Constrained and Unconstrained Optimization
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Lecture – 53
Feasible Direction

Now, in continuation to my previous class on feasible direction methodology Zoutendijk feasible direction methodology, today I will tell you more on that, and I will do the example on that part.

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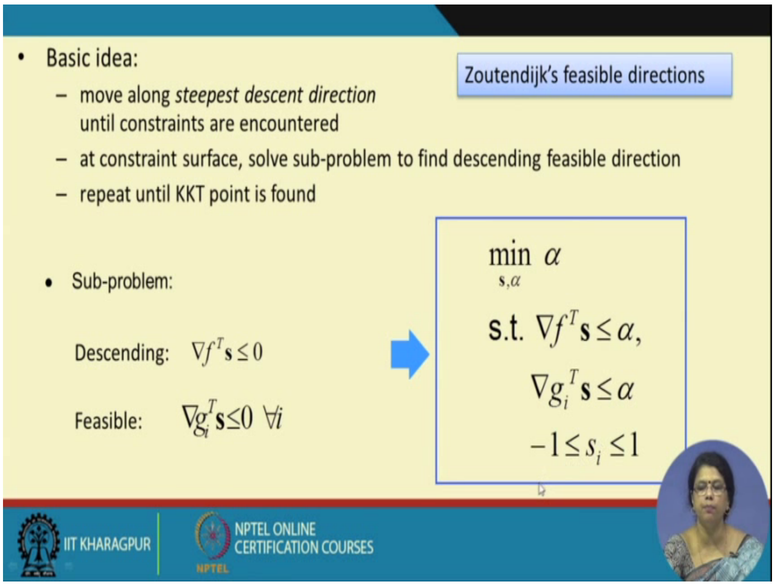

Now, Zoutendijk feasible direction technique you have realized that we are looking for the descent direction for the minimization problem and ascent direction for the maximization problem. Now what is the basic idea of the Zoutendijk feasible solution direction method is that, we start from a point and we move to the next point in such a way that we will remain within the feasible space and not only that we move in the descend direction for the minima.

Let me consider the minimization problem for my class. Now and we repeat the process again and again unless we reach the KKT point that is the basic idea. That is why I repeatedly said that descent direction can be explained if S is a direction in n dimensional space, where S is having different components like S 1, S 2 up to S n, then grad f dot S if we just take the inner product of 2 vector that must be negative for the descent direction because always the descent direction and the gradient of the objective function makes obtuse angle and similarly if we consider the feasible direction there also, I just showed you with figure that the gradient of individual constraint and the feasible direction makes again the obtuse angle that is why the inner product of 2 will be the negative value.

Rather non positive, it can be as well, it will be 0 when we are at the boundary of the constraint that is why if we just want to consider the values for these that just you see both are negative, that is why if we consider a one variable alpha which is the maximum of these 2 negative values then our target would be 2 minimize this alpha because we want to find out the direction as well as the step length in such a way that we will remained in the feasible space.

If we want to maximize, then what will happen one point I am starting I will get the direction, but it may happen I will be out of the feasible space that is why to remain negative at the most we can go up to 0 that; that means, up to the boundary of the feasible region I have to minimize the alpha value that is why in optimization for mod we can just write it down in this way and in other way in other thing is that the direction S i individual direction, we are considering the normalized value that is why always the magnitude will be one if we just take the norm of this vector it will be one.

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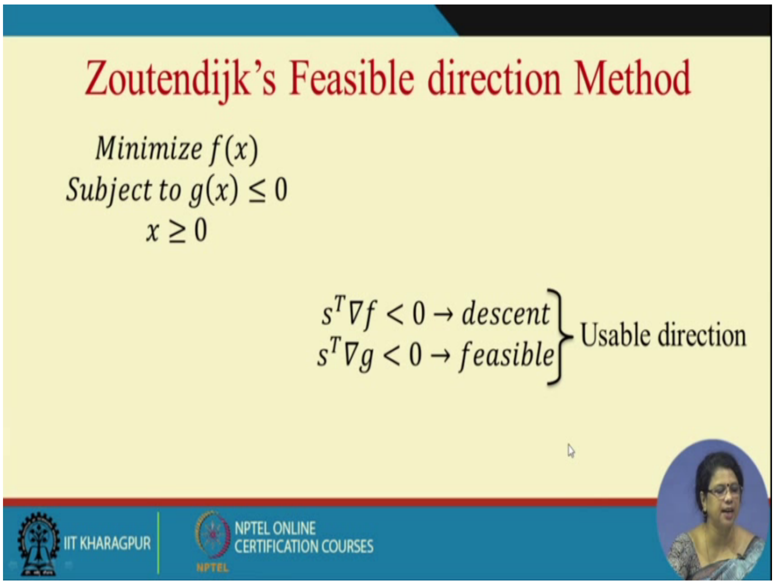

That is why it will in between minus 1 and 1, but Zoutendijk considered the feasible and the descent direction together as the usable direction, that is why our target is to for that the Zoutendijk methodology to find out the usable direction from a point.

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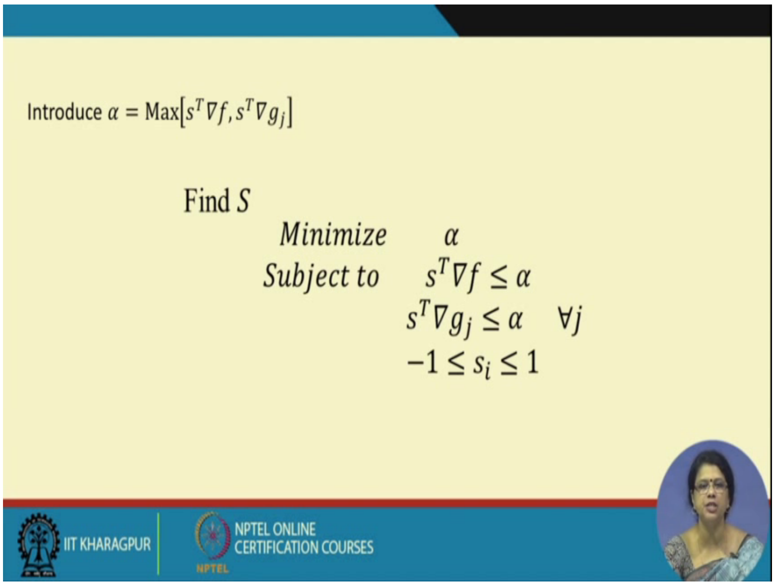

That is why the whole method starts from a initial guess point and it moves to the through the usable direction up to a certain in step length and stop there. Check for optimality if it is not so, again from that point it moves to the next usable direction with the optimal step length go there. If it gets optimal solution fine otherwise repeat the process again and again it has been seen that if we apply different technique this is the one of the direct method. In the direct method, we are using the differential calculus here because you see we are considering the gradient of the function because we are considering the function is continuous in nature; gradient of objective, gradient of constraint we are using those there are different other methodologies as well projection method these are there, but we have seen with minimum number of steps very quickly we can get the convergence, that is why the Zoutendijk feasible direction method is very popular method in non-linear programming problem.

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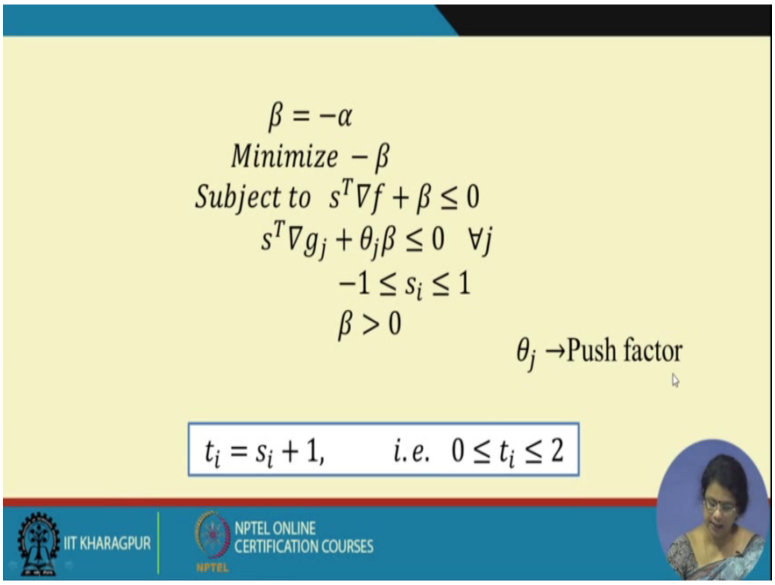

Now, the previous optimization model if we consider as you know this is a linear programming problem let me just elaborate this thing with 2 variable case.

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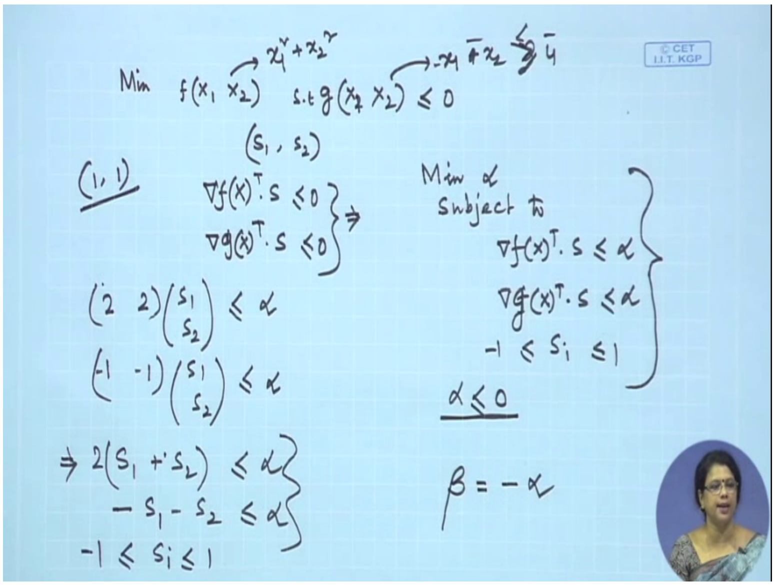

Where we are having the function f which is having 2 variable x 1 x 2 and function g 2 variable x 1 x 2 less than is equal to 0 minimization subject to. Now we are looking for the direction S 1, S 2 as I said that grad f at a point X; X is my starting point T dot S must be less than is equal to 0 grad f grad g at a point x must be less than is this is the usable direction.

From here we are getting the optimization model by considering alpha is equal to max of grad f x dot S and grad of g x dot S we get grad of f x less than alpha, this is g alpha and individual S i minus 1 and 1. Just look at the model this model we are having where alpha is always negative, if alpha is 0 then g x is equal to 0; that means, we are in the boundary of the feasible space that is why if alpha equal to 0 this is a linear programming model because the function at any point if I just elaborate this, we will get for example, we are having grad f 1 as x 1 square plus x 2 square and we are having a point 1 1 from there we are searching from a usable direction.

Then the first equation will become 2 x 2, x 1 at point 1 1, it is 2 and grad f the another 1 2 x 2 at point one; it is 2 S 1 S 2 less than is equal to alpha and for the g let me consider x 1 plus x 2 greater equal to 4 that is why we can have 1 1 S 1 S 2, oh; I am making it minus minus minus that is why minus minus less than is equal to alpha. Just you see we can consider S 1 plus 2 S 1 plus S 2 is less than equal to alpha minus S 1 minus S 2 less than is equal to alpha alpha negative. This is a simple linear programming problem, but linear programming simple as algorithm will not allow you to have the alpha as negative that is why we are considering one variable as beta; beta is equal to minus alpha.

Then once we consider that just you look at the model we can see minimization of minus beta where beta is positive, just to we will place minus alpha is equal to beta and we are we will consider one theta that is the push factor I told you the theta is nothing, but the angle of the descent feasible cone, and this angle we are we want to make to 0. Theta is always positive that I showed you with the example theta is always positive, but at the optimal point theta is equal to 0, at the KKT point theta is equal to 0 our target that is why theta is positive everything is positive here, we can solve with linear programming very nicely.

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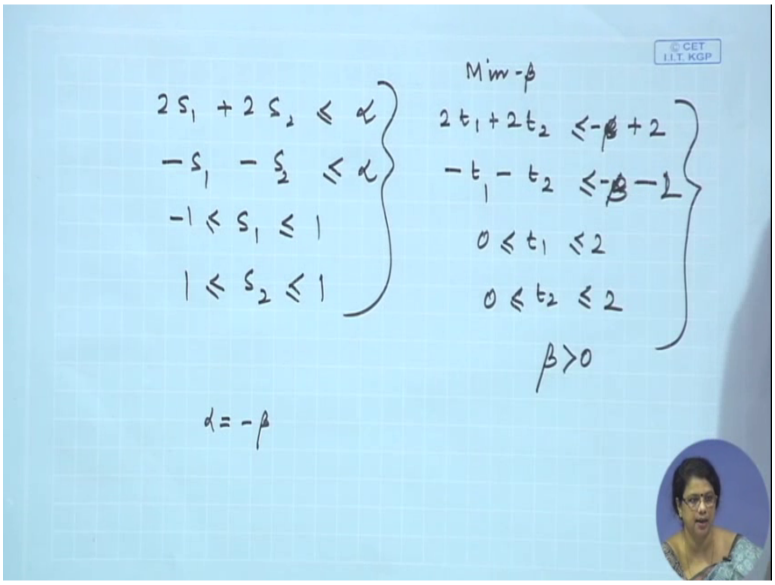

But here we are making certain transformation from S I; from S i we are making S i is equal to t i minus 1, if we just substitute it here the previous equation just to see we were having 2 S 1 plus 2 S 2 less than is equal to alpha minus S 1 minus S 2 less than is equal to alpha, S 1 in between minus 1 and 1, S 2 in between minus 1 and 1. After this transformation, it will become t 2 t 1 plus 2 t 2 less than is equal to alpha plus 2. Minus t 1 minus t 2 less than is equal toit will become alpha t 1 minus 1 plus 1 minus 1 t 1 will become in between 0 to 2 and alpha minus 2 because one from one here from one that one and t 2 will be again in between 2 then you see if we just maximize sorry if we just minimize beta where beta is equal minus alpha, then my job is done; that can be considered as a simple linear programming problem that is the now theta should be there as a push factor.

Student: Minimization of minus beta.

Minimization of minus beta as we consider alpha is equal to minus beta that is why that is correct. Now one thing is that I did not consider the push factor theta here because we may consider we may not consider in the sense that theta will be always positive that is why in general case we can consider theta as equal to one, otherwise I can consider one term here as minus theta beta that can be considered here nicely plus theta beta, all right.

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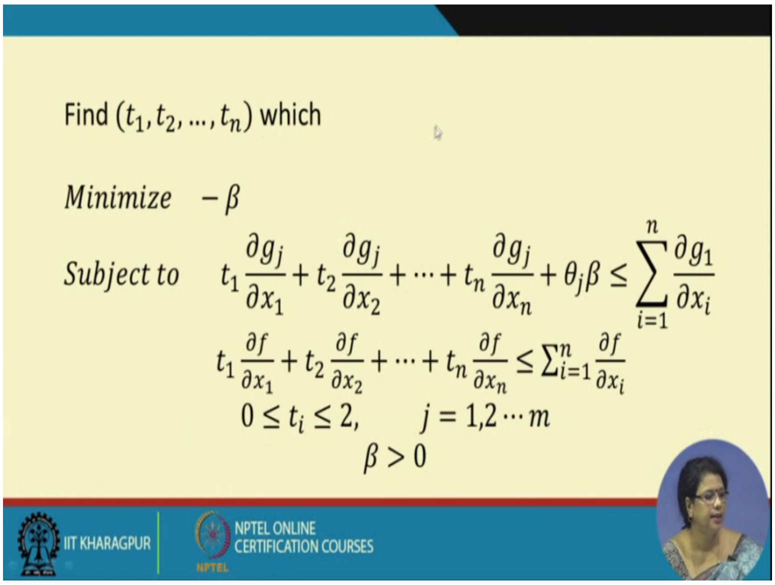

That is why whatever I have written just you see the same thing has been written in elaborated form t 1 del g, j x 1, etcetera and in the right side because of the term S i is equal to t i minus 1 because of one this part will go in the right here also it will go in the right in the similar manner and j will be there are n number of constraints.

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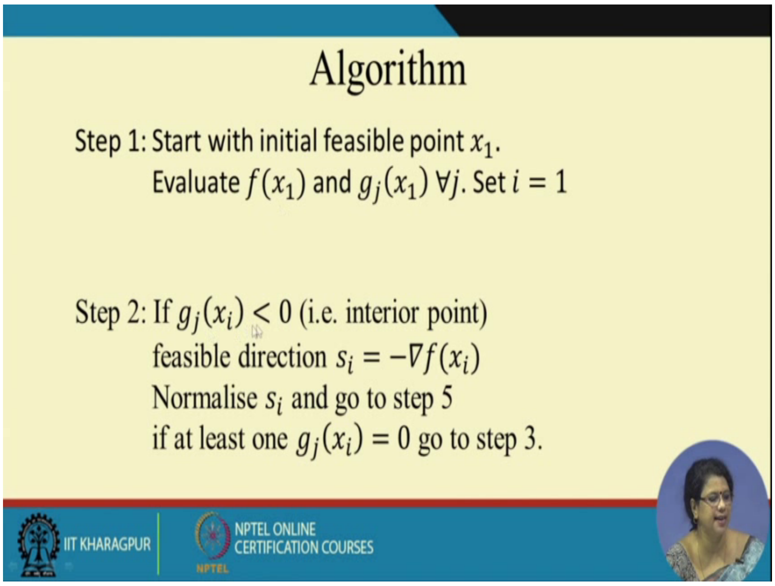

Let us apply this technique for the for one of the non-linear programming problem the Zoutendijk feasible direction. Just I will tell you the algorithm briefly first very quickly we will start from a point x 1, we will see whether the x 1 is the interior point of the feasible space or outsides point that is more important because we cannot start a point which is out of the feasible space. Zoutendijk method will not allow us always it should be the interior point; that is why, we will check whether x 1 is in the interior or not and we will evaluate the value for f x 1, if g j is less than 0 interior otherwise exterior.

We know the feasible direction is minus del f x i that is starting i is equal to 1 and we will get S i, we will normalize S i; that means, we will make the we will divide with the this is a vector, we will divide the vector with the magnitude, we will get the normalized vector the norm of that vector will be 1 we will get that and we will see that at least one of the constraints which should satisfy with that point with equality; that means, that point must be at least on the boundary of the one of the constraint, that is necessity of this method.

Then we very quickly we will get the optimal solution. If you see that at least one of this on the boundary then we will go to the next step otherwise what we will do? We will make hit on one of the constraint set how really we can do it? We know S i we do not know lambda i that is the step length we will make the step length in such a way that we will be at least of the boundary of the constraint, that we can do it we can calculate it; I will show you in the next.

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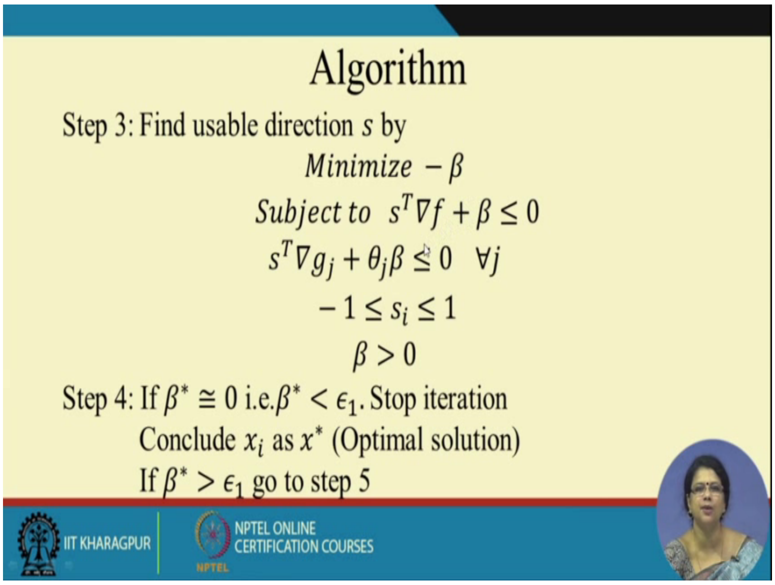

We have to make that point at least one of the constraint boundary, then we will go to step 3. In the step 3 we will find out the usable direction and we will find out beta what is our beta? Beta is equal to minus alpha, we are trying to get alpha is equal to 0 as I said that is why if beta is 0, we can stop our iteration. Instead of 0 if beta is a very small value where epsilon is any pre-assigned value given to us, then we can stop our iteration otherwise we will move to the next.

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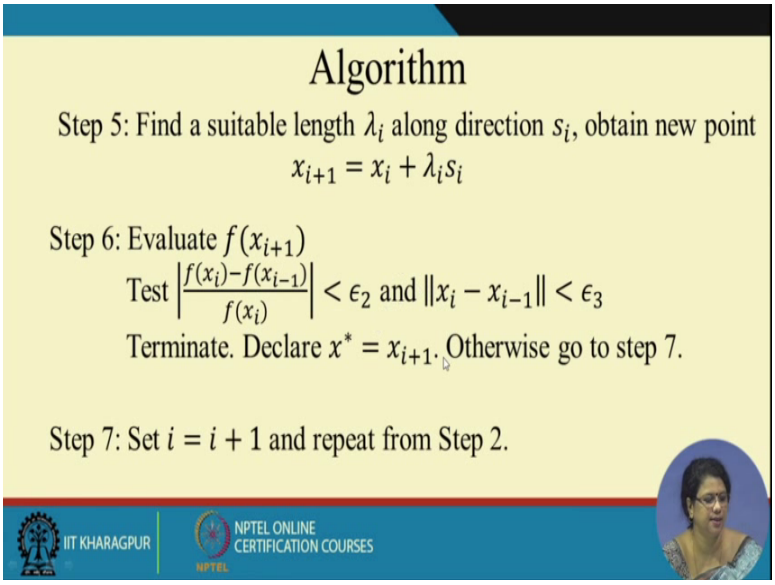

We will proceed further and further and we will find out the step length suitable step length that you know with the optimal step length we will proceed; we will check for optimality and repeat the process.

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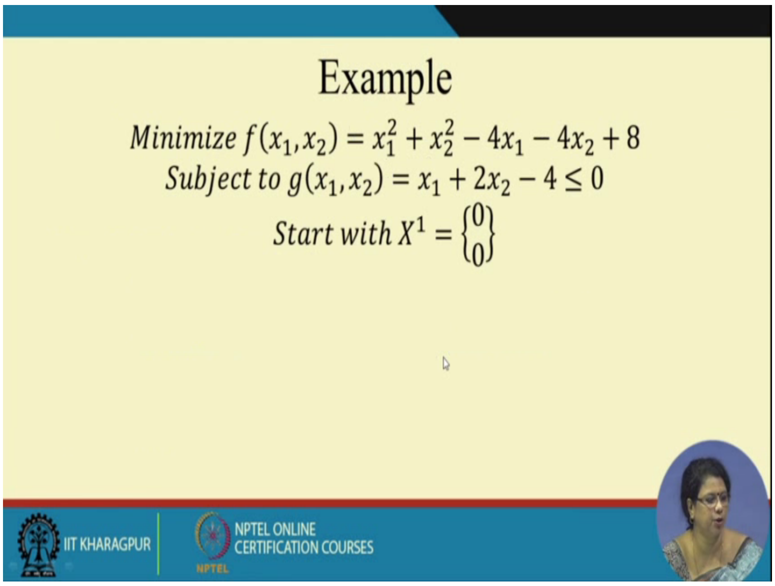

But I will show you entire thing with this example how really we are doing it just look at these we are starting from the point x 1 that is why if we consider grad of f.

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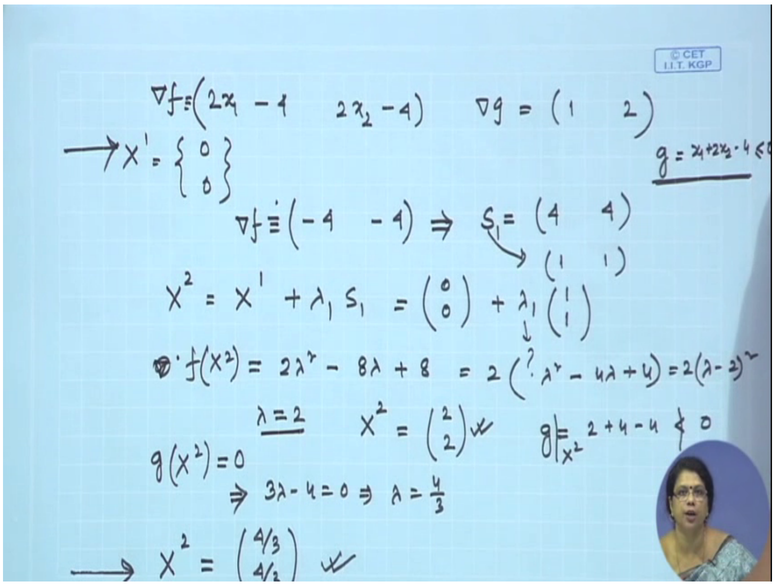

We are a getting 2 x 1 minus 4, 2 x 2 minus 4 grad of g 1 2 all right. Now my point is given as x 1 is given as 0 0, if we consider this we see that grad a value is coming as minus 4 minus 4 that is why we can say S is the direction which is 4 4 because that is minus of grad f 4 4, but I want to make S in between 0 1 1. So, that the modulus is one that is why I will just make it as 1 1, there is no harm to it. We are not really I say it that we are not really interested for the magnitude of x, we are interested for the direction of S vector all right. That is why I can say S is equal to 1 1 that is why if I want to move from x 1 to x 2 what how we can write? X 2 is equal to x 1 plus lambda one let me write down S 1 where S 1 is my 0 0 plus lambda 1 1 1, we can write it all right that is why my point is this one, but we do not know what is lambda one that is another problem for us we do not know what should be the step length. If I just see the point 0 0 our constraint has been given g is equal to x 1 plus 2 x 2 minus 4 less than is equal to 0 that is why 0 0 point is the interior point.

Now, what is the necessity? Our necessity is that in the next, we will reach to the one of the boundary of the constraint here in this problem, we are having only one constraint we know the direction, but we do not know the step length with the usual process let us find out the step length for it how really we can find out? We will find out in this way that in the next grad of f at x 2 must be 0 because we are finding out the minimum lambda of minimum step length so that in the next first order derivative is 0 second order derivative is positive, all right.

If we just find out grad f no sorry function of x 2 at the point function of x 2 then we will get the function as lambda square plus lambda square 2 lambda square minus 4 lambda minus 4 lambda plus 8. Otherwise 2 into lambda square minus 4 lambda plus 4 otherwise 2 into lambda minus 2 whole square what we see that at lambda is equal to 2, we are getting the minimum value of f if lambda is equal to 2 we can check this the whether we are getting the minimum by going to the next order derivative that also we can check.

But if we consider lambda is equal to 2, then my x 2 will be is equal to 2 2. If x 2 is 2 2 just we see whether g is being satisfied or not; g at x 2 the value is coming as 2 plus 4 minus 4 its not less than 0 that is why if I move to this point the problem is that I will be out of the feasible space I do not want it, what do I want I want I want a point which must be at the boundary of the feasible region how I can make at the boundary of the feasible region.

So, that g x 2 must be is equal to 0, then I will be at the boundary of the feasible region I will find out for what value of lambda x 2 will be at the boundary of the feasible region then if I just put x 2, then I will get 3 lambda minus 4 is equal to 0 or lambda is equal to 4 by 3 all right then my next point will be x 2 would be 4 by 3; 4 by 3. This is the first point, this is the next point and this point is on the boundary of the feasible region. Same thing has been written here just you look at I said this is the direction and we are moving to the boundary of the feasible region.

Now, at this point we will see if I just go to this point whether we are getting optimal or not how we can check it?

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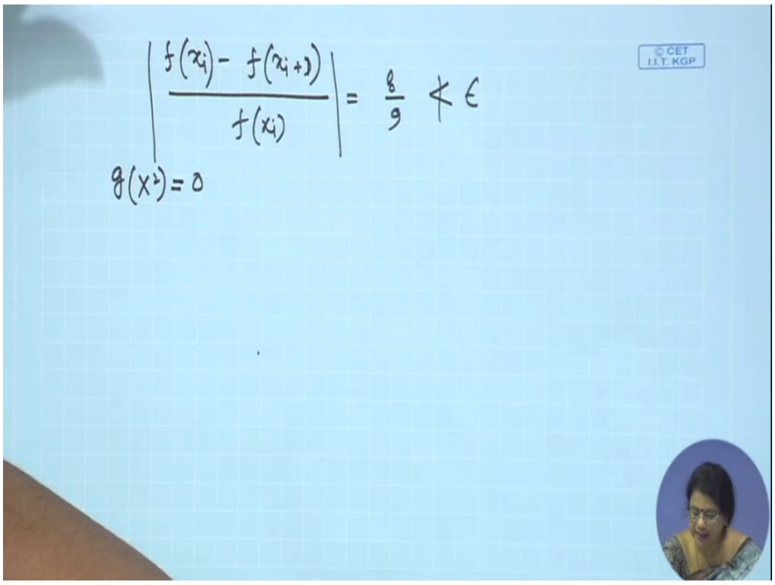

We will see whether f x i minus f x i plus 1 divided by f x i if it is very small, I am getting this value as 8 by 9; this is not very small value that is why this is not an optimal point I will go to the next, not only that we are getting g x 2 is equal to 0 that is why from S x 2, I will move to the next point how I will move? I will move through the usable direction that is the thing because if I move to the usable direction then only I will be within the feasible space. I have started from the interior point then I have moved to the boundary of the feasible region, then you see I have to find out next the usable direction how we can find out the usable direction look at this.

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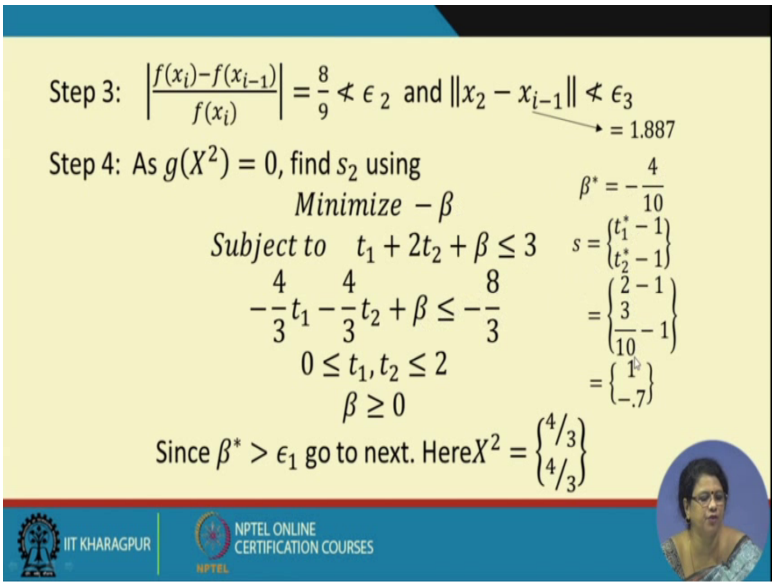

Usable direction is coming as minimization of minus beta and t 1 plus 2 t 2 because my X 2 is 4 by 3, 4 by 3 plus beta less then is equal to 3, minus 4 by 3 t 1 minus 4 by 3 t 2 plus beta less than is equal to by considering theta is equal to 1 again and t 1 t 2 in between this. If we just solve this optimization problem this is a simple linear programming problem, we will see that the solution will come as value of beta will come minus 4 by10 and the value of t 1, t 2 will come as 2 and 3 by 10.

So, that we will get the value form S from here clear now what is the value for beta? Beta is point 4; point 4 is it a small value? Not really we are looking for a small value of beta because beta is equal to minus alpha. Though it is at the boundary x 2 at the boundary still it is not the optimal solution.

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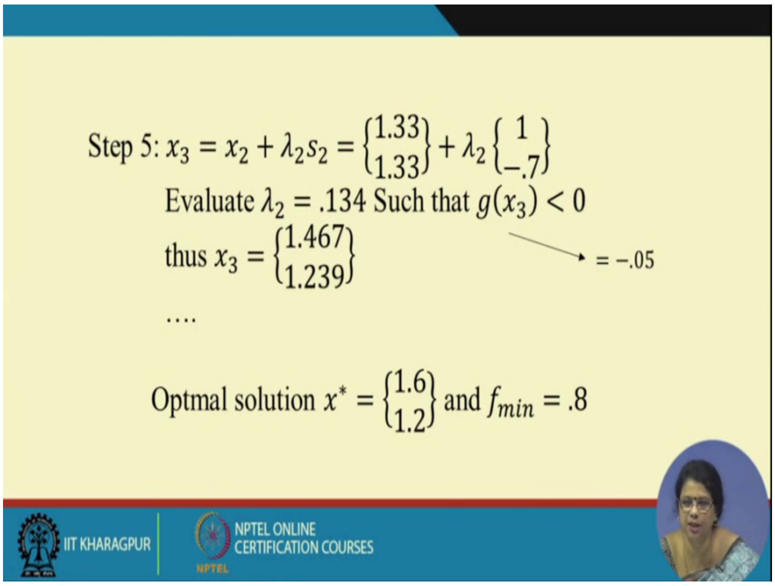

That is why we have to move to the next, how we can move? I will go to x 3 I will find out x 2 plus lambda 2 S 2, where x 2 is 4 by 3; 4 by 3 plus lambda 2, S 2 is this one and from here again, we can find out lambda 2 with a optimal condition that f x 3, we will just consider at what point this is minimum first order derivative is 0, we will find out; we will find out this way and ultimately we will get the value for x 3 this one.

In this way, I will move around the feasible space and ultimately we will get the optimal solution like this. That is the whole feasible direction method; I am just showing you another problem; how we can handle it.

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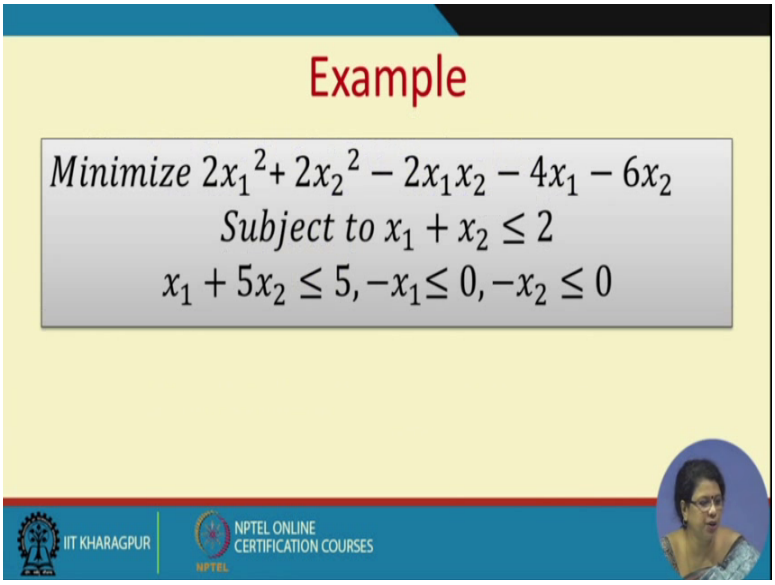

If time permits, I will do a part of it otherwise you just complete it and take it as the assignment, I will supply you the solution for it the problem is this one minimization of 2 x 1, x 2, etcetera.

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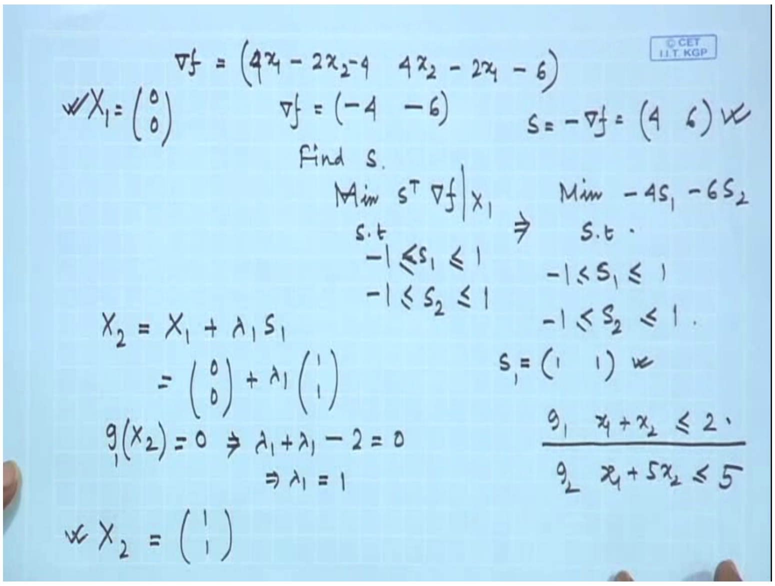

Let me find out first the grad f, this is the first part of it. Grad f is equal to 2 x 1 minus 2 x 2 and its 2 no its 4 x 1 minus 2 x 2, and the next 4 x 2 minus 4; 4 x 2 minus 2 x 1 minus 6.

Now, my starting point if I consider the first point as 0 0 all right. If I consider that what is my grad f value then? Grad f value would be minus 4 minus 6, if I consider this one then what will be the S value? S value would be minus grad f that is equal to 4 6, I have to normalize this one I will get a value for S. This way we can do in other way also we can find out S how we can find out? I can find out this way as well that we will find out find S such that minimization of S T grad f subject to minus S 1 would be minus 1 and 1 S 2 would be minus 1 and 1 at point x 1 that is also another way we can do we can find out the direction find S.

Then we will get the optimal problem from here minimization of minus 4 S 1 minus 6 S 2 subject to S 1 minus 1, 1, S 2 minus 1 1 we can get it and from here, we will get the value for S as 1 1 optimal solution. Now you may ask that this is the linear programming problem negative value is not allowed that is why you can transform it from S 1 to t 1 as t 1 is equal to S 1 plus 1; that way you can solve this linear programming problem, you will get the value for t 1 t 2 from there you can calculate S 1 S 2 this is other way we can do it.

Our target is to move to the next that is why from x 1 I will move to x 2 how I will move to x 2? I will move to x 2 as x 1 plus lambda 1, S 1 this is my S 1 0 0 plus lambda 1 1 1, I want to remain within the feasible space my constraints are x 1 plus x 2 less than is equal to 2, x 1 plus 5 x 2 less than is equal to 5, if we consider if I this is g 1, this is g 2. If I want to remain in g 1, then you see I have to take the value for lambda 1 as 1 then g one x 1 will be is equal to 2, then I will be what I said that I want to be g one x 2 must be is equal to 0.

That means how much lambda I have to take for it that is the question, that is why lambda 1 plus lambda 1 minus 2 must be equal to 0, it means lambda one must be is equal to one that is why in the next we are moving to x 2 to 1 1, this is my second this is my first all right what I have to do? I have to move to the next I will go to x 3 how I will go to x 3? X 2 plus lambda 2 S 2 where S X 2 is 1 1.

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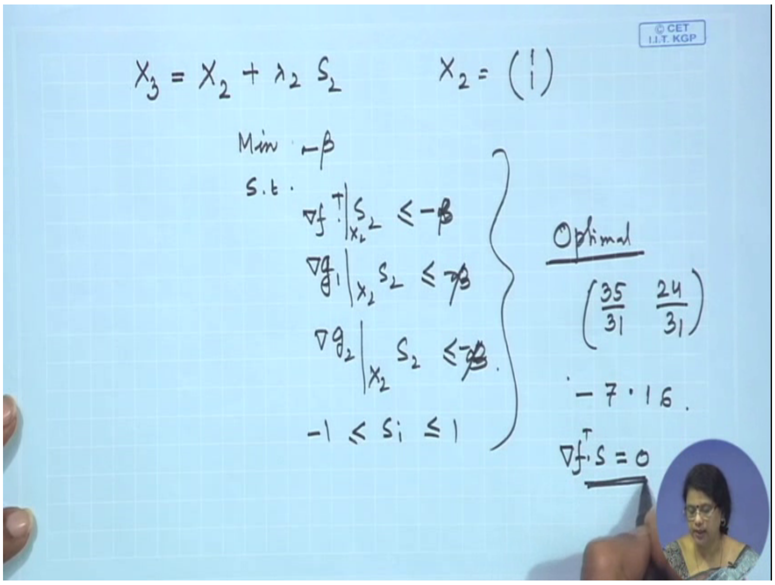

Now, what will be S 2? S 2 will be the direction that is usable because already I am at the boundary of the feasible space that is why I have to solve minimization of minus beta such that I have to just to do the calculations, such that grad f T S 2 must be less than is equal to alpha at x 2 grad g 1 x 2 must be less than is equal to alpha, grad of g 2 at x 2 must be less than is equal to alpha and S i must be in between minus 1 and 1 and the everything you do you will get the value for t 1 and t 2 and do the calculations further.

Ultimately, you will get the optimal solution as optimal solution you will get do not consider is alpha consider is minus beta. Optimal solution you will get as 35 by 31 and 24 by 31 and corresponding functional value, you will get as minus 7.16 and you will see at this point grad f dot the direction is will come as 0, that is your beta will come as 0 almost 0 that is that the beauty of that Zoutendijk feasible direction method in this way you can solve linear programming problem by remaining within the feasible space.

Thank you.