Constrained and Unconstrained Optimization

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Lecture - 55
Penalty method

Today, let us concentrate on the method; that is the exterior penalty method, and we are naming it as well the penalty function method. Now again we are, let me just summarize what I have said in the last class.

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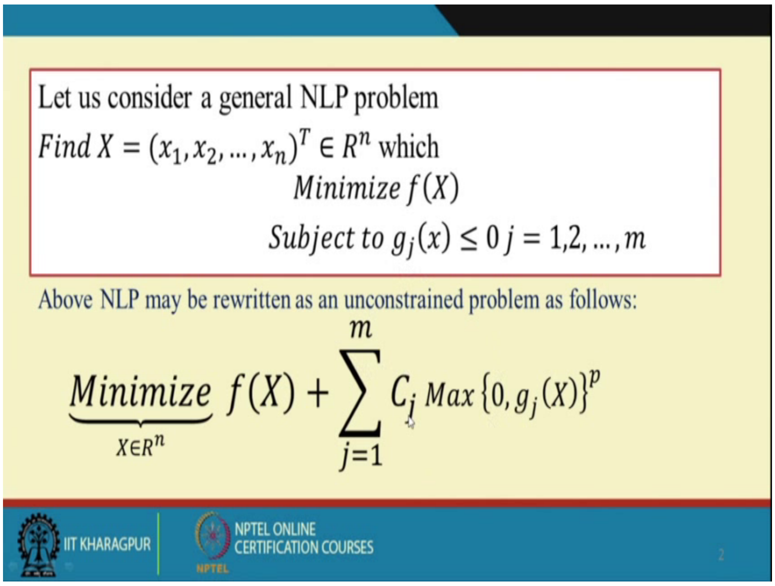

And I will just do further on the same line. Let me consider one general non-linear programming problem, minimization of f X subject to j X less than is equal to a 0. Here we are considering a sequence of unconstraint optimization problems by changing the value C j.

Now, if we have more number of constraints, just like Lagrange method, we will construct the unconstraint problem. This way minimization of f X summation of C j max of 0 g j X to the power p. Now P is positive integer value here. Now C j values these C j has set in property, you have seen for exterior penalty function method; that is C j is approaching to infinity, if we are approaching to the boundary of the feasible region. Whereas, the penalty function; that is P X is approaching to 0 if we are moving towards the boundary of the feasible space.

Here the term, the penalty terms C j into C j into P j has the effect on the penalty value. Now the C j value is always positive here.

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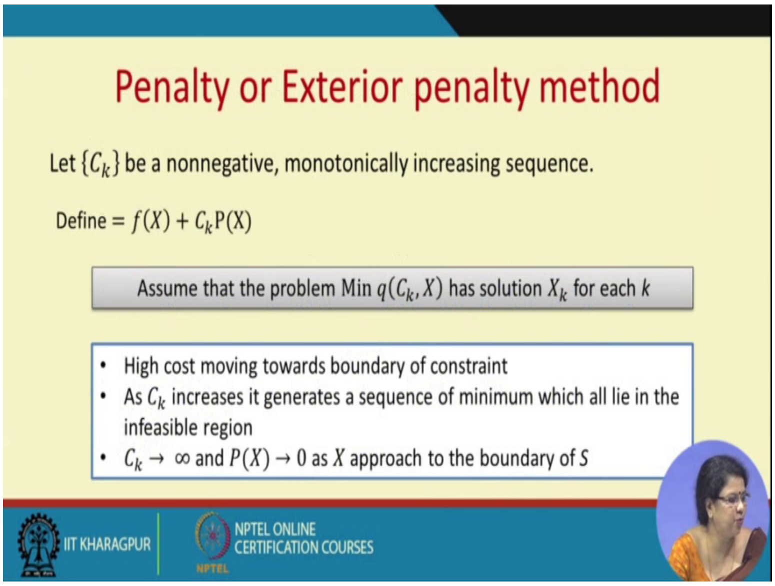

Let us see what does it mean, how the unconstraint objective function is behaving in this situation, and what is the property of the P X. Is it the penalty value is decreasing or increasing, if we move from outside, the feasible space to towards the boundary of the feasible space. Now we will see the properties of penalty function, objective function as well as the objective function of the unconstraint problem; that is why we are naming this one as q C k X.

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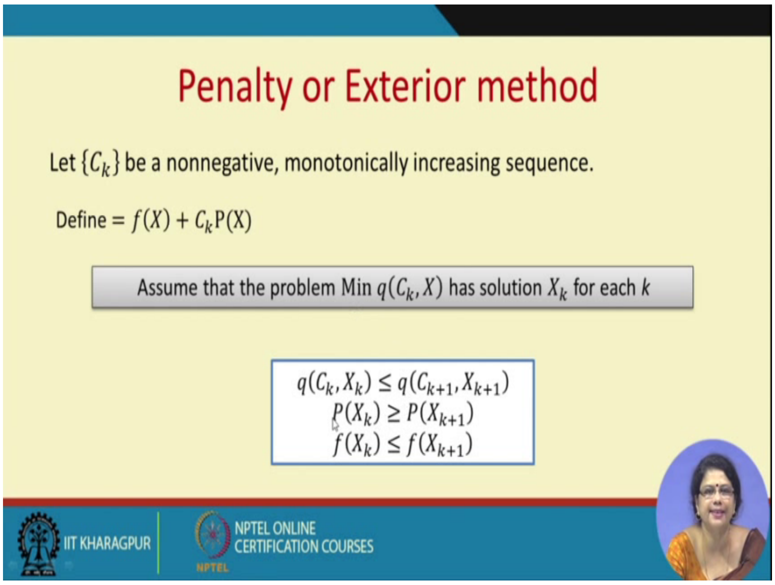

And we will just prove that q value is increasing through the iterations, that is why you see we need to prove that q C k X k is less than is equal to q C k plus 1 X k plus 1.

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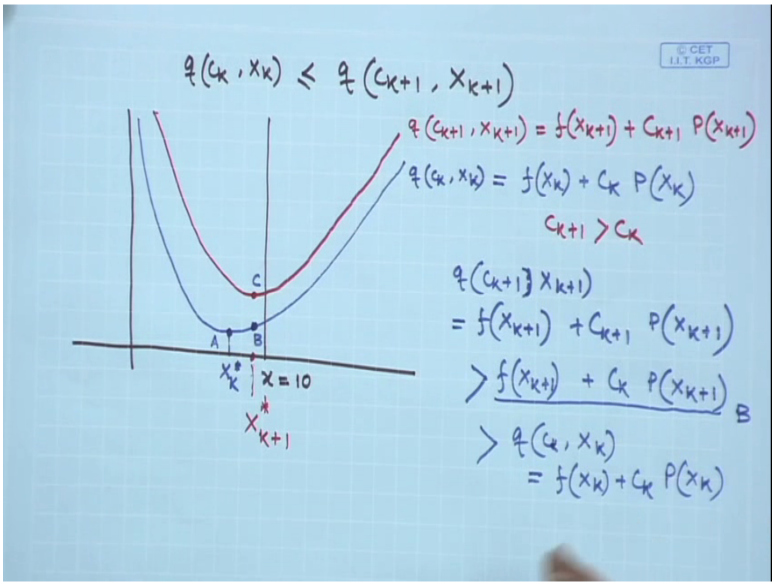

Let me consider the graphical picture of it. Now if you remember we have considered the previous problem X is equal to 10 alright, and for C k say this is the graph of q C k X k here, this is q C k X k, q C k X k, means this is equal to f X k plus C k P, P at point X k alright. This is the function say we are getting the optimal solution here, because we are searching for the minimum of this unconstraint optimization problem; that is why this value would be is equal to X k star.

We are having another, if we just do tech C k plus 1; that is more than C k. If we consider this case, then the next function could be this one alright, where this function would be q C k plus 1 X k plus 1, because k is changing to k plus 1 this is equal to f X k plus 1 plus C k plus 1 P X k plus 1, say this is the optimal solution for this, that is why this value is coming as X k plus 1 star. And as you have failed that the exterior penalty function method, this is moving this way. This optimal we are getting a sequence of optimal solutions, and that sequence the limit point of the sequence is the optimal solution of the original problem, that is the method for this case.

That is why if this is C k plus 1 this way. Now we need to prove this one. Let me start from the calculation, let me start from the value of q C k plus 1, I need that, this is better than this q C k plus 1 X k plus 1 is equal to. Just you see the calculations f X k plus 1 C k plus 1 P X k plus 1 alright. Now C k plus 1 is higher value than C k, because the methodology demands that; that is why we can say this is greater than f X k plus 1 plus C k P X k plus 1 ok.

Look at this value in the graph, where is the value of this one, we are getting the value, at this one is the value, if I consider this as A, this as B, this as C, thenB corresponds this value. Just consider instead of star, this is X k plus 1 clear. Am I clear? Now what you see, this B value the functional value is higher than the A value, is it not? Because this is the curve and this curve is giving the optimal solution at A; that is why any point other than A, would be the higher value than A, that is why B is higher than A, that is why we can say this is greater than q C k X k that is equal to f X k plus C C k P P X k, then it is proved.

The first condition is proved, that always the q value will be the higher value, if we just move through the iterations, but that does not mean that we are getting the reverse solution for the minimization problem, the higher value will guide us to get the higher optimal solution; that is very much clear from the graph, whatever optimal solution we are getting in the previous situation the next, in the next situation the optimal solution will be higher than the previous situation, as the value of q is increasing, it is happening this way. Now let me just prove the next property that P X k greater than equal to P X k plus 1, and the other case as well. Let me draw the graph; otherwise I can put the graph this way. Hopefully it is clear to you, and let me do the calculation with this alright. Is that clear?

Now, here I can say, if I just consider one point at the top here as d, then what is the value at A, value at A would be f.

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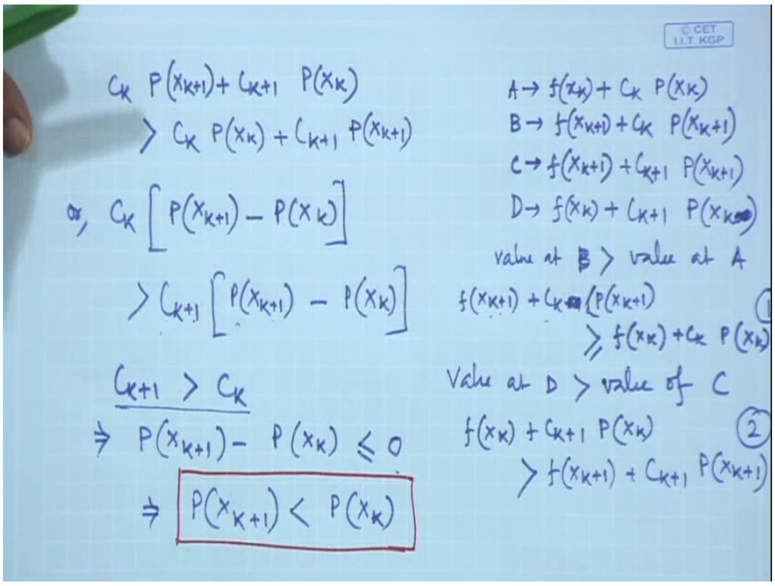

X k plus C k P X k value at b would be f X k plus 1 plus C k P X k plus 1, because B is in the same curve. Similarly the value at C would be f X k plus 1, that is the optimal solution in the next iteration, that is why this is we are getting for C k plus 1, that is why P X k plus 1. And what about D. D is on the curve of q C k plus 1 X k plus 1, that is why at d at point X k, that is why I can say that f X k plus C k plus 1 P X k plus 1 alright. Now one thing is that value at C is higher than the value at A, there is no doubt about it.

Student: d will be P X k.

See this is P X k at point X k alright. Now you see value at C is higher than the value at A; that is why we can say that f X k plus 1 plus C k plus 1. Sorry P X k plus 1 is greater than value at A f X k plus C k P X k. Let me call it as 1; equation number 1 alright. And in other case value at d is higher than A B this not C. This is B alright, that is why we have written. So, and value of D is higher than value of C is that clear. Not clear. Now you see A and B both are on the blue curve.

Student: B a C k. So, that is (Refer Time: 11:56).

That is why value at B.

Student: Value at C, no

No this is value at B I am considering the value at.

Student: Value at C k, it will be C k.

C k, it should be C k value at B alright. Then it is now f X k plus 1 plus C k P X k plus 1 greater than equal to f X k plus C k P X k. Now for d and C this is the case. Now d is f X k plus C k plus 1 P X k greater than f X k plus 1 plus C k plus 1 P X k plus 1. Let me call it as 2. Now you add 1 and 2 together, if I just add 1 and 2 what we are getting, you see we are getting that this part will cancel, f X k plus 1 f X k will cancel each other; that is why only we are having C k P X k plus 1 plus C k plus 1 P X k is greater than C k P X k plus C k plus 1 P X k plus 1, or we can have C k together P X k plus 1 minus P X k is greater than C k plus 1 P X k plus 1 minus P X k alright.

From here we know that C k plus 1 is greater than C k alright. Then from here what we can say that P X k plus 1 minus P X k must be less than is equal to 0 alright. Just you consider this way, because we are getting C k plus 1 minus C k and here P X k plus 1 minus P s k 2 terms differently; one term is positive C k plus 1 is greater than C k; that is why C k plus 1 minus C k will be always positive, that is why the other term must be negative. What you conclude from here. We conclude from here P X k plus 1 must be less than P X k, mind that second result alright.

That was the first result. Now it is the second result for us. Now let us move to the third result alright, from 1 we are getting, for having the third result from 1 we are having f

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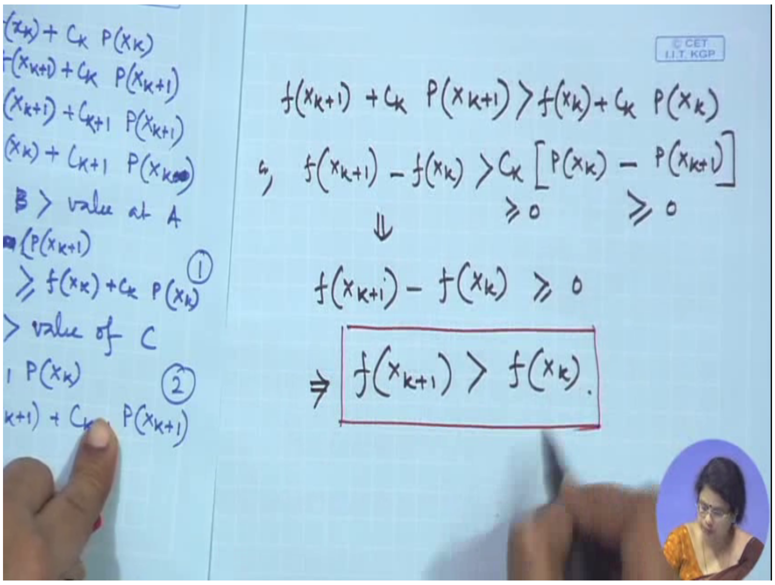

X k plus 1 plus C k P X k plus 1 greater than f X k plus C k P X k. Is that correct. Now from here if I just consider X k plus 1 minus f X k or greater than C k P X k minus P X k plus 1, is it not.

Now, you see P X k is the lower value than P X k. Just now we proved it, that is why this value is always positive. Now individual sequence are always positive that is why from here we can see that f X k plus 1 minus f X k is always positive. What is the case then f X k plus 1 is greater than f X k always. This is the third result, as I showed you alright. This was the first, this was the first result, this is the second, and this is the third result for us ok.

Now come back to the screen as I said that these are the three reasons for exterior penalty function method, and we have proved it. Now we can say that from here as I repeatedly said that we know the property of q. We know the property of q means we are, now that the optimal solutions of q are always moving as C k is going to infinity rather P is going to 0 alright. And if I just have this sequence, and if I just find out the limit point of that sequence that will be the optimal solution of the original problem.

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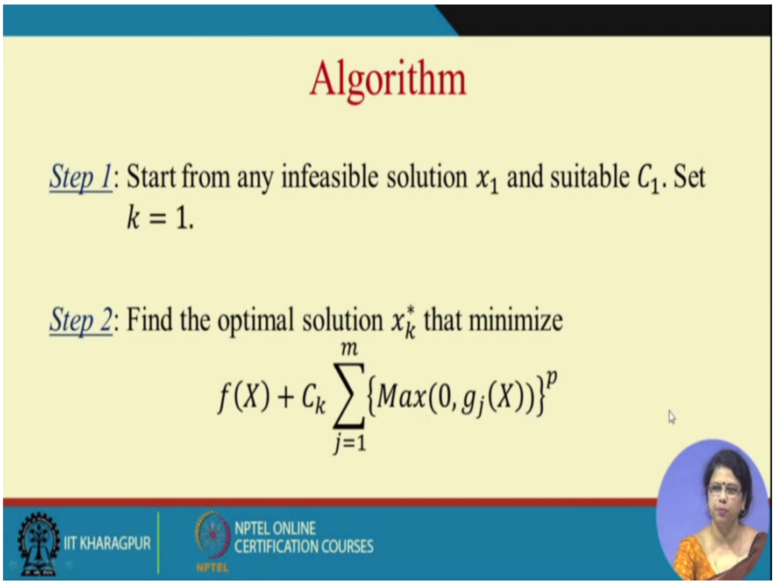

Let us apply this method in different problems. For that we need the algorithm to solve the problem. For algorithm the first step is that, always well start from the infeasible region; that is why we will check 1 point that is in the infeasible space, and from there we will select a suitable C 1, and we will find out the index k is equal to 1. Then the corresponding unconstraint optimization problem would be f X plus C k max of 0 g j X to the power p. We will just construct this unconstraint problem, then we will find out the optimal solution for that unconstraint problem.

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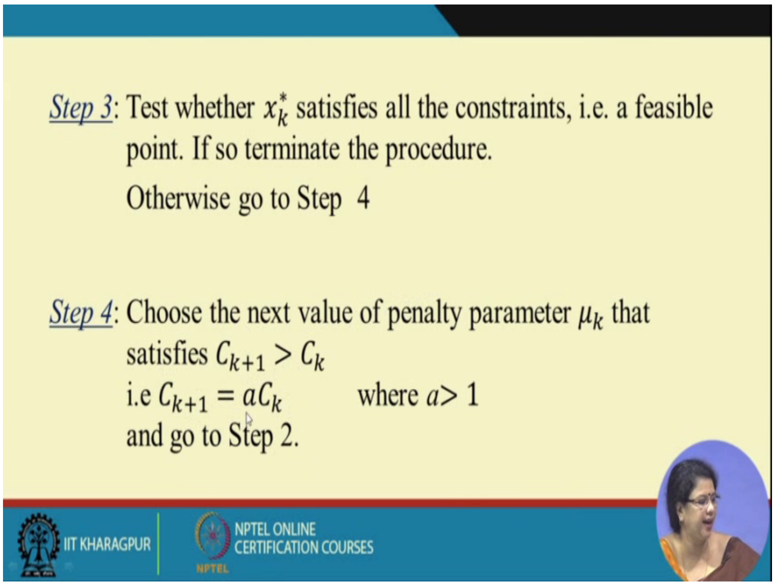

Then test whether that optimal solution satisfies all the constraint or not. If it satisfy all the constraints; that means, we are in the feasible region, that is why the corresponding optimal solution can be declared as the optimal solution, approximate optimal solution of the original problem. Now in the process, it may happen that we are moving C not very uniformly.

So, that even if we just move uniformly by changing the value of P with some step length, it may happen that, we will miss the boundary of the feasible space, that is why we will have certain q with the minimum value will be within the feasible space, and the algorithm says that if that point is within the feasible space, declare that is a optimal solution, that is why I say that is a approximate optimal solution of the original problem, but fortunately if you are in the boundary of the feasible space, then certainly that is the optimal solution of the original problem, otherwise not alright.

Now, this is. Now in the next we have to selective, this is not the optimal solution, select the another C which is the higher value of the previous C alright, that is why consider C k plus 1 is equal to a into C k alright, and go to step 2 means the corresponding unconstraint optimization problem you have to solve it, then

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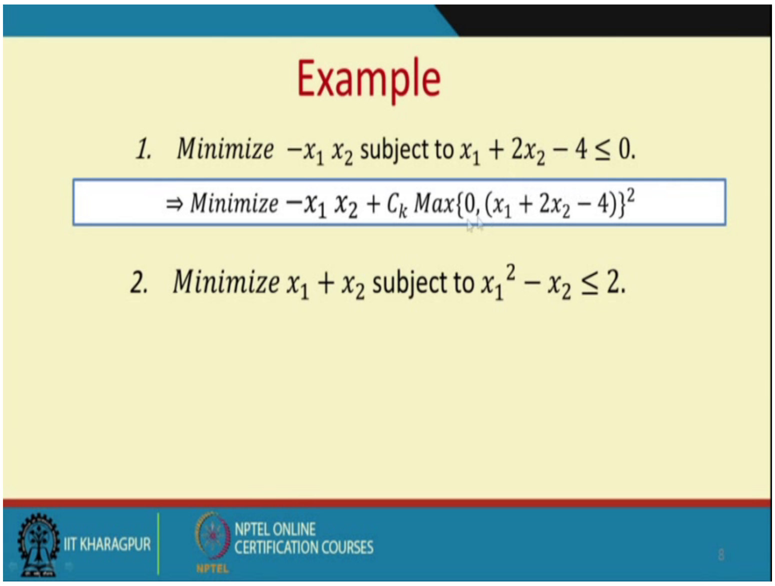

Let us solve both the problems one by one. The first problem says us that this is the unconstraint optimization problem; that is why let me find out the value of P X

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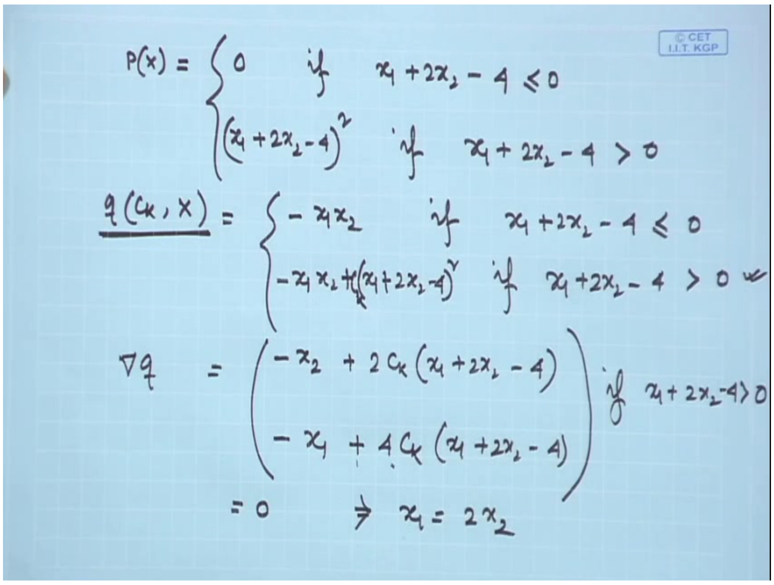

P x, I think say this is 0, If x 1 plus 2 x 2 minus 4 less than is equal to 0; otherwise the value is x 1 plus 2 x 2 minus 4 whole square, if x 1 plus 2 x 2 minus 4 greater than 0 alright. Let me now construct q q C k x, this is equal to minus X 1 x 2. If that is the original problem for us, and this is equal to minus x 1 x 2 plus x 1 minus plus 2 x 2 minus 4 whole square, if x 1 plus 2 x 2 minus 4 greater than 0 ok.

Now, we need to find out the minimum of this, how we can find it out. We can go for any process as you know unconstraint method; otherwise, since we know this is continuous function. We can find out the gradient of this function that is quite possible, that is why let me find out the gradient of C k x, then what would be the gradient, gradient value would be, let me consider only the case where the point is in the feasible space, infeasible space alright. For a specific C we will consider the case where q value is in the infeasible space, that is why if we just consider the gradient, the first 1 would be x minus x 2 plus 2. I forgot to put the C here that must be C k 2 into C k x 1 plus 2 x 2 minus 4 and minus x 1 plus 4 C k x 1 plus 2 x 2 minus 4 ok.

This is the gradient. Now this is the case if x 1 plus 2 x 2 minus 4 greater than 0. Now gradient must be is equal to 0, if gradient is equal to 0 we are getting from here, the condition that x 1 must be is equal to 2 x 2, because this value is greater than 0. If we just multiply this 1 with 2, then it will be minus 2 x 2 plus 4 C k, this 1, and this is minus x 1 plus 4 C k, this 1. If we just equate, both will cancel and we will get x 1 is equal to 2 x 2 alright.

Now, from here if we just solve this is equal to 0 vector, then we will get from this 2 equations, we will get x 1 star is equal to 2divided by 1 minus 8 C alright.

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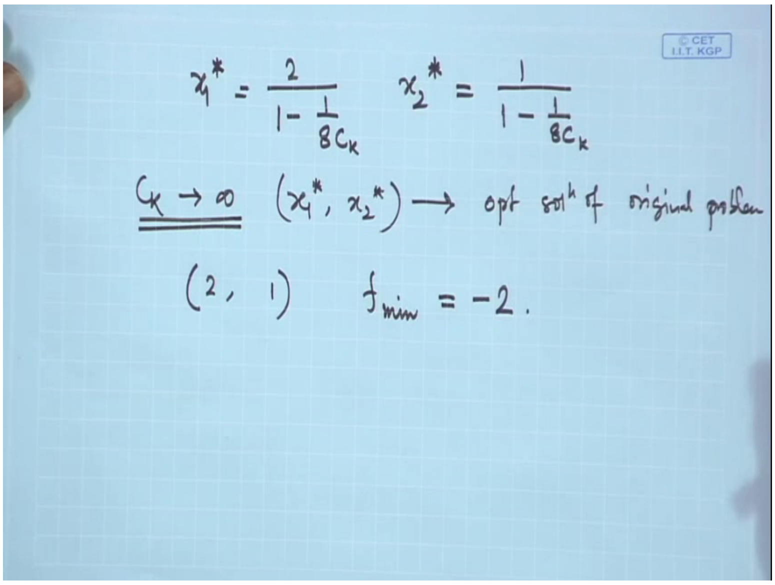

Whereas, we will get x 2 star is equal to 1 by 1 minus 1 by 8 C alright. This we are getting from this 2 here C is equal to C k, we will get it, what is the condition for the exterior penalty function method. The condition says as that if C k tendding to infinity, then x 1 star x 2 star will tend to the optimal solution of original problem.

This is the understanding, this is the guideline for exterior penalty function method that is why if we consider C k tending to infinity in this case, then the optimal solution would be 2,1 alright, and the corresponding functional optimal would be f min would be, is equal to minus 2. This way we can solve the problem. Now I have applied the first order necessary condition, and we can check the second order condition also to have q as minimum, that also we can find out the second order derivative, say (Refer Time: 25:48) second order rather the hessian matrix of q, and if we say that it is positive definite at this point, then we can declare that would be the solution of the original problem. Let me move to the next problem, problem 2.

Here the function q would be x 1 plus x 2 plus C k etcetera. Again the same thing is that P k P x we can define. Now if it is in the feasible space that would be 0, if it is not in the feasible space, then it will be the corresponding x 1 x 1 square minus x 2 minus 2 square. Now let me find out the gradient.

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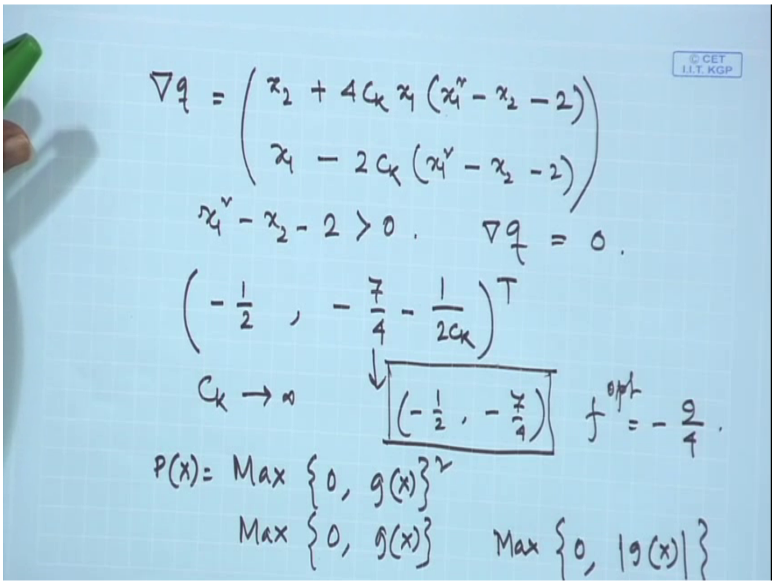

Of the function q this will be is equal to differentiation of q, with respect to x 1 that is why this is equal to x 2 plus 4 C k x 1 x 1 square minus x 2 minus 2, and if we just differentiate q with respect to x 2, then we will get x 1 minus 2 C k alright, x 1 square minus x 2 minus 2 ok.

Now, for this case, if this is less than 0 x 1 square minus x 2 minus 2 less than 0, what we are getting. we are getting that there is no solution at all, because from this case we cannot have the solution. Now if x 1 square minus X 2 minus 2 greater than 0, and if we just equate q is equal to 0, then we will get the tuple as minus half comma minus 7 by 4 minus 1 by 2 C k transpose alright.

Now, is C k tending to infinity, this point will tend to minus half minus 7 by 4 and corresponding optimal solution, we will get x 1 plus x 2 that is why this is minus 9 by 4.

Now, what I suggest to you that we have considered the penalty function as P X as max of 0 g x to the power square, instead of that what I suggest to you, that you find out the solution for the same problem by consider at g as g x 1 case, or consider the penalty function as 0 mod g x that also you find out, and for both the cases you see whether you are approaching to the same solution or not, that exercise you need to do, because there is no single penalty function, penalty function for exterior penalty method can be different, and for different cases whether we are approaching to the same solution, if we change the penalty function, the q function will change the pattern of q will change, but still we need to check whether we are reaching to the same solution or not, and with that I am concluding the exterior penalty function method. In the next class I will do the interior penalty function method for you.

Thank you very much.