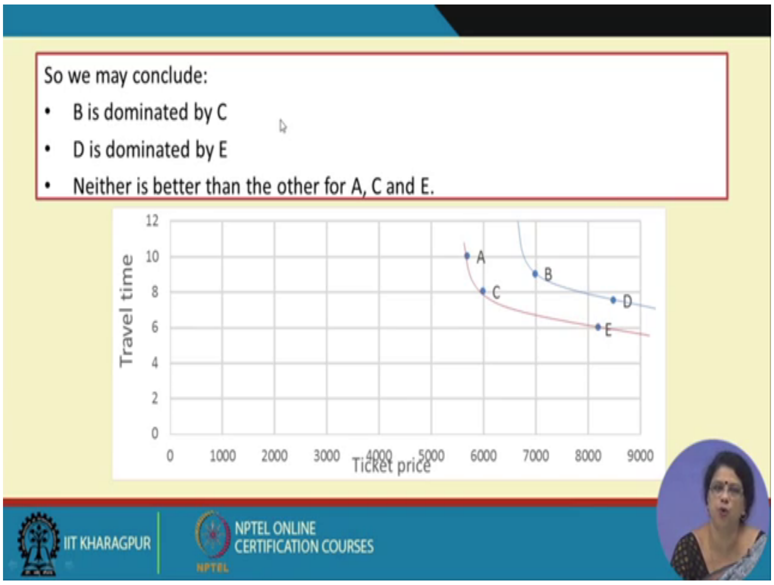

If we just again come back to the figure you see whatever is said that has been given with the lines, as a rank since we are going for the minimization that is why again I am saying that A C and E. These 3 options are not really comparable that is why we are putting together in one rank and B and D will there these are not comparable, but these are not prefer with respect to AC and E. That is why in this subject this multi objective optimization problem, you could realize that this is the situation day today we are facing.

But this is our subject where we are not really interested to find out the optimal solution, we are calling it as a Paratoo optimal solution we target. Now the word I attired that is the Paratoo optimal, instead of optimal there this has been invented by the scientist in the Paratoo. And in his name only the name has been given the optimality has been given as Paratoo optimality. And our target is to find out that which are Paratoo optimal to each other.

Now, in the structure if you if you just go, just we talk about the dominance here. Because you see A B C D E in dominance strum you will see that we will say that A is B is dominated by A B is dominated by C B is dominated by E, but B and D both are incomparable both are equally acceptable to us. Similarly, AC A C and E these are dominating B and D, but A C and E these are incomparable. No one is better than the other one that is why in the dominos structure all are in the same level. And B and D are in the same level that is why we have summarize the situation in this way B is dominated by C D is dominated by E neither is better than the other for A C and E.

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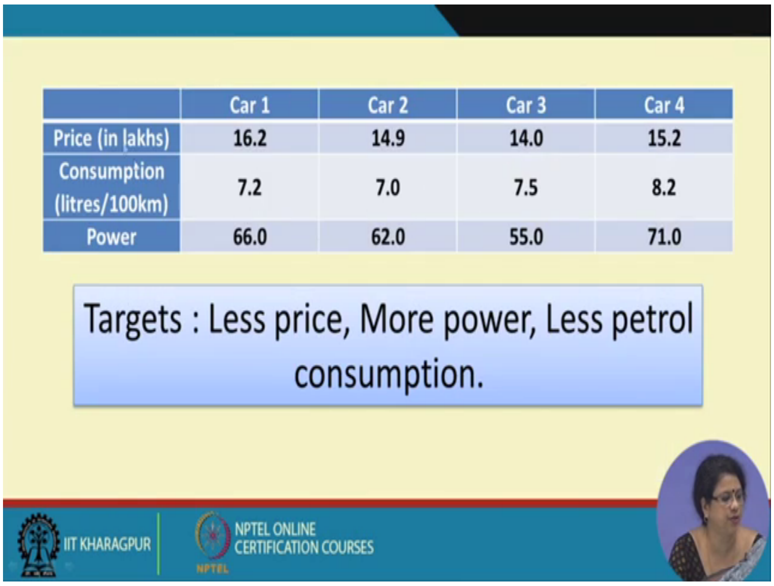

Let us go to the next situation. You see we are having 4 cars, now car one car 2 car 3 and card 4 4 models are there. Now we have to take a decision which car is more preferable to me. Now there are certain criteria to select the car model one is the price in lakh another one is the consumption that is very much important for us whenever we are considering the mileage of a car and what is the power of the car. That is also very much important that is why if I just look at these 3 criteria, and with respect to that we have a table where we have given the options of different cars. If we target less price more power and less petrol consumption rather the mileage will be more in that case.

If I ask you which car to be selected with this dominant structure, try it out. I would say that you the way I have said you just think about it and summarize that which car will be the best car for us, if you cannot take any best option go for the Paratoo optimality go for the dominos structure. Go for the ranking of the group of cars that is also quite possible. And we do every day without knowing multi objective programming theoretical. That is the basic idea of multi objective programming.

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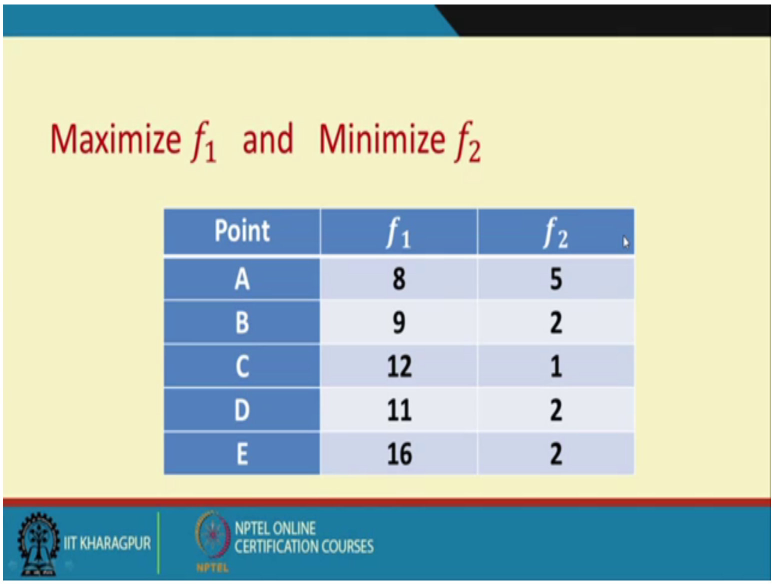

Point	f_1	f_2
A	8	5
B	9	2
C	12	1
D	11	2
E	16	2

Now, you see we have a data set, we are having the data that there are 2 objectives f 1 and f 2 and there are 5 options ABCD and E, we have to maximize f 1 and minimize f 2 that is our target.

Now, you see one thing is clear here when we are dealing this kind of problem, you have come across the situation of linear programming non-linear programming. Everywhere you have seen that we were having several options in front of us. You have come across the situation where the feasible space is consisting of infinite number of points. Infinite number of feasible points means infinite number of alternatives in front of us. And in finite infinite number of options are there in front of us.

But here in this situation you could you just notice that we are having very much finite number of options only ABCD or E. Anyone or a group of objects we have to select that is why the situation is a little bit different than the previous situation, but the again the optimization problem, we have to maximize f 1 we have to minimize f 2. Now in this situation how really we deal with if I tell you just you just do it just, I showed you the problem of car example there you were having 4 options 5 options or 4 options there of 4 car models.

Now, if I ask you draw it graphically very difficult for you because in 4 dimensional space you can visualize, but you cannot draw. That is why those kind of problem how to deal I am just showing you. Because through figure through graph you cannot do it graphically you cannot solve you have to solve it manually, that is why I am I will tell you how to prepare the dominal structure when we have several objective functions together and we are having finite number of alternatives in front of us.

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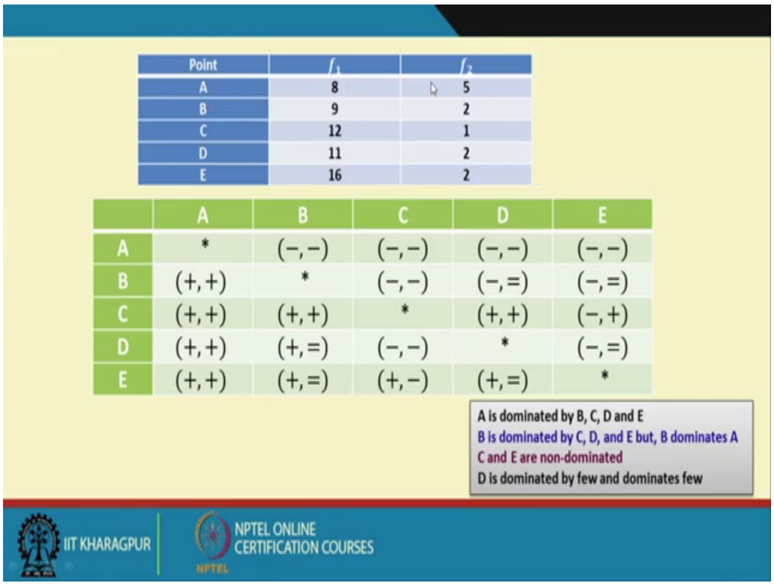

Now, what we do we just prepare the table in this way. I will just prepare the table for you, then I will explain the thing in detail you see we are having 5 options A B C D E.

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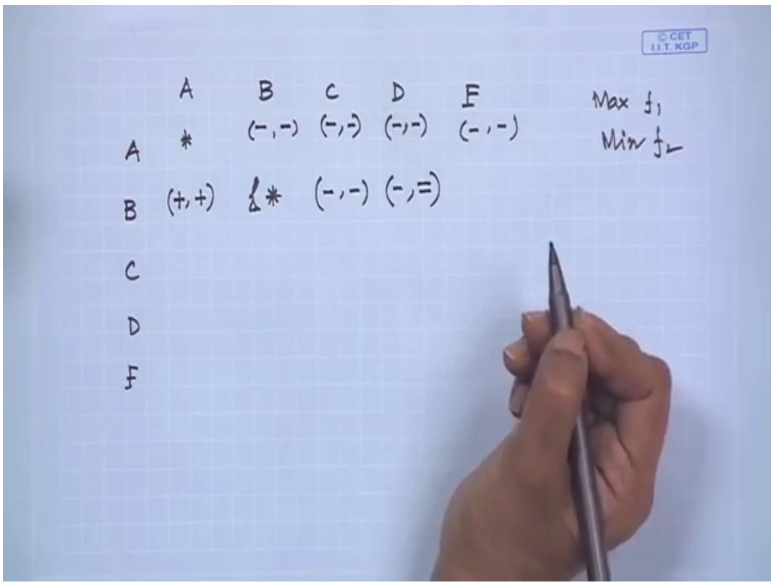

Similarly, A B C D and E what we have targeted. We have targeted maximization of f 1 and minimization of f 2. Now look at the data set this data set. Just compare A and A, if I compare A and A nothing can be saved because both are equal to me. If I compare A and B, I will just put a tuple is a combination of few elements together here I am putting A tuple of 2 elements within the paralysis. One is for the preference for A with respect to B with respect f 1 and A with respect to B with respect to f 2. If A is it that A is preferable to B with respect to f 1 not really because we are maximizing f 1. And the value of A with respect to B, A is 8 and B is 9 that is why with respect to f 1 A is not preferable to B, I am putting the minus sign there. Now let us C for f 2 with respect to f 2.

We are trying to minimize f 2 for A the value is 5 and for the B the value is 2. That is why A is not preferable to B with respect to f 2 again. That is why I am putting minus, but if we just compare B with respect to A what we could see that with respect to A, f 1 B is preferable to A and with respect to f 2, B is preferable to A as well in this way if I just compare A and C. Just you look at a data 8 and 12 with respect to f 1. That is why A is not preferable with respect to f 2 be A is not preferable. And similarly we can D do for a D A E we will get the same result a will not be preferable than B C D E at all right. If I go B with respect to A we have seen this is preferable for both the objective functions. Now, B with respect to C.

If we do maximization of A f 1; that means, B with respect to be nothing B with respect to C. Then not preferable and C is also not preferable. In this way I will put I will just complete another component in the matrix for you, then I will just summarize everything through my screen. Now B and D just you see B and E with respect to f 1 A B is not preferable, but with respect to f 2 both are coming in the same value 2 and 2 B and D. That is why we will put equal to in this way if we just do the entire table just you see if I just do the entire table with plus and minus combination then and equality combination then from here we can summarize the entire situation. In we can put everything at a glance we can take a decision which one is preferable which one is not preferable.

Now you see if you have if you are having 4 options instead of A B C D E 5 options, but if we have 4 objectives. Then we will have a tuple of having 4 elements together. That is why with plus minus and equality we can put all together in front of us. And from here we can summarize that A is dominated by B C D E. Because all are having minus look at B, B is dominating A, but B is dominated by A C dominated by B because one of the option is minus though one is equality. That is why B is lower prefer than D. And similarly for E and C and D, if you just compare C and E if we compare you see C and E these are non dominated to each other. Because all are having either plus or equality if both are equal preferable. If plus preferable that is why we cannot really compare in general here C and E. And what about D D is dominated by few and dominating few this is the situation. That is why which one will be put as the rank 1, if I ask you then.

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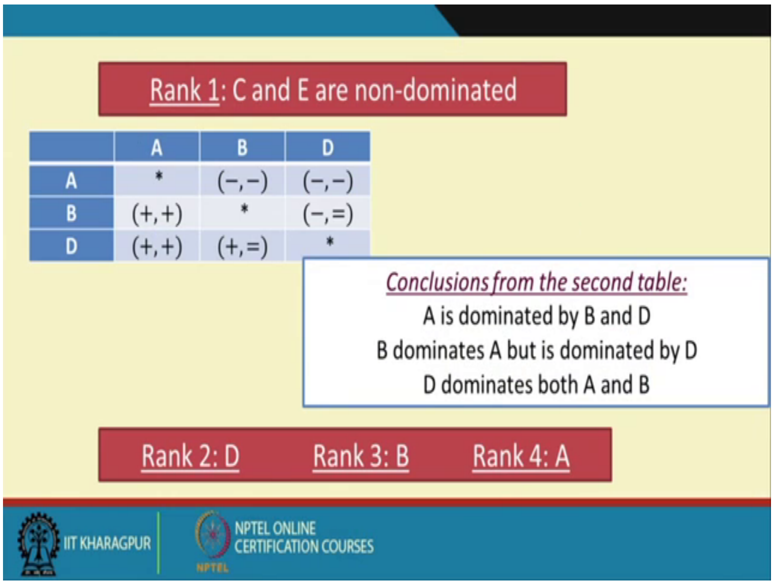

C and E both are non-dominated to each other. That is why I will put in the rank one C and E together. Now C and E I am keeping aside and I am again comparing A B D in the similar manner, and we could see that A is dominated by B and D that is why A must be in the last rank. And within B and D if we just compare D is more preferable than B because in B there is 1 minus. That is why I will put D here and a here that is why if I just summarize everything together we could see that we could group A C and E together. In one rank second rank D third rank B and the fourth rank A, but nothing I could say this is the optimal solution. I would say that C and E are Paratoo optimal solutions. We will see we will see that C and E are non dominated solutions we are we Paratoo optimal solution non dominated solution official solution these are all we use interchangeably.

Because non dominants conception mathematician still like very much. Because all the time we are talking about the dominance, we are talking about the total ordering partial ordering. That is why the mathematicians preferred the term non dominance. We say it both the solutions are non dominated to each other both are non dominated solutions, but economy is the prefer the term Paratoo optimal solutions. Because the Paratoo he is an economist and he has given the name and according to by his name only people economist prefer the term Paratoo optimality. We will say C and E both have Paratoo optimal solutions in this situation.

But the management scientist people there more habituated that the term that is the efficiency. Efficiency of an off a portfolio option where these a portfolio option is efficient in the market or not that way we they view. That is why the management scientist people they call it as efficient solutions instead of non dominated solutions or Paratoo optimal solutions, but non domination non dominated solution Paratoo optimal efficient solution all are equally means because all are accepted equally in, I am just I would say that C and D both are non dominated solutions both are Paratoo optimal solutions both are efficient solution. This I have shown you the option where we were dealing with the situation we were having very much finite number of alternatives.

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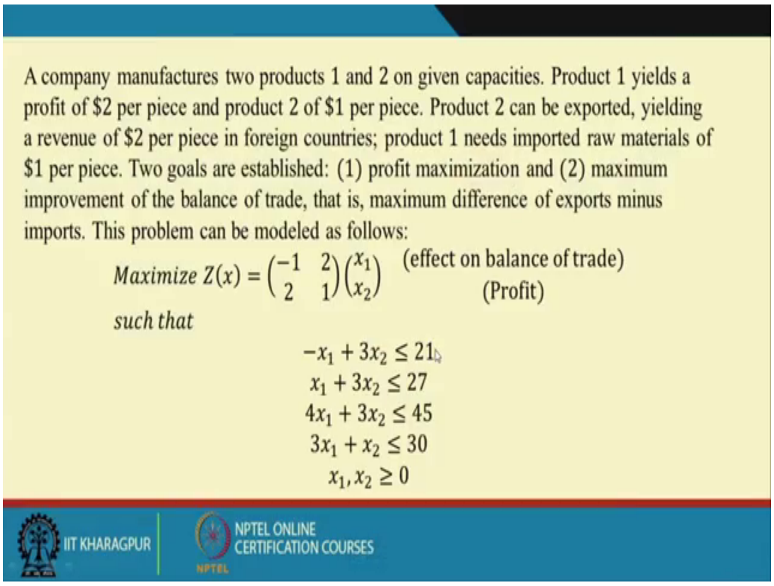

Now I am moving to a situation where we are having infinite number of alternatives. Just read the problem just see the language of the problem, there are 2 products and we need to maximize effect on balance of trade and the profit together and we are having certain constraints it. Now you see here the constraints these are having the linear structure we are having in the constraints set and objective functions again the linear. That is why if we just draw it graphically because you know the linear programming you dealt with the graphical solution.

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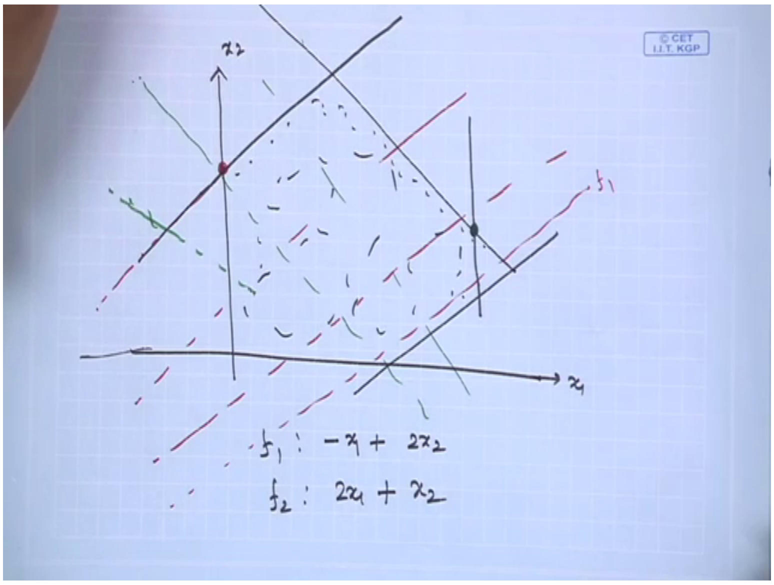

That is why let us try to draw the graph of this problem then we could see that the first one is minus x 1 plus C x 2 less than is equal to 21. That is why if this is x 1 this is x 2 one line this could be this way ok.

The second one x 1 3 x 2 27 that is why one line will be this way and less than equal to that is why certainly. This region we are considering this way if we draw 4 constraints together.

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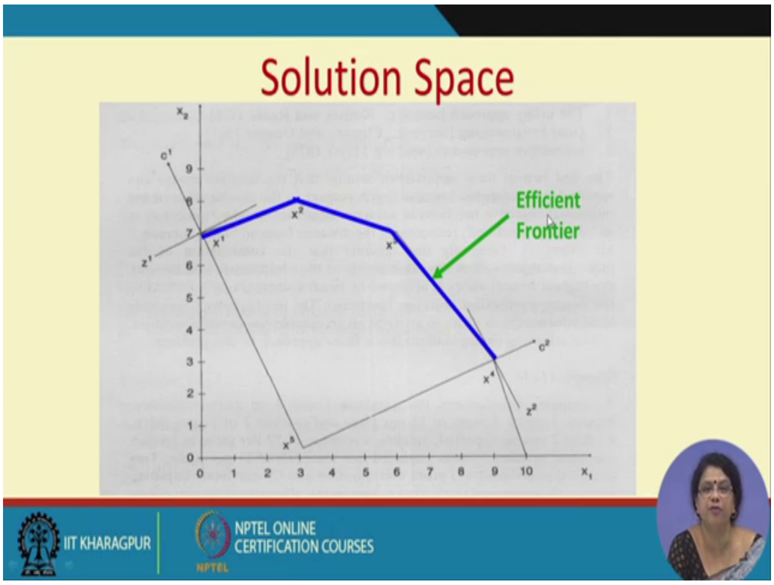

Then we will get this picture these are all the 4 constraints together. And this is the feasible space for us. Now you see if there are 2 objective functions one objective function is minus x 1 plus 2 x 2, this is one objective function. And another objective function is there that is 2 x 1 plus x 2. If we just draw it this is the feasible space for us all right.

Now, minus x 1 plus 2 x 2 it means that that is the yeah, we could see that this is the slope of the f 1 all right. The red color this is the slope if it moves this way, then ultimately here I will get the optimal solution because maximally the objective function can move this way after this it will be out of the feasible space. This is for function one if we do for function 2. The similar thing then we could see that no this is not 2 x 1 plus x 2 that is why it would be this way.

If we could see this way if the objective function is moving, we will see the optimal solution will come here all right. In between just what just you see in the in the screen it is there this is the first objective function this is the second objective function. This is the situation, for you now if I ask you which solution then we will take we will maximizing both together we are not getting a single option we are getting an option one for first objective function this point x 1 and for the second objective function x 4 look at more about it.

You see when we were moving we were just think that we are walking through the edge of the feasible space. I am moving from x 4 to x 1 what we could see that second objective function is reducing further and further if I move through this line. Through the edge of the physical space my second objective function z 2 value will not improving it is the value will be minimizing, because if I move this way other way x 1 to x 4 objective functional value will be increasing, but if I move from x 4 to x 1 the objective functional value will be decreasing, but in the other case if I just looked at the first objective function if I move from x 4 to x 1 first objective functional value is increasing.

Whereas if I move from x 1 to x 4 through the edge of the feasible space if I just walked through the edge of the feasible space the z 1 value is decreasing. That is why through this edge z the one is increasing z 2 is decreasing at every point; that means, I cannot have a single option where both the objective functions are maximum, I can select any one if I select this one then I have to compromise with z 2. If I select x 4 then I have to compromise with z 1. Similarly, if I just go at point x 2, I have to compromise z 1 a little bit, but z 2 more. That is why I cannot really take a decision from x 1 to x 4 if I just walk through the edge of the feasible space then which will be more preferable to me, I will say both are equally accepted to me.

Because if I improve one objective functional value other objective functional value for that we have to really compromise. That is why always there will be a trade off within this line. That is why all this points in the in the space these are equally accepted to me. That is in other way I will say all points are non dominated to each other Paratoo optimal to each other efficient to each other. That is why we can say that the points which are at the edge of the feasible space from x 1 to x 4, these are non dominated solutions or Paratoo optimal solutions or efficient solutions. Now I hope you understood the situation though I have writ10 it as efficient frontier. We call it as Paratoo frontier as well and we say non dominated frontier as well the same thing.

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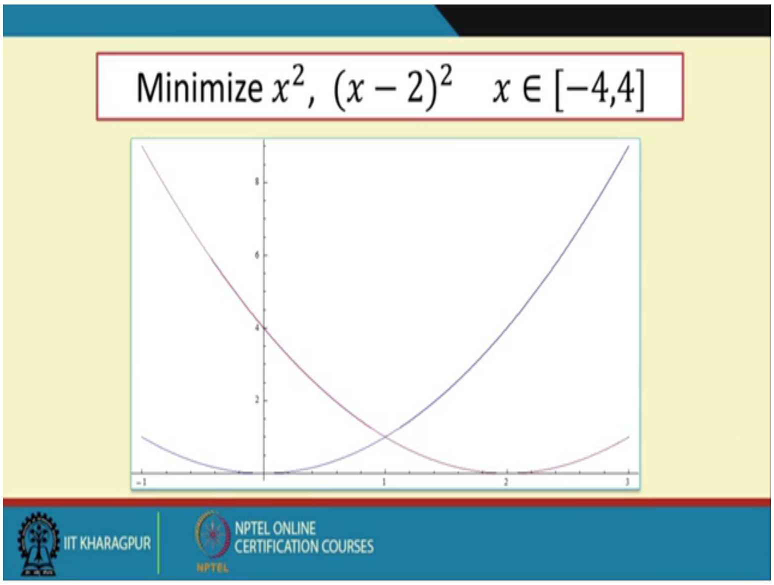

Now, let us consider another objective function. This is the un contestant non-linear programming problem that x is moving within minus 4 and 4, but minimization of x square and minimization of x minus 2 square, if I just draw it this is x square and this is x minus 2 square all right.

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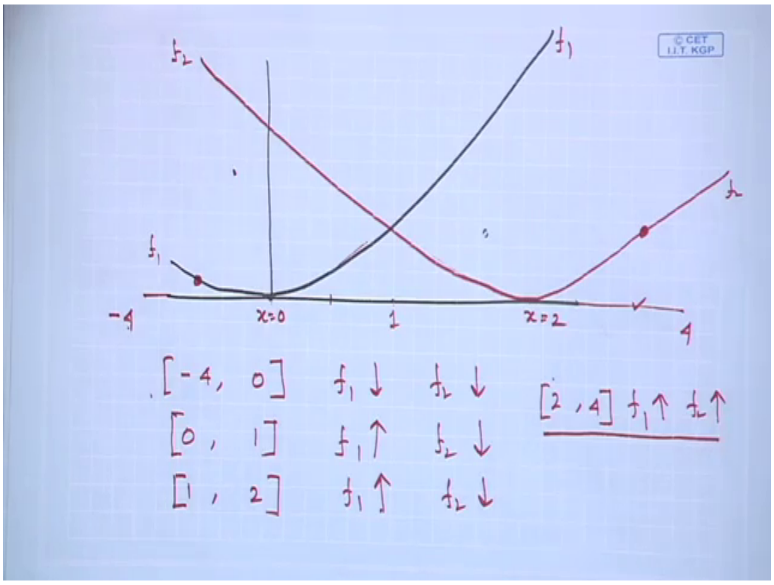

Now, what we could see that from minus 4 to 0, just let me draw it for you this is x square this is x minus 2 square all right. This is 0 x equal to 0 and this is x equal to 2 and x is ranging from minus 4 and plus 4.

Let me just move through the x axis if I move from minus 4 to 0, what we could see that this is my f 2 this is my f 1, f 1 is decreasing as well as f 2 is decreasing from minus 4 to 4, minus 4 to 0 both f 1 and f 2 decreasing all right. Now this is the point this point is as 1. If I move through 0 to one we could see that f 1 is increasing, but f 2 is decreasing all right now. Similarly, if I move from 1 to 2 f 1 is increasing, now 1 to 2 f 1 is increasing and f 2 is decreasing and from 2 to 4 f 1 is increasing and f 2 is increasing once again I am repeating. From minus 4 to 0 f 1 is decreasing f 2 is decreasing all right from 0 to 1 f 1 is increasing this is the f 1 line f 2 is decreasing this is my f 2 curve. From 1 to 2 f 1 is increasing, but f 2 is decreasing again from 2 to 4 both are increasing this is the whole pattern. If I ask you that within minus 4 and 4 where is the tradeoff; that means, one is increasing and other is decreasing I cannot have a single option here. From minus 4 to 0 if I want to minimize both f 1 and f 2 certainly I will select this option, there is no doubt about it from 2 to 4 if I ask you to minimize f 1 and f 2 both together for this point I will select f 2 that would be the minimum value, but within 0 to 2 there is no single option if I select this point then f 1 is f 1 is lower, but f 2 is higher.

Similarly, here also the same within this space I would say that there is a tradeoff. Because within this range both the values f 1 and f 2, if f 1 is I prefer f 1 then I have to compromise with f 2, if I prefer f 2 I have to compromise with f 1, f 1 that is why I would summaries I would say that from 0 to 2, I am unable to take a decision I will say from 0 to 2 all are equally accepted to me. That is why the non domination will be there Paratoo optimality will be there from 0 to 2 and that is the non dominated frontier or efficient frontier or Paratoo optimal frontier.

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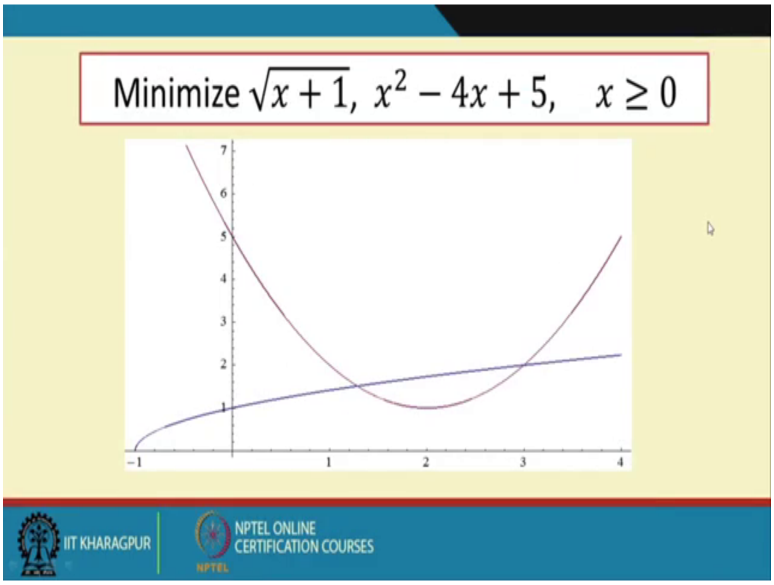

Now, from 0 to 2, I am giving you another example for you that minimization of root x plus one and x squared minus 4 x plus 5. And this part I am keeping as an assignment for you. If you try it out, we to which part would be the where the tradeoff of the both the objective functions at there you just try to find out because in x axis x is there in y axis f 1 and f 2 both are there. And you find out which one where the tradeoff is there for both objective function, then you can declare that is the non dominated frontier for you.

Thank you for today.