Constrained and Unconstrained Optimization
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Lecture – 56
Penalty and barrier method

Today, I will talk on interior penalty function method or web area method. This is another technique indirect method for solving non-linear programming, constraint programming problem.

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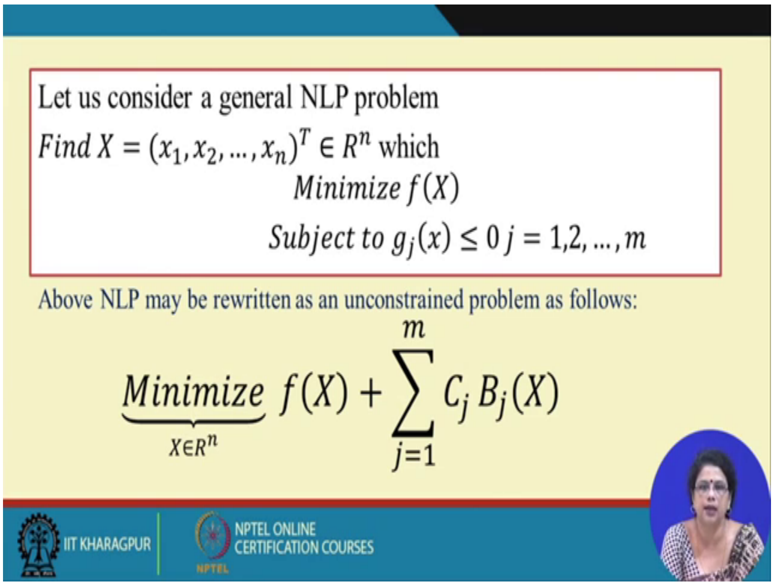

Now let us consider again the non-linear program; that is minimization of f X subject to g I X g j X less than is equal to 0 is a set of non-linear constraints are there. Now this barrier method or the interior penalty function method. Here we are considering again the penalty, penalty of not getting the optimal solution. We have to bear it and we will to try to minimize the penalty function. Now in, that is why this kind of methodology, we are just converting the constrained non-linear programming problem into unconstrained non-linear programming problem.

Now, for minimization type of problem we are considering the penalty function as positive penalty function, because we are trying to minimize the objective function as well as we are trying to minimize the penalty function. Now in the penalty term, there are 2 components; one is the penalty parameter and another one is the penalty function.

Now, we are concerned about the penalty term, in minimization problem, we are minimizing the penalty term and for the maximization problem negative of the penalty function, we have to maximize, because penalty maximization of a objective function does not mean that maximization of the penalty function, as well penalty is something that is not desirable. We are incurring penalty, because the corresponding point rather the corresponding. See corresponding location of mine of my algorithm; that is not optimal, that is either within the feasible region for the interior penalty function, and for exterior penalty function. This point will be outside the feasible region, that is why we can just convert this constraint minimization problem as unconstrained minimization problem, as minimization of a objective function summation C j B j, where B j is the penalty function.

In interior method we call it as a barrier function. Why we are calling it as a barrier function. I will just discuss later on, but this is a penalty function. If the point is not in the optimal or rather the boundary of the feasible region, this barrier functional value will be positive. Now, C j will be the corresponding penalty parameter, as a whole we are having a summation of all penalty terms. Now as many number of constraints we are having that many number of penalty terms, we will have, that is why we will have this is a unconstraint problems, and this can be solved either through the differential calculus approach by considering the first order necessary condition, second order optimality condition, or we can choose any region elimination technique as well, for solving this kind of problem, but using the region elimination technique, you know there are certain restrictions to it, because in the region elimination technique the function must be unimodal, then only, but this in this case.

Once we are having f X as a non-linear function, that non-linear function can have different modes, different peak points, minimum peak or maximum peak. We are trying to find out the global minimum or global maximum for the corresponding problem. Similarly for the function g X, that is also non-linear, when we are adding these 2 terms, we are not sure, we cannot expect even always the function will have unimodal within a range of definition, that is why region elimination technique. If you want to apply, then we have to break the intervals in to different parts, and in different parts we need to check the unimodality of the function, then only we can use the region elimination technique. But in general this kind of non-linear programming problems, the objective function as well as the constraint, these are all continuous in nature; that is why, but it uses the simple technique, we know that is the differential calculus approach that is the classical optimization techniques we can adopt it.

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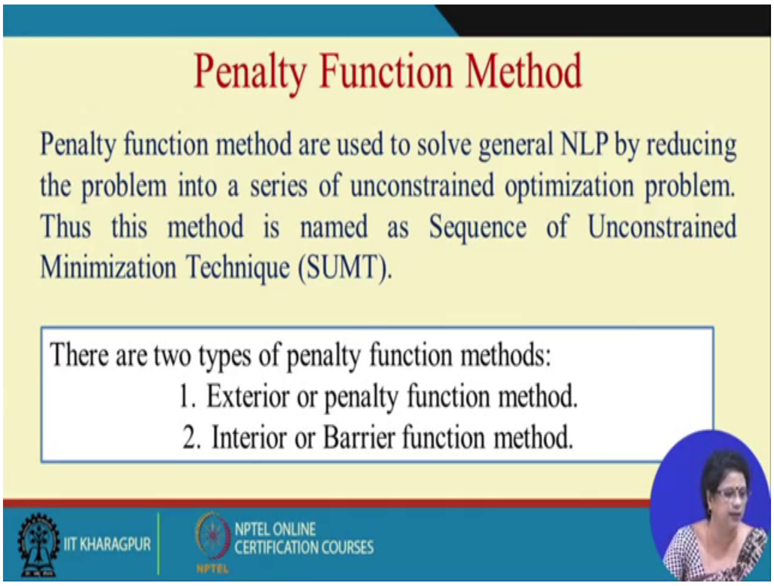

Now, this we know that for the exterior penalty function method, there is a penalty term and we are naming it as a penalty function, but for interior penalty function method, this is the barrier, but we are having a sequence of unconstrained optimization problem, that is why this kind of penalty methodologies is being termed as SUMT, that is a sequence of unconstrained minimization technique and this is for the minimization problem. If we have the maximization problem, then we will have the sequence of unconstrained maximization technique. This methodology is being named as termed as.

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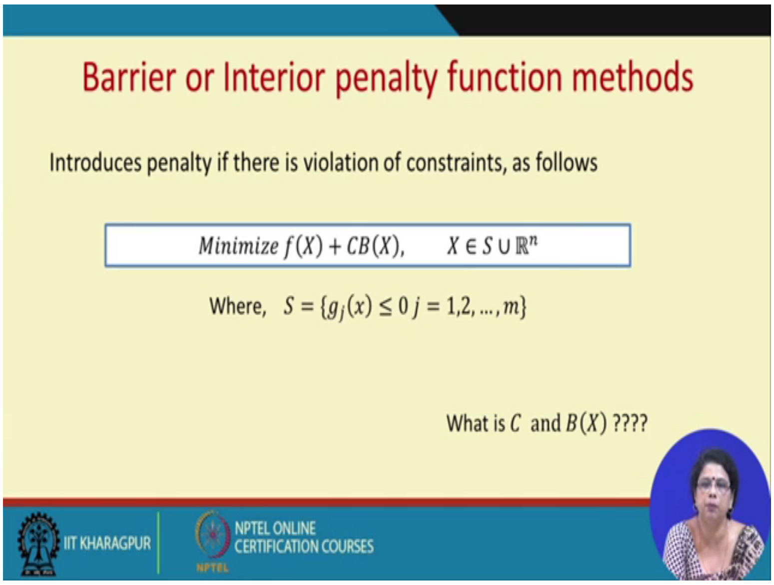

Now, there are different penalty function you must have realized in exterior penalty function method. Similarly here also we are having different kind of barrier function, but the barrier function should have certain property, if you say that we are right, if we just make the unconstrained object objective function as f X plus C B X, then we need to know what is a property of B X, what is the property of C, how C and B X are behaving. 1 thing is clear that it is, the process is moving we are having a sequence of unconstrained optimization problem, but all the problems are being defined within the feasible region, that is why the behavior of C and B must be examined. For this case, because the behavior will not be same as exterior method.

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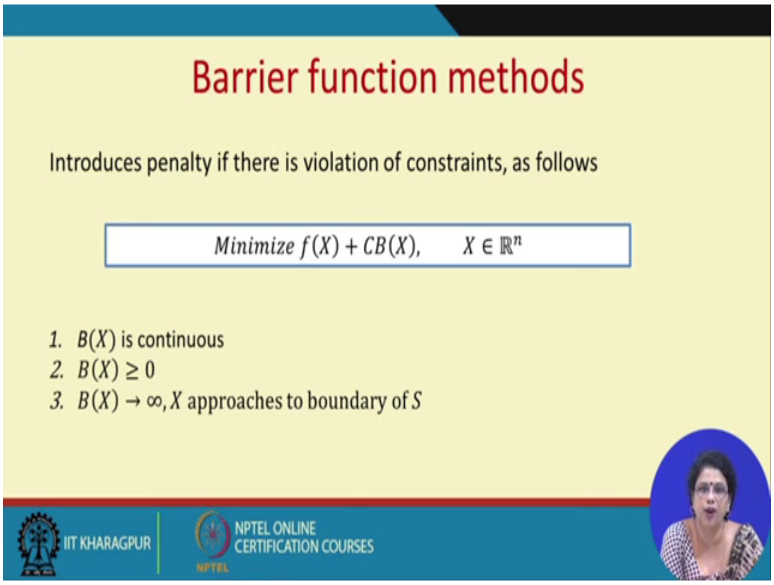

Now, if I want to get the property, if you ask me what is a property of B X, then again I should mention the same just like your n exterior penalty function method that you are barrier function must be positive throughout and barrier function is continuous. Since g X is continuous bear B X is the function of g X that is why B X is continuous B X is positive and B X is tending to infinity as X approaches to the boundary of s s is the feasible region of the. Now, you see whenever we are writing minimization of f X plus C B X. Here we are not considering X within the feasible space, you must have been, must have seen that we are considering X is in R n, R n, we are considering, because for the ponds where the ponds are outside the feasible space in that case, B X will be 0, but once it is within the feasible space then B X is positive all right, but there is a trained of B X as well just like your P X as you have seen.

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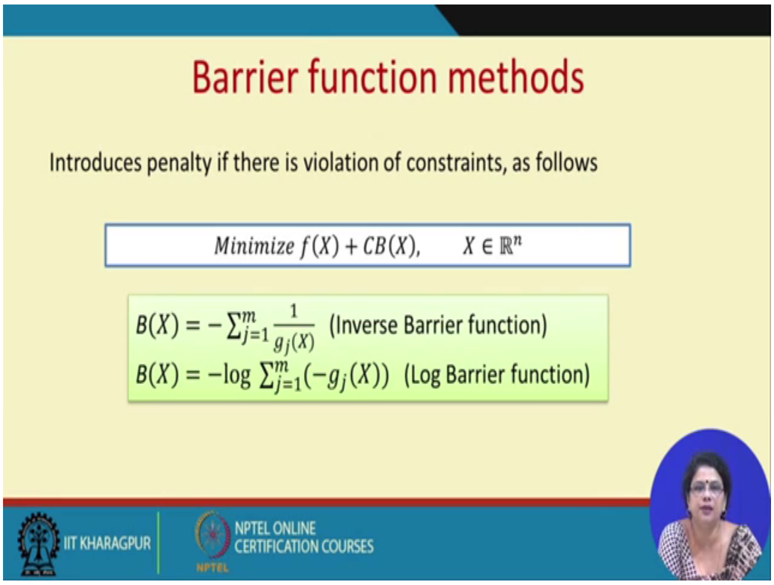

Now, B X is been something that is being there measured as a violation penalty, because of violation of constraint. Now, one barrier function can be considered as 1 by constraint function, if we are having 1, can 1 constraint functions g X, then barrier function can be considered as minus 1 by g X, You see g X is always less than is equal to 0 in general model, we are considering our model is minimization of f X subject to g X less than is equal to 0; that is why within feasible space C X is always less than is equal to 0. But outside the feasible space g X is greater than equal to 0, but if we consider within the, we are within the feasible space, we are starting our sequence from the feasible region only, that is why 1 by g X will be always negative, and if we consider minus 1 by g X, then it will be positive all right. This is one of the barrier function, this barrier function is being named as the termed as the inverse barrier functions.

There is another kind of barrier function that is the logarithm barrier function, there we are considering the barrier function as minus log minus g X; that is also gives you B X as positive, because g X is negative that is why. Now this summation is being considered, because this summation gives you the value for all constraints together, we are considering all 1 by g X for all g j X that is why this way we will considered the barrier function.

Now, if you remember I was talking, I was just explaining one problem that minimization of f X X greater than equal to 5. Now once were considering f X equal to X as objective function X greater than equal to 5 as a constraint, then if I just make it in the form of, in the form of g X less than is equal to 0, then I will write down it as 5 minus X less than equal to 0. Now you are object, your feasible region is X greater than equal to five. Now for X equal to 6, now value will be negative ultimate 1 by g 1 by g X; that is why 5 minus 6 that would be negative that will be minus 1 and another minus is there with B X; that is why it will be the penalty, you need to incur, that is 1 for X equal to 7 penalty, you need to incur as 1 by 2 in this way.

But you see if we considered, if we are just reaching to the boundary of the feasibility region by considering X equal to 5 though X equal to 5 as the optimal solution of the problem; that is a, advantage of this methodology is that we will get the unbounded solution, because 1 by X minus 5 and X equal to 5 it will give you infinity. That is why this barrier function has the disadvantage that always it makes a barrier within the feasible space by constructing the unconstraint function that is why this method is being named as the barrier function method that is the barrier we are considering within the space.

Now, let us consider this 2 barrier functions in a interior penalty function method.

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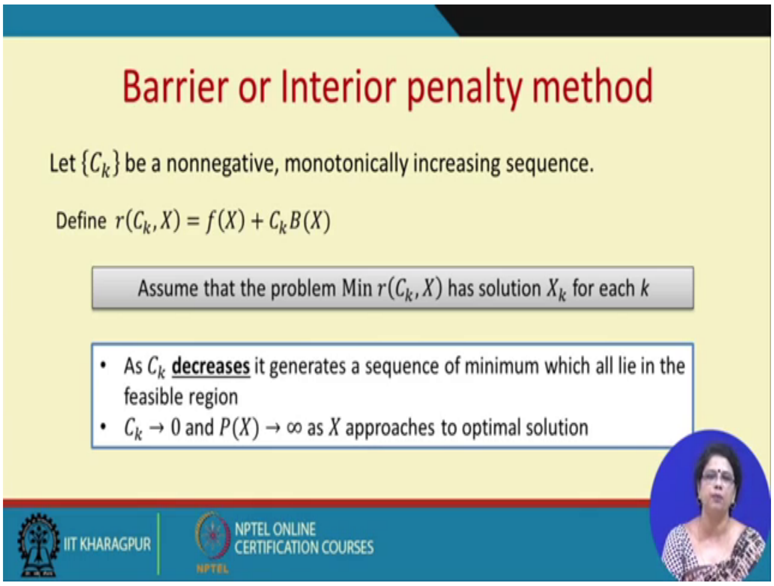

Now, if we again consider the, there is a sequence of C. Now we got the impression what should be the, what should be the case, what should be the property of P X, that is the barrier function, but we do not know what is the property of C. How C behaves within the feasible space, if I want to move towards the feasible space, not exactly the boundary of the feasible space, at least towards the feasible space. Then whether C value will increase or decrease, as we have seen for the interior penalty function method, the value of C is always, if we just increase it, then we will reach to the boundary of the feasible region, but in this case, they will see if we just reach, if we just decrease the value of C we will reach to the boundary of the feasible region. We will have a field with certain examples, but for different C we will get a sequence of unconstrained optimization problems, that is why if we consider k as the index of iteration starting from k equal to 1, then C 1 will be considered such a way that we will be within the feasible space ok.

Now, C 2 will be considered C 2 must be lesser than C 1, then only we can move towards the feasible boundary of the feasible region, that is why in this method instead of increasing C, we have to decrease C, but there is another convention, you will see in the literature that in the barrier function method, people are not considering the constant term, that is the parameter barrier parameters C k as C k. They consider 1 by ck that convention is there, why it is. So, because to make ck as similar as the exterior penalty function method as we are saying, that we are trying to decrease C k, if I make ck as 1 by ck, then we can say that we are increasing 1 by ck that is why that convention is there.

Otherwise if C k tending to 0, we will see the barrier function will tend to infinity, because if g X is equal to 5 minus X less than is equal to 0, then 1 by g X is equal 1 by g X at X equal to 5 is infinity; that is why this barrier function that is a P X its not P X, that is B X, that B X will go to infinity if X approaches to the optimal solution, that is the understanding; that is why for this case if I want to solve the non-linear optimization problem, then we have to select k C in such a way that we will remain in the feasible space all right. And this selection is not really so easy why, because this selection is the C value is very high, then in that case we will be far from the feasible space, what should be the optimal value for C, what is the suitable value for C, that is also a question, because we are assuming certain C value, these are all the disadvantage. There are different methodologies available for non-linear constraint programming problem, that because of this drawbacks all right.

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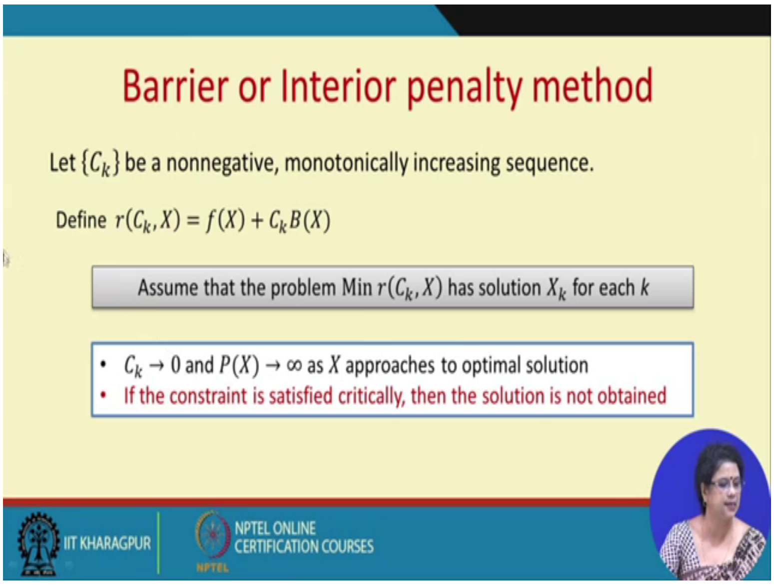

Now, let us just try to solve the problem, and it has been written if the constraint is satisfied critically, then the solution is not obtained, the solutes the problem will be unbounded, objective function will be unbounded; that is the.

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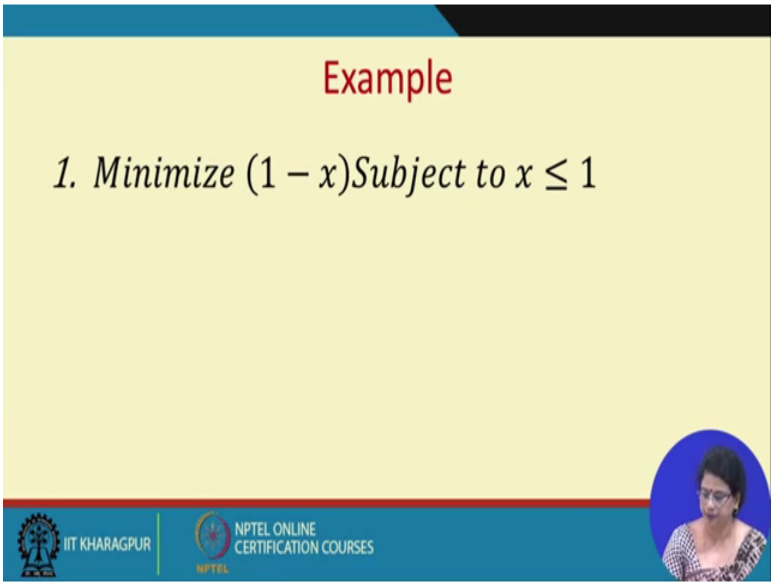

Now, let us consider one problem minimization of 1 by 1 minus X is subject to X less than is equal to 1. Let us consider it if this, then let me make the unconstraint, problem as C k X is equal to 1 minus X minus C k, and let me X use the logarithm barrier function, then ln X minus will l minus X minus 1; that means, it is 1 minus X all right, by using the logarithm barrier function.

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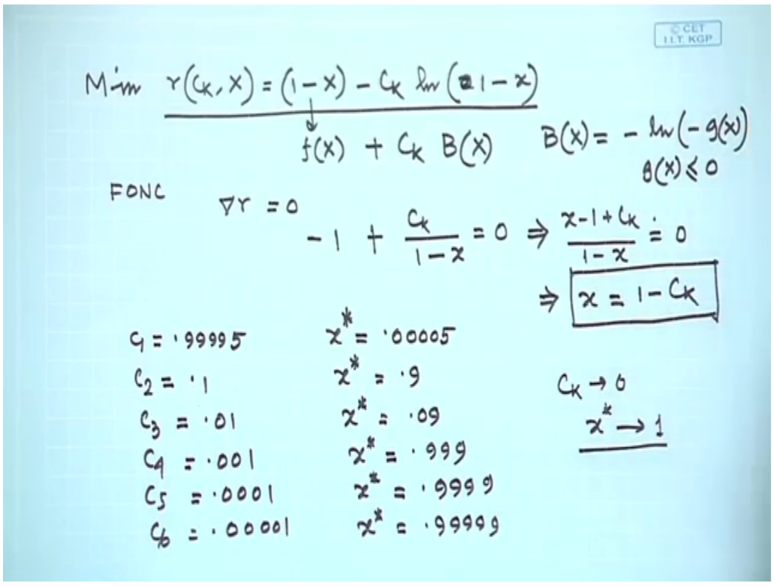

Now, we have to minimize this objective function where this is the function f X plus C k barrier function, where barrier function is equal to minus 1n minus g X, where g X is of the form g X less than is equal to 0 clear.

Now if I want to minimize let us consider the first order necessary condition, then grad of r must be is equal to 0. Now there is only 1 variable here, that is why we can see that this is equal to minus 1 minus C k divided by 1 minus X, and because of 1 minus X this is plus this is equal to 0 all right, this is the first order condition if I just take X then 1 minus X x minus 1 plus ck is equal to 0. From here we are getting X is equal to 1 minus C k clear, that is why let me consider first k is equal to 1, let consider C 1 is equal to sum value point very high value you will start, 0.995.

Then what will be corresponding optimal solution for this X term must be 1 minus C k, that is why the X star value will be 0.0005. Now I have to increase C 1, I have to decrease C 1, that is why let me consider C 2 as 0.1, then corresponding X star would be is equal to 0.9 all right, if I consider C 3 is equal to 0.01, then X star would be is equal to 0.09 C 4 is equal to 0.001 X star would is equal to 0.999. Similarly C 5, what you see the pattern it is approaching to 1 C 6.

We can go further what we see that if C k is tending to 0, X star is approaching to 1, that is why we can say X is equal to 1 could be the optimal solution for this problem, this way we can solve this problem.

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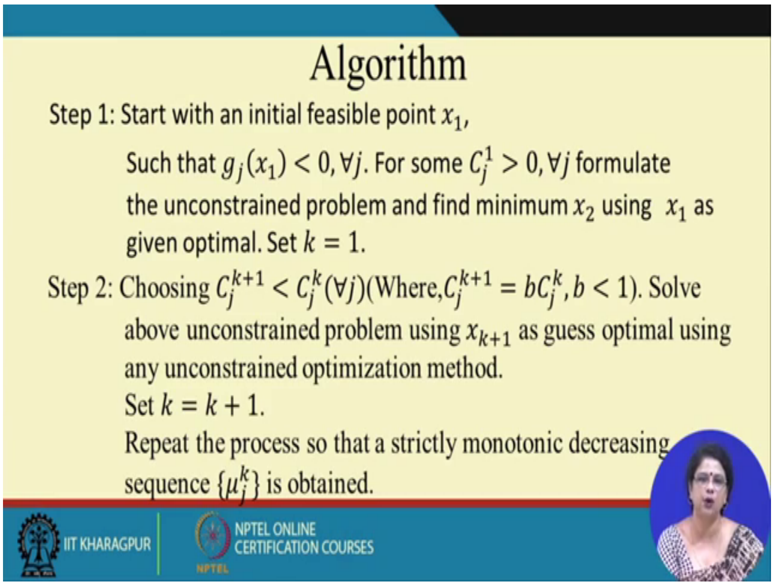

Now, if I say what is the algorithm for solving this exterior penalty function method, sorry interior penalty function method. First of all we have to select a value for C 1 for k is equal to 1 the consideration of C 1 depends on the feasible point that is why we will consider a feasible point within the space. Now let us start the initial feasible point at X 1 such that g j X 1 less than 0.

Now, in this case as it was for exterior penalty function method that we have to consider 1 point which is outside the feasible space, how we checked it. We checked it, we were considering point which was, if there are 3 constraints we were checking at least 1 of the constraint should not be satisfied with that point, but in the interior penalty function method we need to consider that all constraint must be satisfied, then only this is the feasible point, that is why X 1 must be feasible point for all j. Now once we are getting C 1 j formulate the unconstraint problem find the minimum, and check for optimality, check for optimality. The only thing we cannot do, because unless we just process it now, because we are more interested to get the sequence of unconstrained problem.

And we will say that you see we are converging a C k tending to 0, we are converging that sequence the limit point of the sequence is this one. Now that is why we are considering another C; that is lesser than the previous C, then again we are just doing the process, and we are getting the solution this way let me consider another problem.

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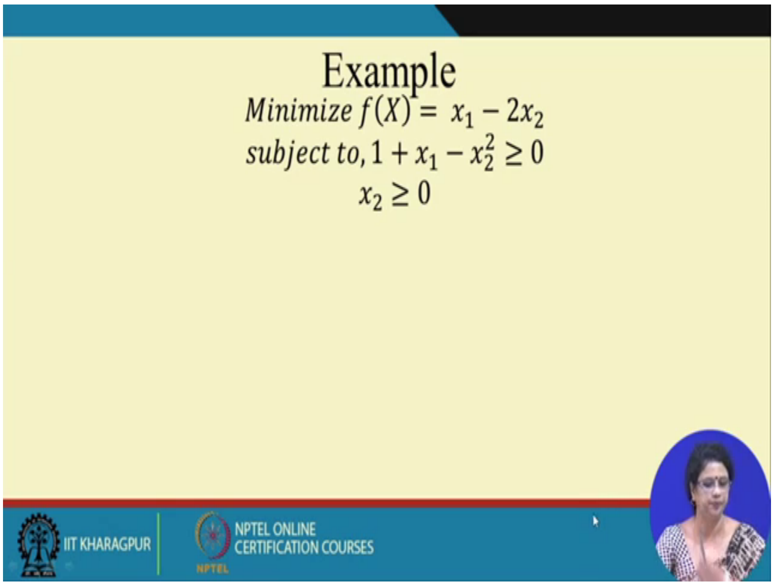

For this X 1 minus 2 X 2 subject to 1 plus X 1 minus X 2 square 0 if this is the problem for us then we can say that, R C k X must be is equal to X 1 minus 2 X 2 minus; if I considered g X g X will be X 2 square minus X 1 minus X 1 minus 1 less than is equal to 0.

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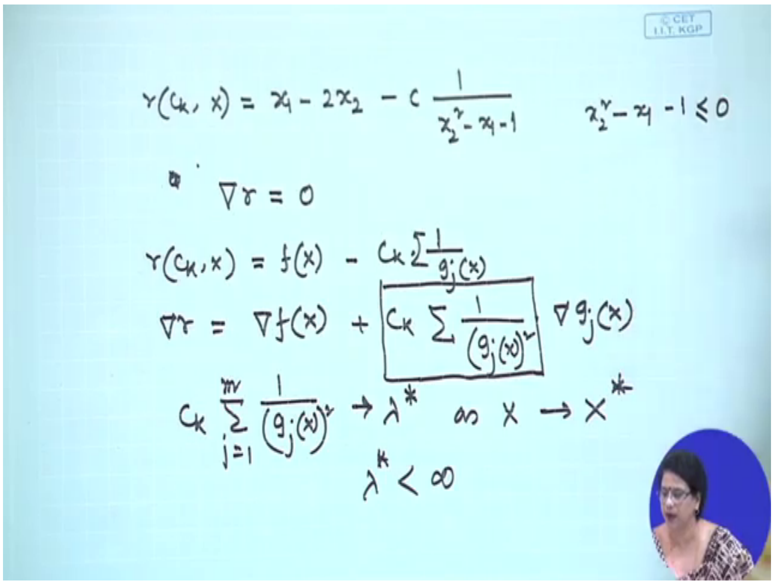

Now, this problem we have already solved, with a interior penalty function method exterior penalty function method, and I am just considering this as C 1 by X 2 square minus X 1 minus 1, and you solve this problem with exterior penalty function method by considering delta r is equal to 0, find out the value for X 1 and X 2 in terms of C then you take C tending to 0, and find out the value of it.

In this process one thing I just I would like to say that you see we are considering r C k X is equal to f X e minus ck divided by summation g j x all right, and what is the grad r, grad r is equal to grad f X plus ck summation 1 by g j X square into grad of g j x. Now you see look at this term you recall the Kuhn Tucker conditions. What was one of the condition in the Kuhn-Tucker condition for inequality constraints that was the grad of f X plus Lagrange multiplier grad of g j x all right. Here also we can say that ck summation j is equal to 1 to m 1 by g j x whole square tending to lambda star; that is the Lagrange that is the KKT multiplier as X tending to optimal point all right. Now what are the other thing is we can say that lambda star is equal to 0 for active constraint, and lambda star not is equal to 0 for inactive constraint here. Also we can say when this lambda star is finite value then we can say that.

We are getting that corresponding j g j corresponding g j would be the this is finite active constraint all right, this is one consideration. Now let me take one another example for solving with the interior penalty function method.

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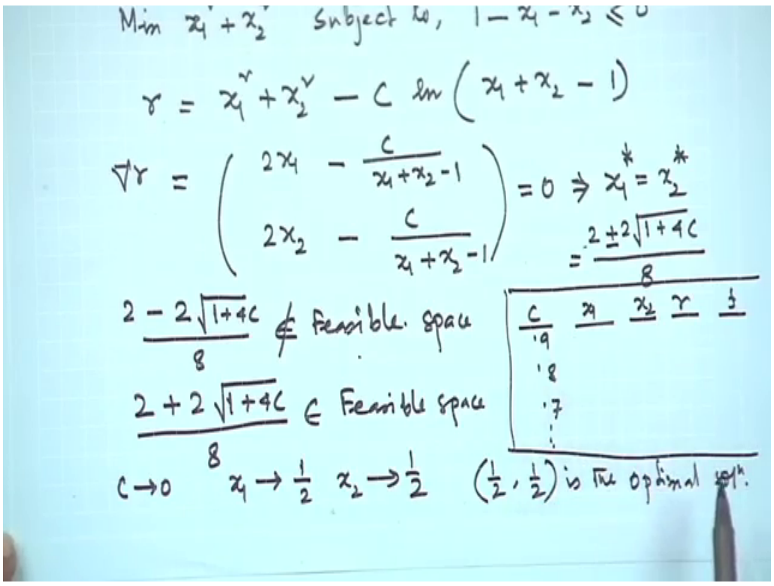

Minimization of X 1 square plus X 2 square subject to 1 minus X 1 minus X 2 less than is equal to 0 consider the barrier function r is equal to X 1 square plus X 2 square minus C ln minus of these, that is why it will be X 1 plus X 2 minus 1 all right.

Consider the grad of r grad of r would be is equal to 2 X 1, if I just differentiate with respect to X 1, this 1, then 2 X 1 minus C divided by X 1 plus X 2 minus 1, and if I just if partially differentiate with respect to X 2, this is equal to minus C divided by X 1 plus X 2 minus 1, this must be is equal to 0. And this gives you the at the optimal stage X 1 star must be is equal to X 2 star, because if we just equate equal to 0, this is less than is equal to 0 this 2 terms from there, we can say that X 1 star is equal to X 2 star not only that, we can say that this is equal to 2 plus 2 root over 1 plus 4 C divided by 8 plus minus, because we will get the condition, like this. If we just substitute X 1 is equal to X 2 here, then we will get this equation with of degree 2 from there, we will get 2 roots of the equation.

Now, one point that is if we just substitute the values here, we will see that 2 minus 2 root over 1 plus 4 C divided by 8 is not within the feasible space, that is why only the point we have, that is in the feasible space 2 root over 1 plus 4 C divided by eight, this is in the feasible space. Now, one thing you can do by changing the value or see you will get sequence of sequence of optimal solutions, because we know X 1 star is equal to X 2 star we can have for different values of C starting from say 0.9 0.8 0.7 X 1 we can have X 2 we can have functional value the r we can have, we can have the functional value as well we can have a table like this, we will get. By substituting the value of C 0.9 we will get 1 plus 4 into 0.9 root of that 2 into this etcetera, then that would be the value for X 1 same is the value for X 2, we can complete this table, this is one way of getting, and we will see that where X 1 X 2 values are converging in the corresponding r value will converge or f will converge that would be the solution other way, what we can do, we can take C tending to 0.

If C tending to 0 what we see that X 1 value will tend to 2 plus 2 divided by 8; that means, half X 2 value is tending to half, that is why we can declare half is the optimal solution same thing, you will get through this table as well what I suggest you just check it, and for the previous problem also just do the same exercise for that problem also.

Thank you for today.