**Constrained and Unconstrained Optimization**
**Prof. Debjani Chakraborty**
**Department of Mathematics**
**Indian Institute of Technology, Kharagpur**

**Lecture - 57**
**Penalty and barrier method**

Now, today I will discuss the general penalty or barrier function method for those non-linear programming problem where we are having inequality constraints as well as the equality constraint. Not only that we will solve a problem I will try to solve a problem we were for the maximization type of non-linear programming. Now, I will start the class with this example today.

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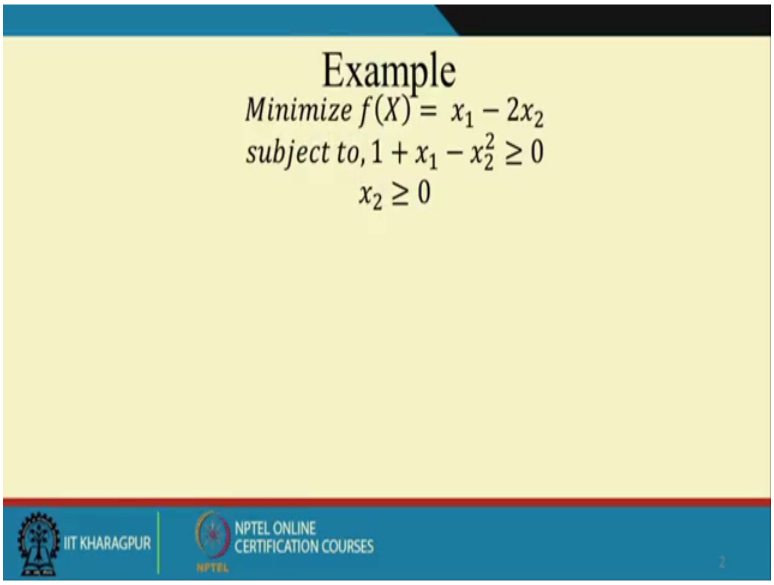


Now, this example I was just mentioning in the previous class and hope you have done the solution of it. Let me just to do the solution for this and you just check it whether you have done the correct one or not.

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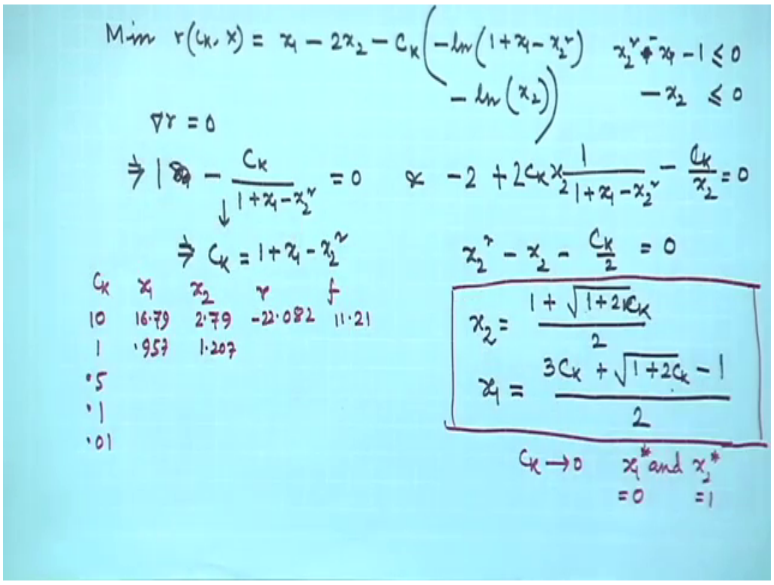


Now, we have to minimize R C k x, we are solving through the interior penalty function method. This is equal to x 1 minus 2 x 2. Now, you see the constraint has been given in the form of greater than equal to that is why I will make it as less than is equal to x 2 square plus x 1 a minus x 1 minus 1 less than is equal to 0 and minus x 2 less than is equal to 0 all right.

Now, we are having the barrier constant that is say C k and the summation will be there that is why I will consider ln minus this function that is why the function will be the original function minus, ln x 2 all right, now this is equal to 0 the first order necessary condition gives you the condition that x 1 minus C k divided by for this case plus x 1 minus x 2 square equal to 0. And we are getting another condition by partially differentiating with respect to x 0 that is minus 2 minus C k divided by 1 plus x 1 minus x 2 square and for this 1 minus 2 x 2 that is why this is plus 2 C k minus minus C k divided by x 2 again equal to 0, we are having two conditions together all right. This is is equal to minus 2 plus 2 C k x 2 x 2, 1 divided by this 1 is equal to 0. This is not x 1, this is 1 minus certainly, this is 1 minus, this 1 all right.

Now, from this from these condition from here we are getting that C k is equal to 1 plus x 2 minus x 2 square all right. If we substitute this value here we are getting 1 equation that is x 2 square minus x 2 minus C k by 2 is equal to 0 all right and from here we are getting 2 values for x 2. Actually this is 1 is x 2, we are getting 1 value 2 values of x 2 1 will be within the feasible region another one will not be feasible region. We are having this one within the feasible region you just check for the minus it is not within the feasible region all right. If this is my x 2, then we can substitute is value and we will get x 1 is equal to 3 C k plus 1 plus 2 C k minus 1 divided by 2 all right.

If we get this two values for x 1 and x 2, then what we will do, we will formulate the table for different values of C k ck x 1 x 2, the value of r the value of f all right. One thing we can do, we can make C k tending to 0. We will get the values for optimal value for x 1 and x 2. This is one way we can get it otherwise we can if we just substitute C k tending to 0. This term will be 0, this will be 0 and we are getting one minus 1 divided by 2, then we will get this value equal to 0 and for x 2 star we will get C k tending to 0 that is why 1 plus 1 divided by 2 x 2 will be is equal to 1.

Instead of that if you consider different values of x 1 x 2 and C starting from 10, if we just get the value we will see the value for x 1 would be 16.79. If we substitute the value for C k here then we will get the value for x 1 this 1 and for x 2, you will get 2.79 all right. Need to ask what then for the r the value will be minus 22.082 and the original value of f would be 11.21 is that clear. How to get this value, this value we will get C k, if we substitute the value for C k here x 1 will come this x 2 will come from here and the r value will come. If I just substitute the value for x 1 and x 2 and C k here and f value theoriginal f will be that is x 1 minus 2 x 2 will be this 1. If we just a do for other values 10 see 0.5 1.01 any problem.

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If C k tending to 0, this is 1 plus 1 divided by 2, that is 1, C k tending to 0 this is 0 and we will get a sequence of this, that would be 957 1.207 like that. If I just summaries everything, here then we will get the value as this one.

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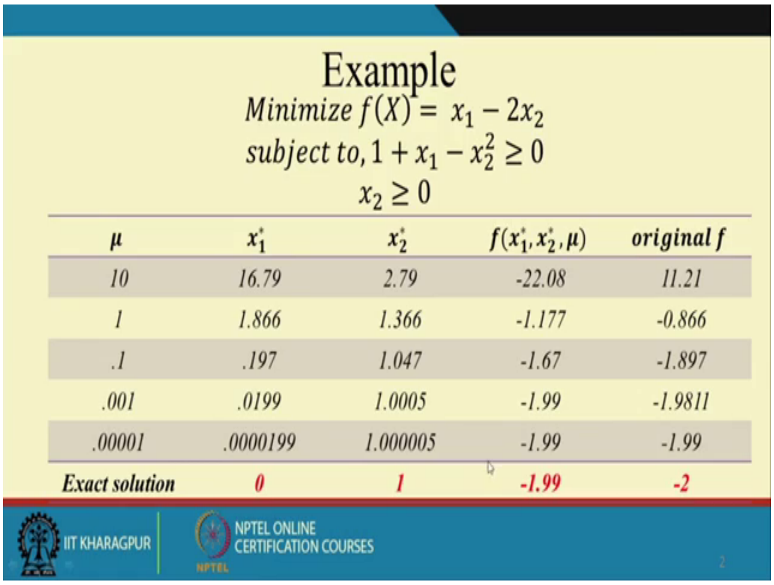


For different value of mu just to see this is converging, X 1 is converging here 0 x 2 is converging to 1 and r value is converging to minus 2 and a value is converging to minus 2. Now, this is the solution of the problem one thing you must have been must have observe that, this is one way the theorem is there for the interior penalty function method. If the function is considered the unconstrained problem is being considered in this way, then if C k tending to 0 x a x k rather the optimal solution will converged to the optimal solution of the original problem that, but it is again I need to mention that I need to mention that for. This in general if the if the consternate constraint is being satisfied critically then we would not get the optimal solution the problem will be unbounded.

Now, let us continue with the general model, general model is that whatever I said that has been summarized here.

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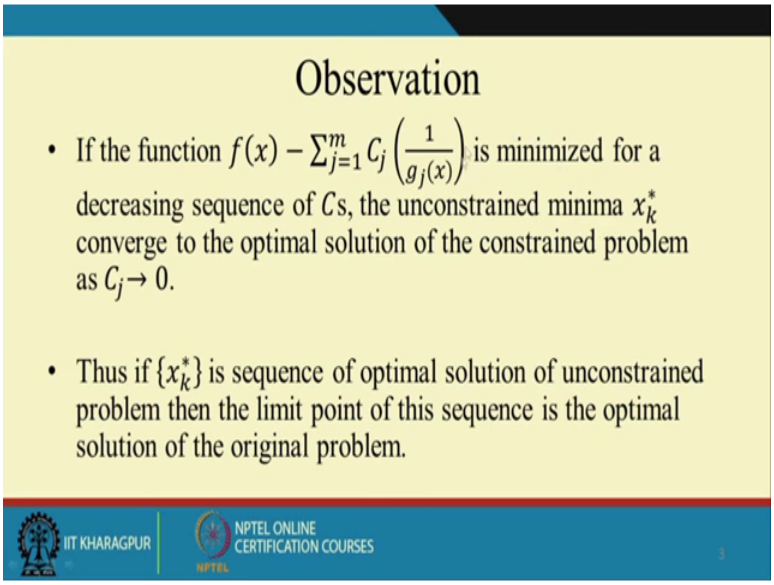


Now, for the general model where we are considering both the equality constraint as well as the inequality constrained. We are having a set of p equality constraints and we are having m number of inequality constraints.

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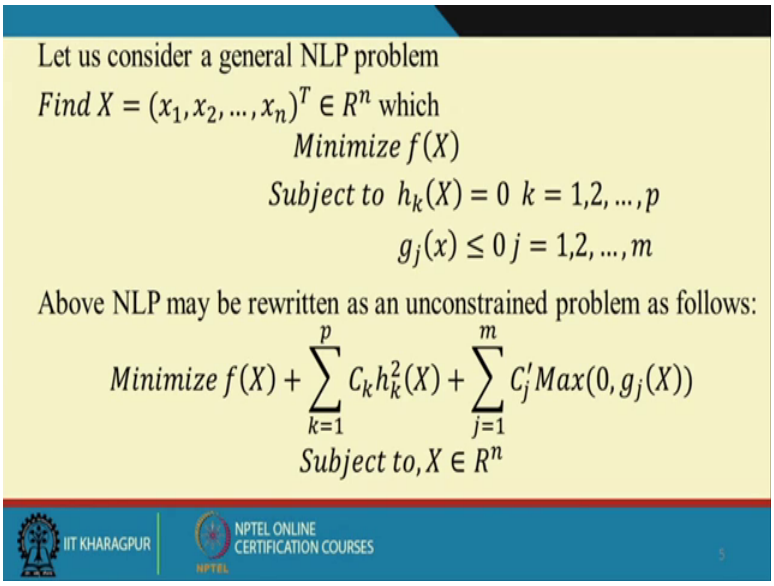


Minimization of f X subject to p plus m number of constraints; how to handle it. Now we know for the inequality constraint we are having the interior penalty function method and the exterior penalty function method now for the constraints of that of the type equality type.

Can you guess what could be the penalty how the penalty will be calculated we expect that for the optimal x the equality constraint must be satisfied equally; that means, for the corresponding x h k x must be equal to 0, but if the solution is not optimal then we will see that if the solution is not optimal then h k x will not be equal to 0. It will be some other value that value will be the penalty value that is why this value can be negative., this value can be positive to make it just like our exterior penalty function method we can take one penalty function as maximum of 0 and h k square.

If this is so then we can have the unconstraint problem like this minimization of f X plus summation C k h k h k square plus C j prime. This we are considering less than is equal to we are considering the exterior penalty function method. We can considered the interior of function as well here now this is the problem. Now you see there are two functions what is the property of this two function this function will be zero if it is in the on the constraint and if it is non zero if it is not on the constraint because this is equality type of constraint and for inequality type of constraint this will be zero if x is not within the feasible region; that means, x is not we cannot, x is not satisfied with all the inequality constraints that is why if we just summarize then we can say that here the terms C k h k square can determine the penalty incurred by the equality constrain.

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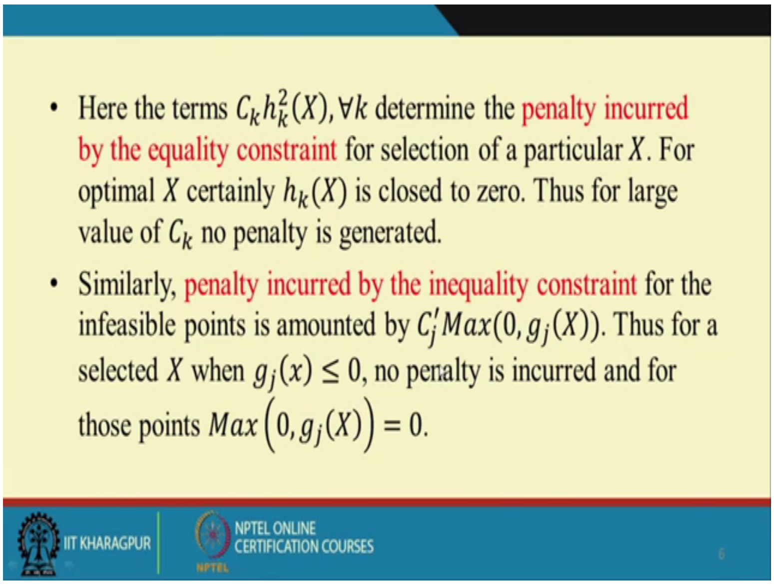


For selection of a particular x and for optimal x certainly h k x is closed to 0 that is what large value of C k no penalty is generated. Now, for the we are making C k tending to infinity that is why we have considered the exterior penalty function method where we are making C j prime tending to infinity as well that is why here also that this is the penalty term incurred and if a x is the feasible within the feasible space then g k g j x is the corresponding penalty otherwise no penalties incurred.

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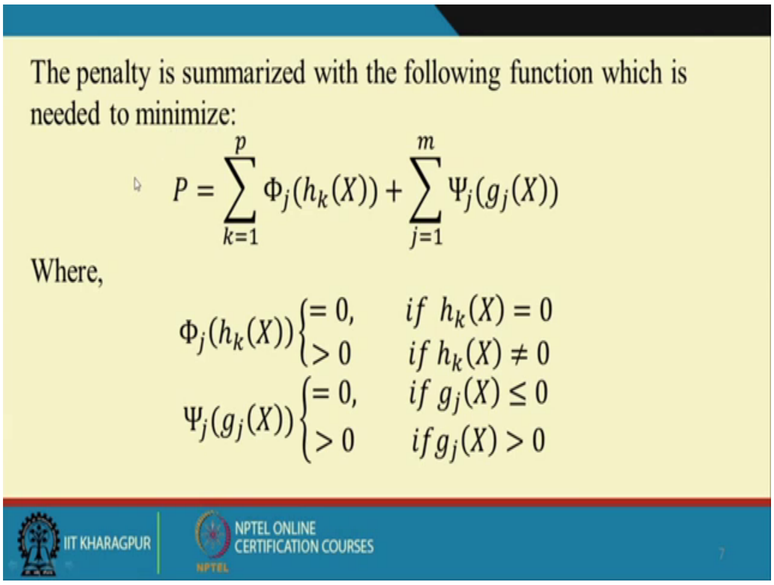


Now, that is why we can say that one function is a function of h k and another one is a function of g j 1 is if it say 1 as psi j and phi j j and another one as a psi j then we can say that the phi j is equal to 0 for h k see equality type of constraints h k x is equal to 0 and this is equal to 0. And if h k x not equal to 0 then phi of j will be greater than 0, why it is greater than 0 because we have considered phi as square of h j h k that is what there is a violation of constraint is there in the positive side or negative side we are considering the square of it always it will be positive. And the psi j psi j again will be positive because g j x we are considering in general case less than is equal to 0 type that is why always psi will be positive when x is not within the feasible space to make the same type phi and psi we considered h k as h k square.

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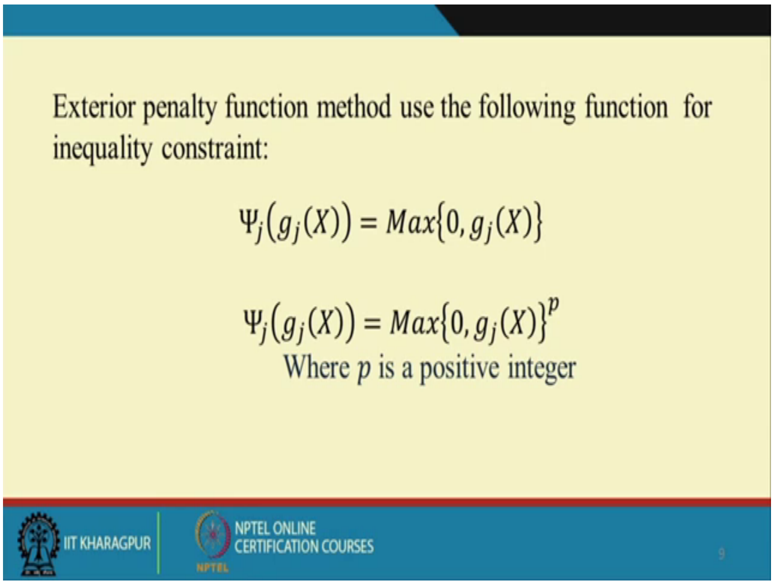


If this is so then we can also consider psi as the inverse barrier function, we can consider the logarithm function just like previous this is a general model and for the exterior penalty function method.

We can have 0 g j x, we can have max of 0 g j x square, we can have max of 0 mod g j x all right that also we can consider or we can consider to the power p where p is a positive integer. That is a general model for us and this is the in general the interior penalty function method the algorithm have already I have discussed and for the exterior penalty function method algorithm already I have discussed the only difference is that here the C values are increasing where as for the interior penalty function method C value is decreasing, then only we can approach to the feasible region that is the difference.

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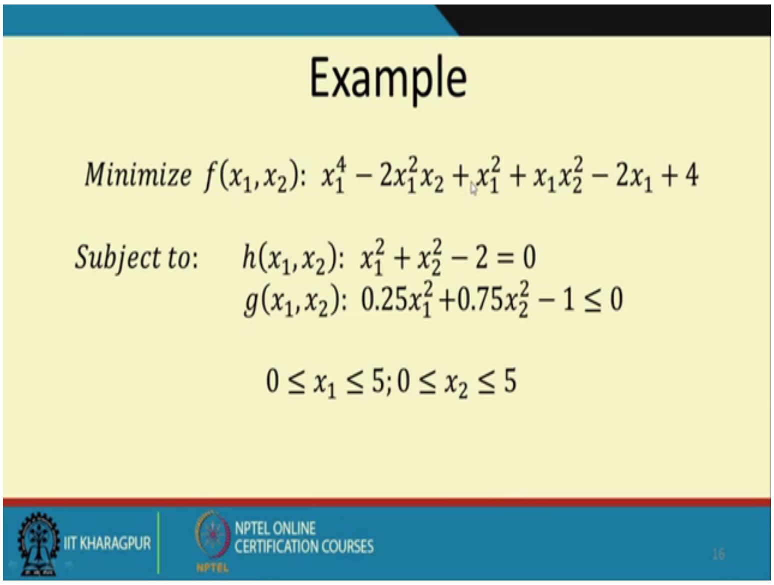


Now, if we have this type of example with us how to solve it that x 1 to the power 4 minus 2 x 1 square you see 1 constraint is of equality type and another constraint is of inequality type. Let me consider now, how to get the unconstrained problem whatever problem we dell till now, that we were not having any equality type of constraints, but here we are having equality type of constraint here h is of equality type and that is why once we are considering the unconstraint problem. We will have h k square is the penalty function along with the penalty parameter we will consider and here we will take maximum of 0 1 this all right.

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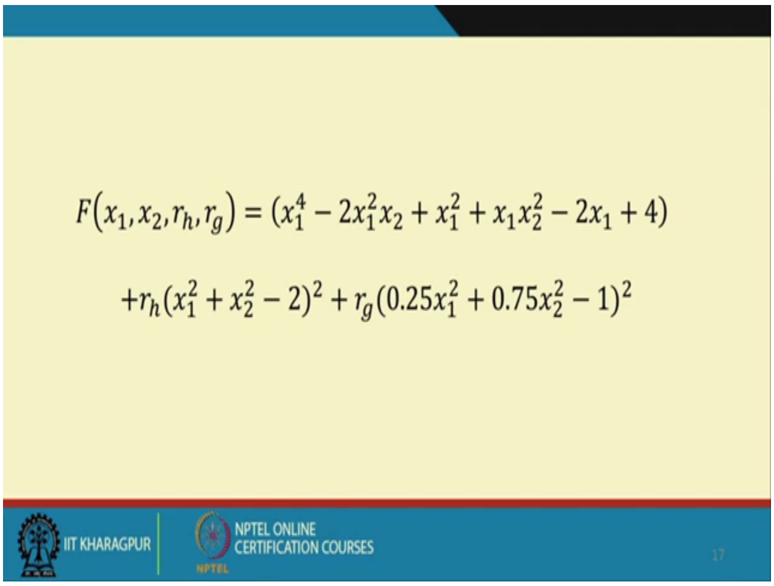

That is why we will have the function this way r h is the parameter. We have considered for the equality type is parameter is the penalty parameter and here r j is for the inequality type. This is the penalty parameter we did not consider max of these, because we have considered those combinations of x 1 x 2 which are outside the feasible space, because this is the extension of the exterior penalty function method that is why the value will not be 0 here the value will be g x square all right.

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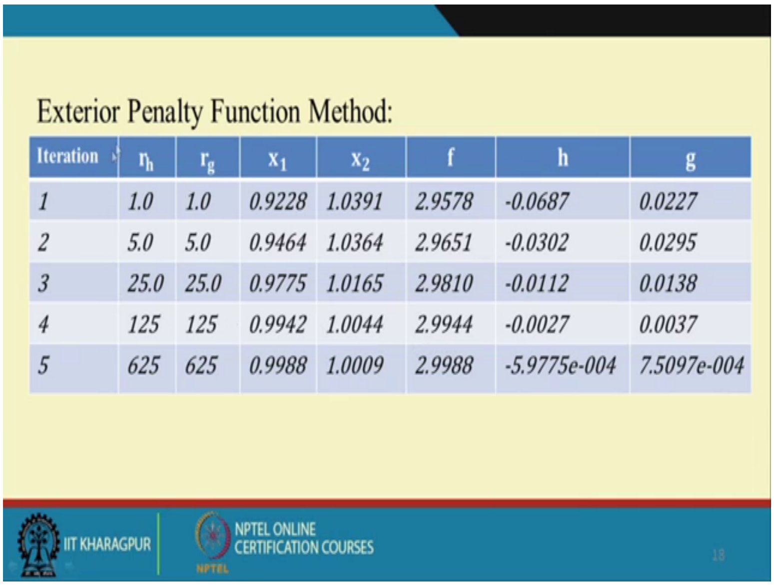

Exterior Penalty Function Method:

| Iteration | $r_h$ | $r_g$ | $x_1$ | $x_2$ | f | h | g |
|---|---|---|---|---|---|---|---|
| *1* | *1.0* | *1.0* | *0.9228* | *1.0391* | *2.9578* | *-0.0687* | *0.0227* |
| *2* | *5.0* | *5.0* | *0.9464* | *1.0364* | *2.9651* | *-0.0302* | *0.0295* |
| *3* | *25.0* | *25.0* | *0.9775* | *1.0165* | *2.9810* | *-0.0112* | *0.0138* |
| *4* | *125* | *125* | *0.9942* | *1.0044* | *2.9944* | *-0.0027* | *0.0037* |
| *5* | *625* | *625* | *0.9988* | *1.0009* | *2.9988* | *-5.9775e-004* | *7.5097e-004* |



Now, if in this case if we consider different value of r h and r g by considering the first order necessary condition we can have a series of values equal to 0 then you see this table is converging to x 1 as 0.99 and x 2 as 1. Whereas the where the objective functional value is converging to 2.99, otherwise from the first order condition if you just consider the r h tending to infinity or r j tending to infinity you will get the same value all right. Let me consider another problem that is of maximization type.

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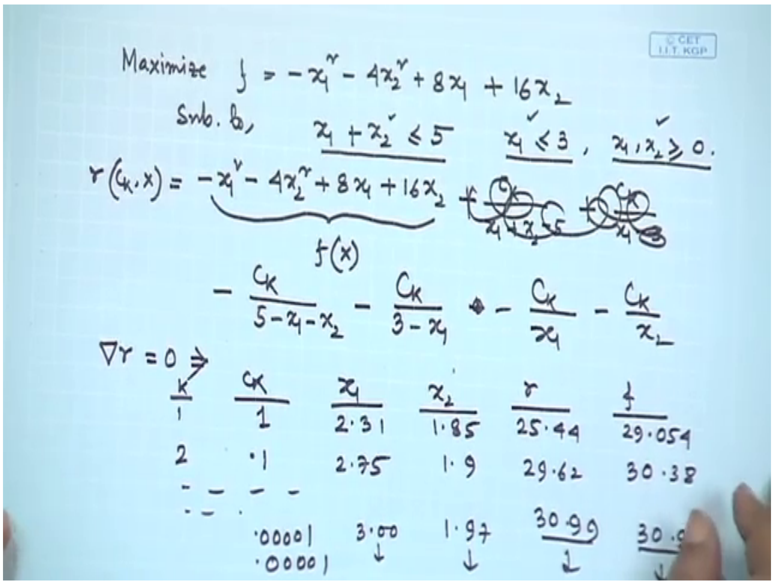


Say the problem is plus 16 x 2 subject to this is are maximization type of problem the constraints are x 1 plus x 2 less than is equal to 5, x 1 less than is equal to 3, x 1 and x 2 greater than equal to 0. Now, we want to apply the interior penalty function method for this and we will consider inverse barrier function for this that is why the r C k x will be is equal to minus x 1 square minus 4 x 2 square plus 8 x 1 plus 16 x 2.

Now, this is f x and the function will be minus say C k divided by x 1 plus x 2 minus 5 minus C k divided by x 1 minus 3 one thing. I must say that you see here the constant is of less than equal to type, we have to make the constraint of type greater than equal to because this is a maximization problem all right that is why the greater than type of equation that would be 5 minus x 1 minus x 2 here it will be 3 minus x 1 that is why this value will come as plus all right 5. No, let me write down with the proper form if I just write down with a of proper way then C k divided by 5 minus x 1 minus x 2 all right.

If you want to make it is less than equal to type if you want to remain in this way then you have to take the barrier function as positive value plus C k divided by that 1 otherwise you have to take this way minus C k divided by 3 minus x 1 plus minus C k divided by x 1 minus C k divided by x 2 from 4 constraints all right. Now, this is one unconstrained optimization problem you find out the value of grad of r is equal to 0. If we equate we will get a relation between x 1 x 2 and C k and for different value of C k if we just substitute we will get the values and we are starting from the 0.11 that is why C k we are considering as 1 as the first iteration k is equal to 1 where considering C k equal to 1 we will get x 1 as 2.31, x 2 as 1.85, r as 25,  44 and the f as 29.054.

Similarly, if we just go to the next iteration r is equal to 2 and by considering C k tending to 0, it will be that is why C k will be this 12.75, 1.9, 29.62 and 30.398. It we just do all the iterations for different values the lower value of this one, we will get that this will be 3.00, 1.97, 30.99 and 30.996. You see both the values are almost same if we just extend it further just do it try this will again come, it will converge here. This way we can solve the exterior penalty, we can solve the problem with the interior penalty function method. Now, what is suggest that you consider another problem to solve that is a maximization of.

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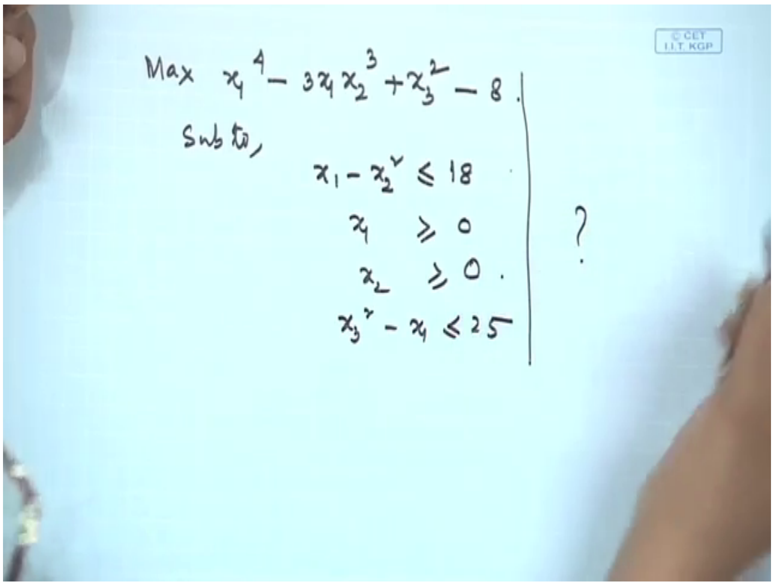


X 2 to the power 4 minus 3 x 1 x 2 cube plus x 3 cube. There are three variables together and the constraints are subject to x 1 minus x 2 square less than is equal to 18 x 1 greater than equal to 0 x 2 greater than equal to 0 and there is another constraint as well that is x 3 square minus x 1 less than is equal to 25 . Solve this problem and get the optimal solution for this.

Thank you for today.