Constrained and Unconstrained Optimization
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Lecture – 54
Penalty and barrier method

Today, we will discuss another procedure that is the indirect method for solving non-linear programming problem. This set of methodologies are being, name as penalty or barrier method.

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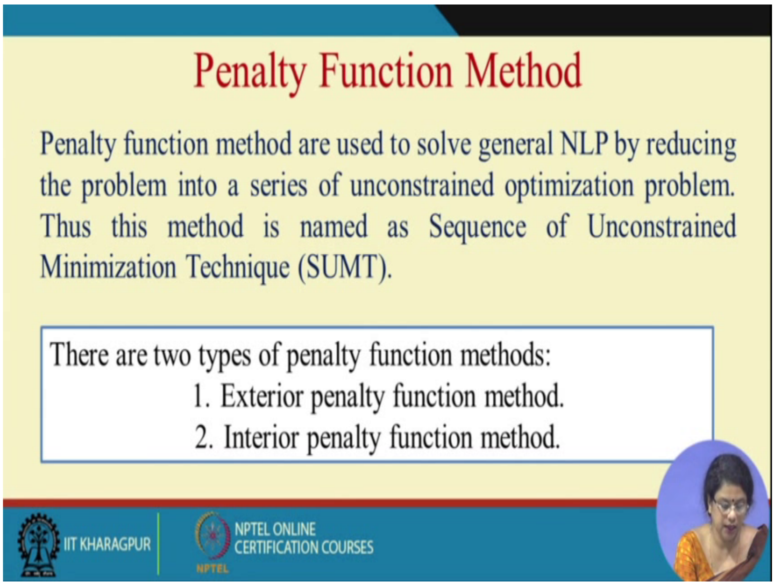

Let us consider a general nonlinear programming problem. Now in a, if we consider a non-linear programming problem with a set of inequality constraints, now with these penalty function method with this series of methodology, what we do, we are converting the non-linear programming problem, where the constrained non-linear programming problem into a sequence of unconstrained non-linear programming, and we solve the unconstrained non-linear programming problem with the methods we know.

Now, from the sequence of solutions, we are predicting, we are taking the decision what would be the, what could be the solution of the original non-linear programming problem. Generally this penalty function method there are 2 types of methods are there; 1 is the exterior penalty function method, and another one is the interior penalty function method. In the exterior penalty function method, then we start the solution procedure from the infeasible region, and from the infeasible space through the sequence of unconstraint optimization problems, we are approaching to the feasible region, and we get the optimal solution. Whereas, in interior penalty function method we start our process from the interior of the feasible space and we will approach to the boundary of the feasible space in the through the sequence of unconstraint optimization technique.

Now, this series of methodologies; that is the penalty function method is being named as sequence of unconstrained minimization technique. Now if we consider the minimization problem, then this is the minimization; otherwise if we consider the maximization problem, then there will be a sequence of maximize, maximized unconstrained optimization problems.

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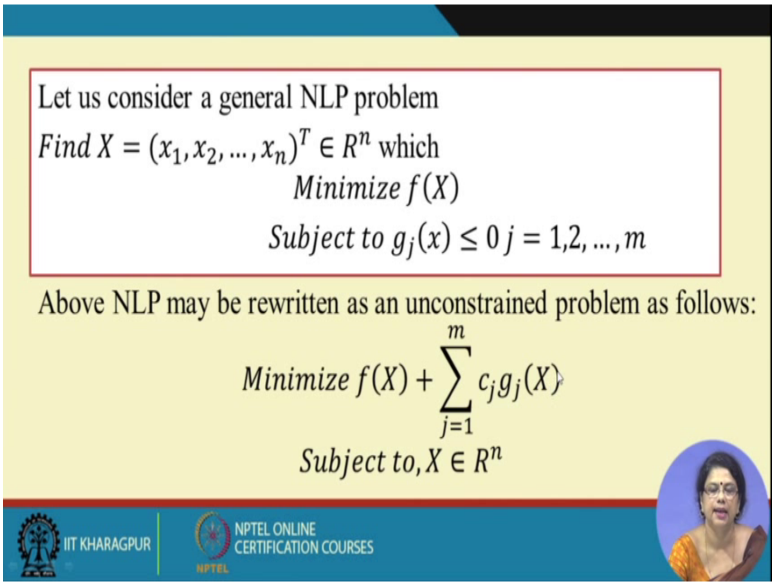

Let us consider a general non-linear programming problem, where we are having one objective function which is non-linear in nature, and the decision variable is n dimensional space. Now the constraints we are having m number of constraints, the basic idea is that, just our Lagrange method we convert this constraint problem into an unconstraint problem, like this, f X plus summation c j g j X, here c j a c j are the constants, which are the positive constants we convert in this way. Now this g j X we consider in this way once we are considering the Lagrange method, through the (Refer Time: 03:32) condition. We know we can get the solution for this problem.

Now, but this interior exterior penalty function method, they do not consider as g j x they consider as a function of g j X, there is one set for the exterior penalty function method, and there is another set of functions of g j X in interior penalty function method.

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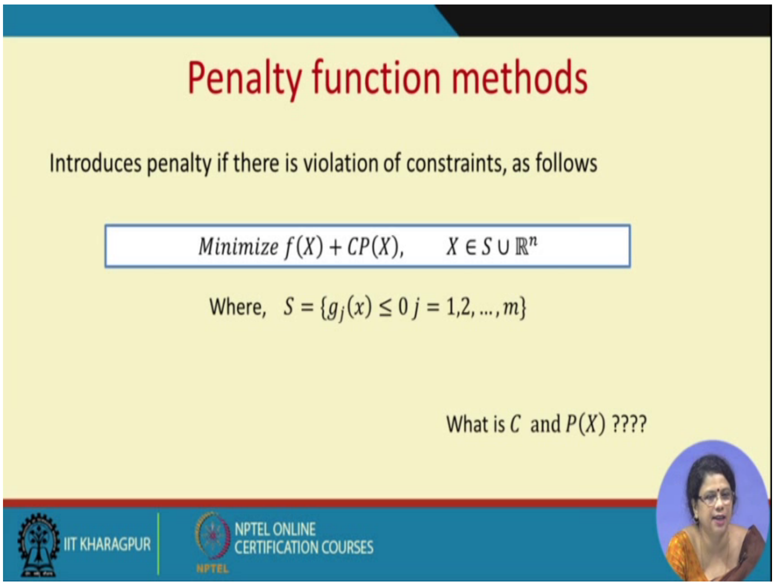

That's why let us a first discuss the penalty function method, where we have the form f X plus CP X, where PX is the penalty function, not only that PX has a function of g j X now, what is penalty function method. As I said we are moving from infeasible region to feasible region or from the interior of the feasible region to the boundary of the feasible region in the process, we are losing, we have to give some penalty value. For example, if we consider a toll road, where cars are going they are only recording the time, when you are entering the toll and when you are departing from the toll, from there they are finding out the mileage of your car.

Now, in the process they have the policy, if they have the policy like this if your speed of the car is 50 mile per hour, then within that, then you need not to give any toll at all, but if your car speed is more than 50 mile per hour, then you have to give a toll of rupees, something; that means, you are giving a penalty for having the higher speed than 50. For example, we can have another process like this, we have a pick up scheduling process is going on; where one few cars are moving through certain region. Now there they are taking the people from their respective residence.

Now, if the reach in time, then they need not to give any penalty, but if they reach in delay, then the car organizer they have to bear certain penalty, that is way we can adapt the penalty value in the process, this penalty is just like that. If we are far from the feasible space. For example, we are having the, we are approaching the feasible space from outside the exterior penalty function method. Now if we are much far from the feasible space, we have to bear much penalty than the situation, where we are very much nearer to the region of the feasible space that way we are calculating the penalty function. Depending on the value of x, the penalty is being calculated and C is one constant, which is a positive constant, and we have to find out the relation of C and PX, how they are really working within this model, we have to find it out what are the properties of C, what are the properties of PX, we are going to find out for the exterior penalty function method, and for the interior penalty function method.

Now, as I said p s is a function of g j x; that is why the value of PX will depend on the value of x, whether it is within s or not that is why in the next we are finding out the Properties of PX.

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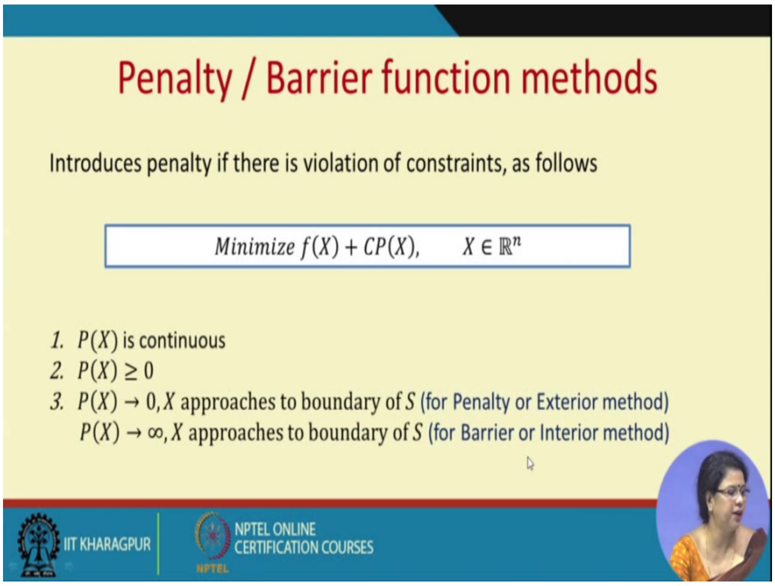

Now in literature it has been said that if we are having one unconstraint, if we just make 1 non-linear programming problem, if we just converted to an unconstraint optimization problem non-linear programming problem, by with inclusion of the penalty function as well as the constant c.

Now, there are certain properties of P X, PX must be continuous one property and PX must be positive that; that is another property next PX tending to 0 as X approaches to the boundary of X S; that is this is for the exterior penalty function method. An exterior penalty function method is being named as the penalty method, and interior penalty function method is being name as the barrier method. Now if PX tending to 0 as x approaches to the boundary of the feasible region S, then this is the exterior penalty function method, and on the other hand if PX approach is approaching to infinity as x approaches to the boundary of x, then it is the barrier or interior method.

Let me take one example for this, let us consider one problem.

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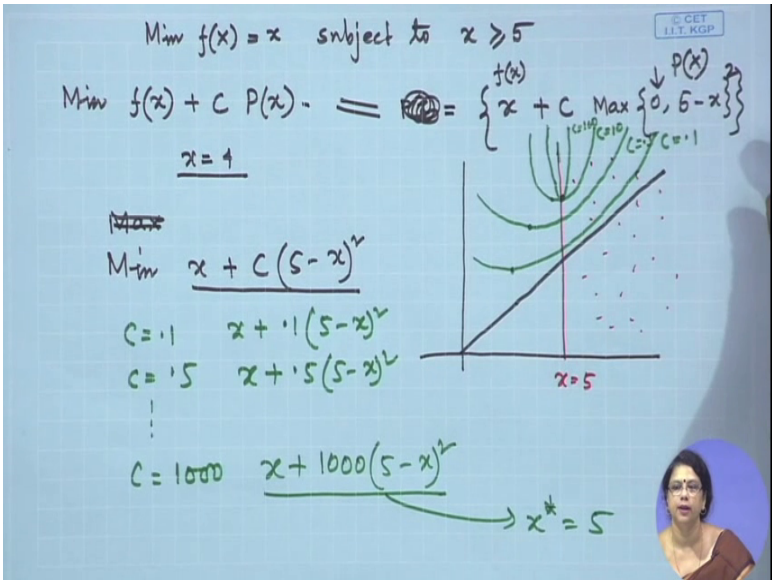

Minimization of f x is equal to one dimensional problem, we are considering subject to x greater than equal to 5. Now problem is very simple one that is why the solution of this problem must be 5, because 5 only give you the minimum value of f x. now in the penalty function method what we do? We convert this method as f x plus c p x, how really what does it mean, how PX can be formulated. Now we may consider PX as this way, x plus c x is the objective function, max of 0 5 minus x, what does it mean? It means that if this is the feasible space for me, then this is the objective function f x is equal to x, and x greater than equal to 5 that is why if we consider, this is equal to 5, then this is the feasible space.

Now, you see if the value of x is 6, then what is the value of this, this is minus 1 and max of 0 and minus 1 would be 0, that is why there is no penalty if we are remaining within the feasible space, but you consider the other way. If we consider x is equal to four, then the value of max PX would be max of 0 and 5 minus x; that means, 1 if I am, if the value of x is 3, then the penalty value would be 2; that means, you see if we are approaching far from the feasible space, the penalty value is increasing ok.

This way we can consider a penalty function, that is why you see if we consider this penalty function; that means, the penalty function is nonzero, if we are out of the feasible space and the penalty functional value is 0, if we are within the feasible pace, that is why the penalty value is 5 minus x. now you change the penalty function, if I consider the square; that means, we are incurring more penalty for it that is why this we are incurring high cost for the situation, where I am out of the feasible space than the previous situation. I can make this power s square q 4 according to the need, how much penalty really I want to consider all right.

Now, if this is the situation, then you see for a point x is equal to 4, if this is my starting point, then what is the objective function. Now objective function would be this, is not p x, this is f x plus c p x, this is my f x f x plus c p, this is equal to this all right. This is only PX clear. Now if this is. So, then the function is maximum, sorry minimization problem this is minimum of x plus c 5 minus x square.

Now, for different value of c. Now this is an unconstrained problem, previously the objective function was linear objective function, but here the objective function is non-linear, if I just draw this function, we will see for different value of c, we will get this way; this value for c may be 0. 1, and for the next we can get the value for c is equal to 0.5; say now in this way we can construct this, and we will see as the value of c is increasing, we are getting the functions like this, what is the significance of this. significance of this is this 1 for c is equal to 0. 1, the minimum value could be here, here for c is equal to 0. 5, the minimum value could be here, for c is equal to say 10, c is equal to say 10, the minimum value may be here, and for c is equal to 100, the minimum value is here, because these are all the unconstrained functions, and for this case, this is unimodal finding out minimum through the processes process. You have already learnt it is not a difficult for you.

We can find out for different value of c. We can have the function like this 0.1 5 must x square for c is equal to 0.5. The function would be 0.5 5 minus x square as the value of c will be c 1000. We will see the function will be x plus 1000 5 minus x square all right, and we will see that this minimum will approach to the minimum is equal to 5. That is the beauty of the process; let us consider the penalty function in other way.

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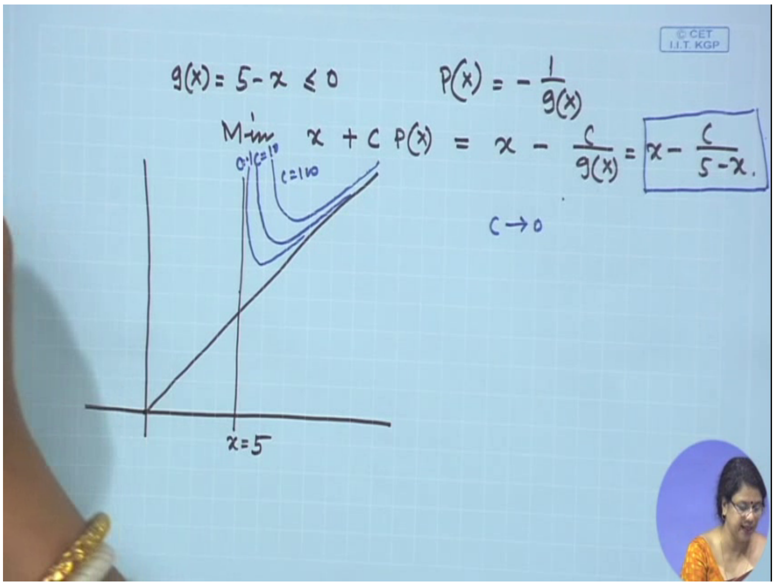

Let us consider the penalty function as instead of g x, if here my g x is 5 minus x less than is equal to 0, let me consider the PX like this, PX is equal to minus 1 by g x, then the function would be the unconstraint optimization problem would be minimization of x plus c into p x. Here it would be x minus c by g x or x minus c by 5 minus x clear.

Now, if I just draw this function again, let me find out the, this is the objective function c is equal to x this is x equal to 5 etcetera. Now we will see that if we just draw this function for different value of c, we will see that for c is equal to say 100, the function would be like this. Now for c is equal to ten, the function will be like this, and if the value of c is going to 0, the minimum will just approach to the value x is equal to 5, this is another way we can consider the penalty function.

Now, let us see what really its happening for both the cases. For this case when we are approaching from the outside of the feasible space, as the value of c is increasing that c is going to infinity, we are approaching to the optimal solution and in the other case, if c is approaching to 0, for this is for say 100, this is for say 10, this is for c is equal to 0.1. If c is approaching to 0, we are approaching to the optimal solution, what I mean that the optimal solution, because the minimum we are just looking at the optimal solution of individual unconstraint optimization problem I will speak more on this now.

Now, just I wanted to say that previous 1 is the exterior penalty function method, and this 1 as the interior penalty function method.

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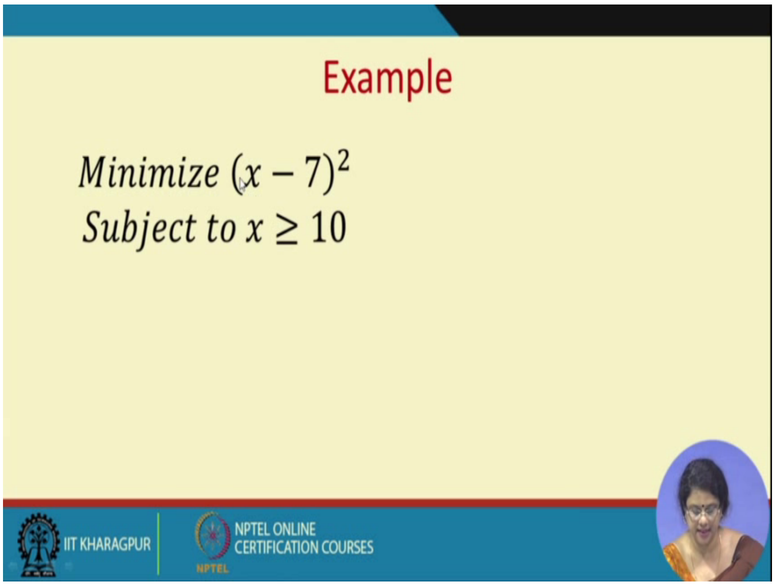

Now this is one of the example x minus seven whole square, and subject to x greater than equal to 10 always. Once we are having the minimization problem, let us construct the constraint as g x less than is equal to 0.

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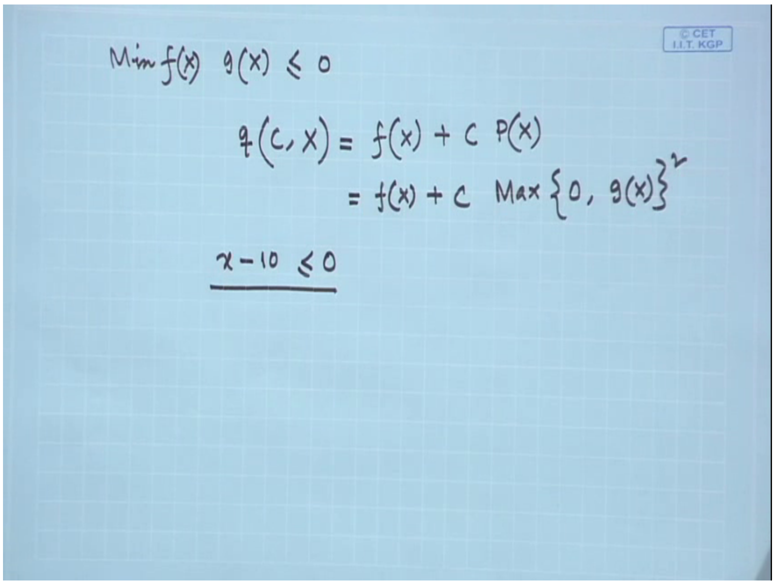

Minimization of f x, then for this problem if we consider the unconstraint optimization problem; say q; that is the function of c, and x this would be is equal to f x plus c PX the penalty function.

Now, for this problem you see we have considered the penalty function as f x plus c max of 0, and the value as g x, and we have considered the square, and if we just found that for c lower value of c.

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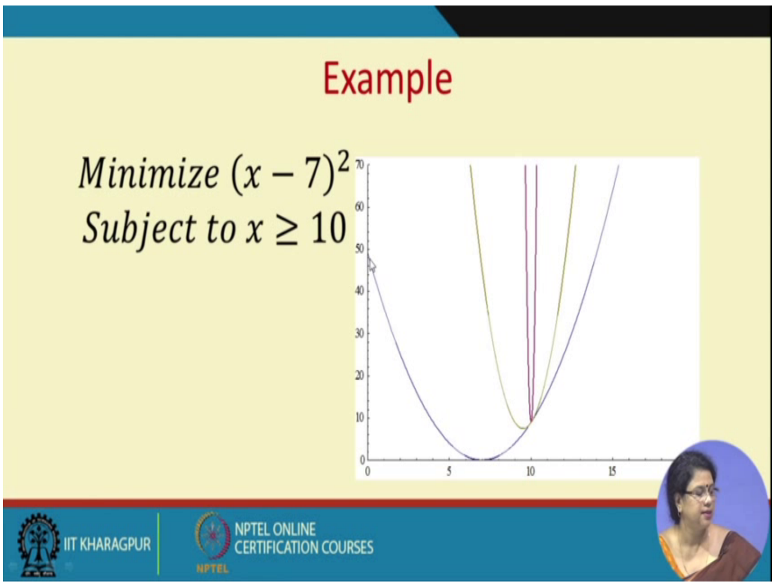

This blue colored graph is the graph, and if I just find out the minimum, here is the minimum for c higher value of c than the previous one. This is the gray colored graph, and the minimum is coming here, and as the value of c is increasing, we will see the minimum is converging to x is equal to 10. Now let us consider this problem instead of considering x greater than equal to 10. These are the value I have considered, and I just found. Now if I considered the problem in the different way x less is equal to 10 instead of x greater than equal to 10. Now just to think here my g x would be, is equal to x minus 10 less than is equal to 0. Now, if I just approach in this way, what could be the solution of this you can find it out?

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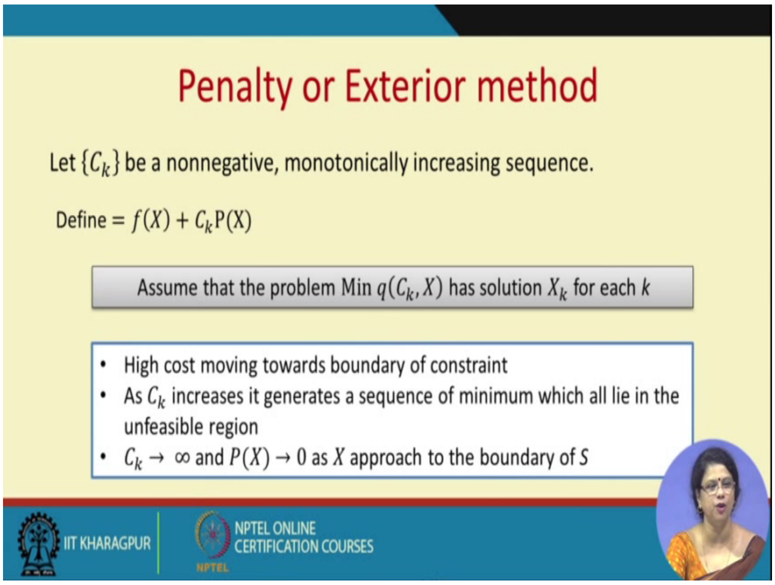

Now, if I just formalize the penalty function method, then we will see that I can say that c k is the non negativity monotonically increasing sequence; that is we are talking about the constant c, and if we define q is equal to function of c k x, and here the index k is for the iteration number. We can start from k higher value here lower value here, or we can start the k from the for the higher value, the k will, the value of x will be for the exterior penalty function method. From the name it is very much clear that for the exterior penalty function method, we have to start from outside of the feasible space; that is why k should be very low value ok.

If this is the case then the problem will be minimization of q, where x k is the solution, optimal solution for each k all right, and where the penalty function can be considered. In this way p is a positive integer. Now in this case, if we considered the penalty function this way always the value of the penalty function will be positive, and the value of the PX is continuous, and PX is approaching where than, then you need to find out if we approach to the boundary of the feasible space you must have been realized. If we are approaching, the x is approaching through to the boundary of the feasible space; the PX value will converge to 0. Whereas, as the value of c k will go to infinity all right, this is the exterior penalty function method.

Now, there are certain properties for these exterior penalty function method. The first is that we have to incur higher cost for, if we just move towards the boundary of the constraint, because c the value of c k, the value of PX is going to 0, but if we are far from the boundary, then we have to incur the higher cost. 1 thing I must say that the function must not be; such that it will make the optimization problem, as a unbounded problem always. We should have the solution of that problem that is why the selection of c must be in such that we will not get the unbounded solution of the optimization problem, that is why the higher cost must be considered, it should be high, but as it should not be too high. So, that we are getting the unbounded solution. Not only that, it should not be too high so that the computational cost will be very high that way.

The next is that if c k increases, then it generates a sequence of minimum solutions, that you must have been realized. Now if we are getting the c value from the lower value to the higher value, we will get a sequence of the minimum values. And from that sequence we will find out the limit point of that sequence, and that limit point of that sequence of optimal solutions will give you the optimal solution of the original problem, and for the exterior penalty function method, as I said as c k tending to infinity PX tending to 0, because x approaches to the boundary of x, and they are certain other properties also for exterior penalty function method, that give values are always increasing p values are always decreasing. Whereas, the a value we are getting better, and we have to prove this fact, and this fact I will prove in the next class.

Thank you for today.