Constrained and Unconstrained Optimization
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Lecture – 52
Constrained Optimization

Now, today again, I am dealing with the multi variable constraint optimization problem. I explained one of the methodology that was a direct method and to solve the constraint non-linear programming problem.

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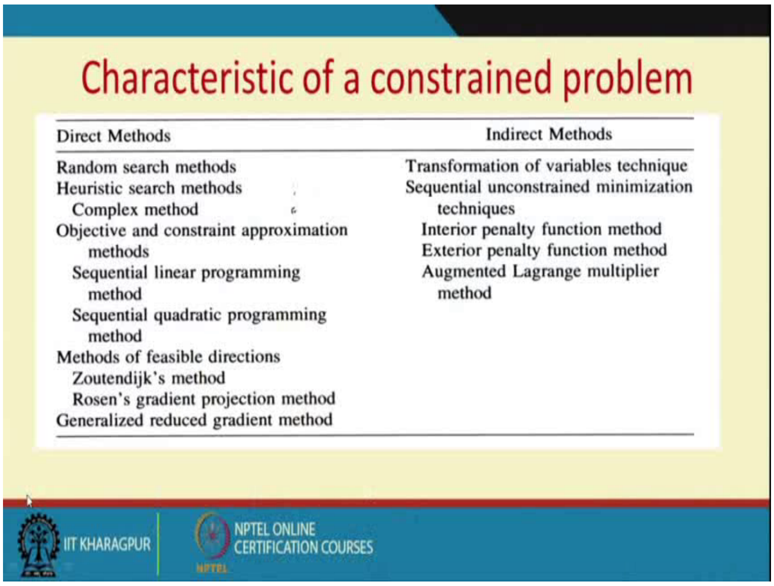

I told you there are several methods available in one of that Kelley's cutting plan method that was the sequential linear programming method we have solved. Now today I will tell you the method of feasible direction that is zoutenijk method, to solve the constraint non-linear programming problem.

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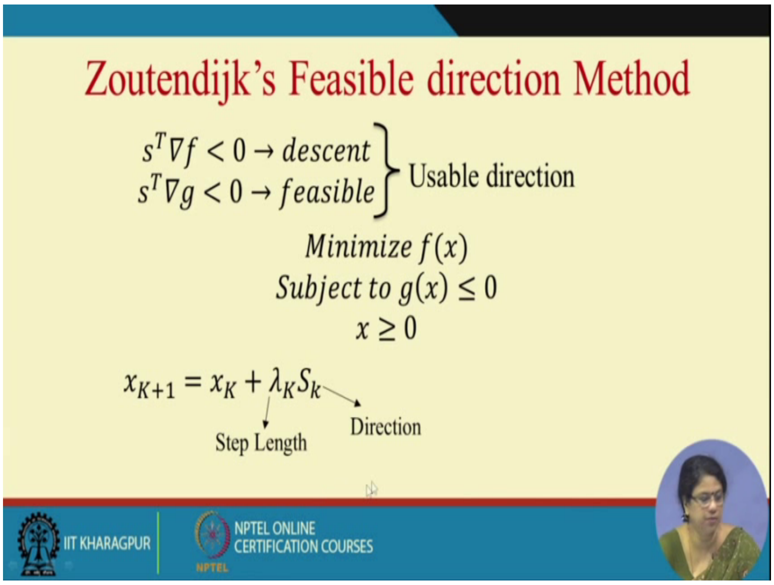

Now, let me tell you the basic philosophy of this feasible direction method first.

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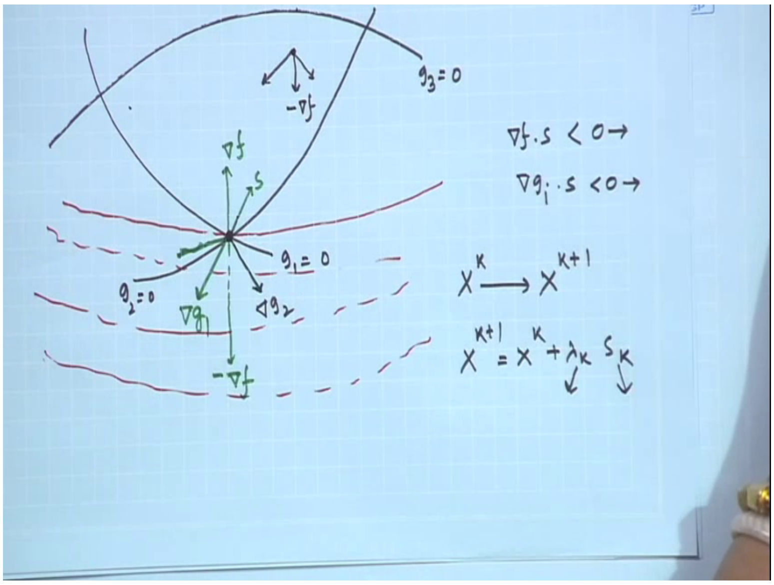

Now, if we have constraints like this g1 g 2 g 3. And this is less than equal to that is why this is the feasible space for us, and this is the level curve of the objective function. Then this is the optimal solution this a graph already, we have shown you before. Now at this point you see this direction is the grad of g 2 direction. This let me do the directions with different color, then it will be understandable. And this direction sorry that should be the this must be grad of g1 that is the normal, if I draw a tangent here at this point this is the normal to that tangent is the gradient direction at this point of g1 and at this point if I consider the grad f this is the grad f direction that is why the opposite this one is the minus grad f direction all right.

Now, from here this is a feasible direction from this point, if I move this way is the feasible direction. That is why if I consider this is one of the direction as s, then we can say that s is making acute angle with this and s is making obtuse angle with grad of g1 or g 2. That is why you had the relation that grad f dot s minus grad f and you had the relation grad g i dot s this is the feasible direction if this is greater than 0 making the acute angle. And for the other thing s is a feasible direction, if this is less than 0 all right, but on the other way if I tell you in the ascent direction grad of f is less than 0, but grad of f dot s is less than 0 for the descend direction it is also true.

The Zoutenijk method of feasible direction based on this fact from any point we will find out what is the descent direction for the minimization problem and what is the ascent direction for the maximization problem. As well as we see that what is the feasible direction from this point. This is the this is the direction is happening for the feasible direction. This is the direction for the ascent direction all right. Instead of this point if I consider a point here, what we can see this is the minus grad f direction all right. And what about grad g1 this is the grad g1 direction this is the grad g 2 direction.

That’s why what would be the feasible direction for the for the minimization problem from here, I will move here through these from main any path from this point, I will I have to reach to this point. That is why the method should be such that it should guide me in such a way that if I start my journey from here it should guide me such a way I will end up here all right. If I start from here I should end up here with many with a shortest path I have to come. That is why the Zoutenijk method of feasible direction it has been said that you cross it from a point you cross it to that direction where you are getting, the direction as a feasible as well as decent clear.

From here, if I this is the guess point of my optimal solution. Then you see I cannot move further within the feasible space with this 2 conditions, but if I start from here I can do it if the problem is of maximization type the optimal solution cannot be here. If this is the level curve of the objective function, then optimal solution must be somewhere here. That is why the method should guide me in such a way that if this this point is my starting point of my journey. For maximization problem I should move through to the through the ascend direction as well as I will go through the feasible direction. That is the basis of the zoutenijk method of feasible direction.

Now, if we consider a minimization problem of type minimization of f x subject to g x less than is equal to 0, and we can see that descend directions since this is the minimization problem then s T grad f less than 0, that is the descent direction and s T grad g less than 0 is the feasible direction refers, direction means what direction means if you remember that I was saying that if I start my journey from x k point. Then I will move to the point x k plus 1 how x k plus 1 would be x k plus lambda k s k where lambda k is the step length and s k is the direction of the corresponding point.

We can have infinite number of directions, but if we consider a single component of n dimensional space this direction can be only in the right. Or in the left that way and the Zoutenijk said that if any direction is descent as well as feasible that must be usable direction for the minimization problem. And for the maximization problem if the direction is ascent as well as feasible that is the usable direction that is the term given by Zoutenijk that we will move through the usable direction, we would not said that we will move through the descent direction or ascent direction we will say that we will move through the usable direction.

That’s why we will move from x k to x k plus 1 in such way the direction must be usable in nature that is the basic philosophy of Zoutenijk method.

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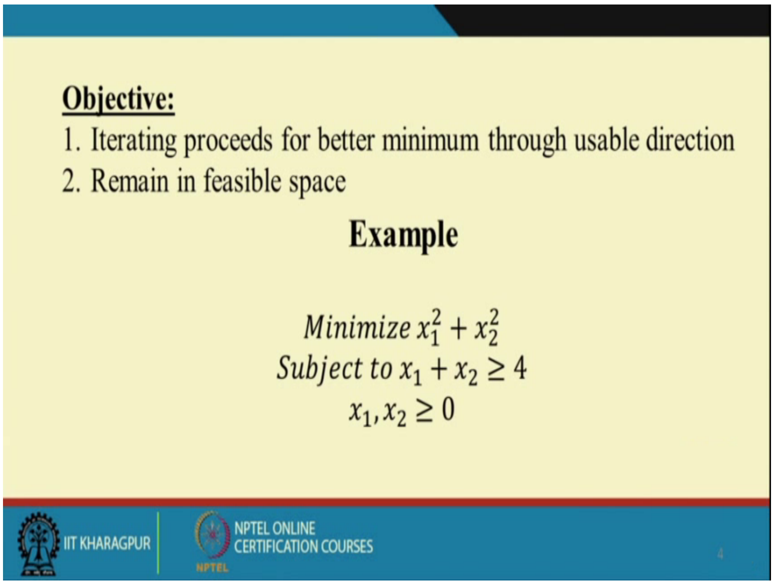

Now, this method is a again and iterative process you must be understanding that we will start from a guess point we will move through the next point, we will reach to the next point through the usable direction again we will go to the next point through the usable direction. How long we will do? As long as we can we will remain within the feasible space. Once we fill that we are out of the feasible space we will stop our journey and we will declare that could be the optimal solution that is the beauty of this methodology.

And another thing is that all the points are within the feasible space that is why we will do in such a way that we have to remain on the feasible space. How to define that feasible space? Once we have the constraint set defining feasible space is not a problem for us because we will say that from this point to this point you move you move to such a point where the point must be within the feasible space the point must be in the usable direction. These are 2 things we will mention in the methodology.

That is the beauty of this feasible direction method. And for explaining more about it I am just giving one example what does what does it mean about the decent direction and the that is a feasible direction. This is the problem for us you must be realizing descent direction if corresponds to the objective function and feasible direction refers to the constraint function that is why we will deal both the functions together.

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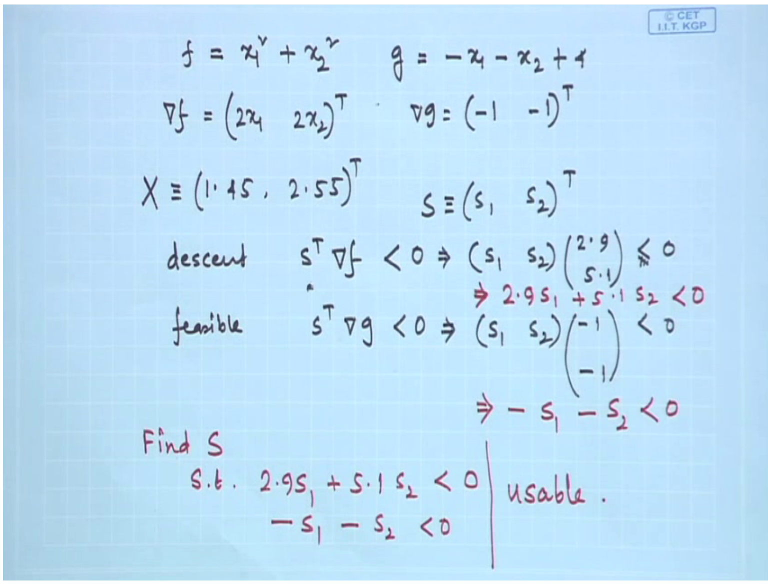

Now, we have the function f as x 1 square plus x 2 square and we will have the function g that is the constraint function as minus x 1 minus x 2 minus 4 we are making it plus 4. We are making it less than is equal to 0. If we do so, then what is your grad f grad f would be 2 x 1 2 x 2 transpose because that is a column vector. Grad g would be is equal to minus 1, minus 1 that is again a column vector that is why we are putting it as a transpose.

Let me take one point that is x, any point within the pace. You check whether this point is within the feasible space or not. If I just add it is it greater than 4 it is coming exactly equal to 4 all right. Let us see which direction is the if I ask you if I start my journey from here which direction will be the descent direction for me because the problem is of minimization time, what you will do? You will find out that you will say s is a direction s direction means what you have 2 variables here. Then I will say this direction will have 2 components s 1 and s 2, again s 1 is related to x 1 and s 2 is relate to x 2 all right.

Then what is the descent direction descent direction would be s T grad f must be less than 0. And what is the feasible direction I showed you with the geometry s T grad g must be less than 0. What we are getting from here that s 1 s 2 at this point 2 x 1 2 x 2 that is why this point is coming 2.9 and this point is coming 5.1 must be less than equal to must be less than 0 no not equal to 0. And this is equal to s 1 s 2, no involvement of x 1 x 2 must be less than 0. What we are getting from here we are getting from here, that descent direction is that direction where (Refer Time: 13:42) point 9 s 1 plus 5.1 s 2 must be less than 0 and here the feasible direction is a minus s 1 minus s 2 less than 0.

That’s, why if I ask you to find out the usable direction you will say find s such that 2.9 s 1 plus 5.1 s 2 must be less than 0. And must minus s 1 minus s 2 must be less than 0. That must be the usable direction all right. That is why for any non-linear programming problem if I give you the point, if you know the problem you can find out which would be the usable direction. From that point very nicely clear. If I just draw the graph of it you will get another beauty here, x 1 x 2 x 1 plus x 2 greater than equal to 4. That is why if I consider this way this is a 4 0, this is 0 4 and this is a feasible space for us all right.

Now, minimization of x 1 square plus x 2 square. That is why my objective function is center at 0 a center a circle, it is moving this way, may be here the solution. Here is the solution for you all right for the optimal solution. Where did you see?

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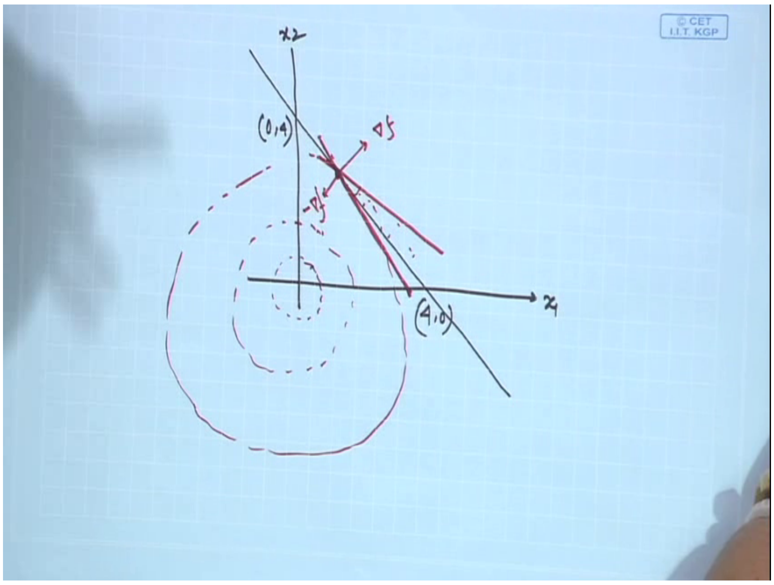

This direction is the grad f direction; these direction is the minus grad f direction all right. At this point if I consider that this is the tangent at this point then certainly oh sorry if I consider the tangent at this point, this way actually at the optimal solution the I wanted to say that descent feasible cone is empty at the optimal solution. If this is not an optimal solution always we will get a descent feasible cone like this. And this angle will help us a lot in finding it defining the Zoutenijk direction feasible direction method this angle will be named as a theta angle as a push factor our aim is to make theta as 0. So, that this feasible descent cone will be empty that is the check for optimality. Later on I will show you Zoutenijk method in the next.

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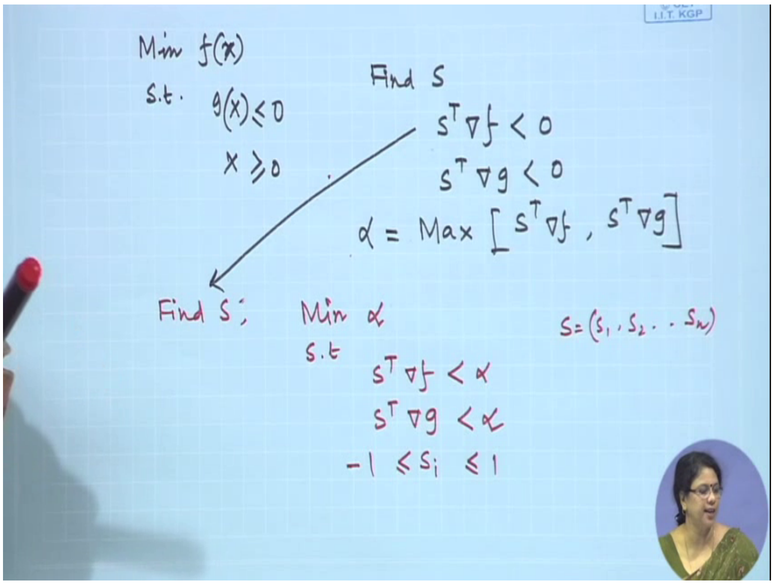

Now, for explaining this thing I am writing the problem once again for you. Minimization of f x subject to g x less than is equal to 0 x greater than equal to 0. Now our target is to find out the usable direction such that s T grad f must be less than 0, s T grad g must be less than 0 all right. Let at let us introduce one, artificial variable alpha such that maximum alpha is equal to maximum of s T grad f and s T grad g.

Then this problem can be written as this way you see. Find s minimization of alpha in the screen subject to s T grad f less then is equal to alpha s T grad g less than is equal to alpha. From here we can say that find s such that minimization of alpha. Why we are minimizing? Minimizing alpha, you see these are all the negative values. We are getting alpha as a maximum negative values, but we want to reduce alpha still it becomes non positive that is negative that is our target that is why we are reducing alpha; that means, we are finding out minimum of alpha subject, to since alpha is max of these that then individually this is less than equal to alpha this is less then equal to alpha. That is why we can write it as grad f less than is less than alpha grad g less than alpha, what did I say I say that s is a direction s is a vector, if we have n dimensional decision when vector then s will have n components.

That's why you must be understanding that for individual sis where s would be s 1 s 2 s n in n dimension individual s, I will be in between minus 1 and 1.

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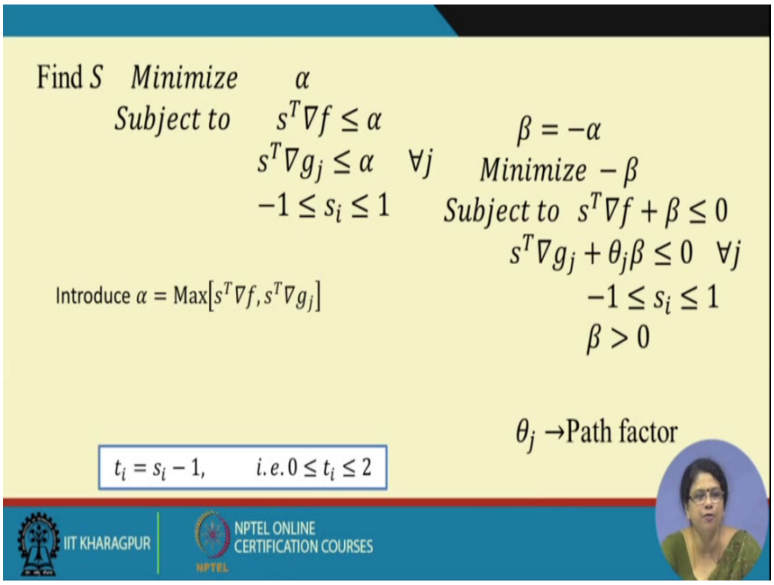

That is the given condition for us that is why we can say that that's why we can redefine the problem as this way. Minimize alpha subject to s i in between minus 1 and 1. One thing you see grad f is a grad f we will get from here a linear function that I just wanted to say. This is a linear function, this is a linear function, this is a linear function. That is why if we want to have the linear programming problem we have a constraint that if I want to apply the simplex algorithm basic constraint in simplex algorithm is that all the decision variables must be nonnegative.

That's why we cannot have any decision variable as negative value. We have considered as alpha as max of s T grad f s T grad g. That is why alpha is negative if I keep as a minimization of minus alpha within the problem it minimization of alpha within the problem itself then I have to declare alpha as non positive, but simplex will not accept it simple algorithm. As I said the constraints are becoming the linear constraints. That is why it cannot be to make it positive I am considering another variable beta, considering beta is equal to minus alpha then beta must be positive. If alpha is negative beta is positive that is why minimization of alpha will become minimization of minus alpha. And alpha if I replace alpha with beta it will become s T del f plus beta less than is equal to 0, s T del g j g j because there are j number of constraints j can be from 1 to m m number of constraints plus beta less than is equal to 0.

Now, here in Zoutenijk method feasible direction method. Here we can consider a factor that is the factor is factor theta. You see alpha is the variable that is the constant value, s is a vector delta f is a vector delta g is a vector, but s is a vector s vector is what that is the usable direction. We are not really interested for the magnitude of that vector we are interested about the direction of s. That is why we kept direction from minus 1 to 1, we have taken the normalized value of s, but we are more concerned about the magnitude of delta f magnitude of delta g. All right beta is something relates to the magnitude of alpha, but you see theta g is theta j is an angle which angle refers to the angle of descent feasible cone that is the usable cone. What is our target our target is to push theta? If theta is equal to 0 means what the constraint is tangent, that is why in other cases always theta g will be greater than 0 it cannot be less than 0, less than 0 theta j impossible.

That's why this push factor we have considered with beta only, by taking theta j as greater than 0. That is why the original problem we had original problem minimization of f x subject to j x less than is equal to 0. Now it is becoming a problem of this kind minimization of minus beta s T delta f plus beta less than is equal to 0, s T delta g j plus theta j beta less than is equal to 0. Theta j theta is again it is having different components for one constraint for the jth constraint the component is theta j for theta all right.

Now, for that is why individual theta j must be greater than equal to 0. In at the optimal stage for one constraint theta j is equal to 0, for other constraints it may not be in that way this is the push factor we are considering, but again you say this is if we consider as a linear programming problem. The problem is again here is that we cannot take s as a negative value. Though we have consider s as a magnitude of the vector for the usable direction, and that is why we took the normalized value from minus 1 to 1, but once it is coming as a linear programming problem all the decision variables must be positive.

Then let us have certain mechanisms. So, that this negative variable will be converted to positive variable, that is why we will consider that one factor t i that is equal to s i minus 1.

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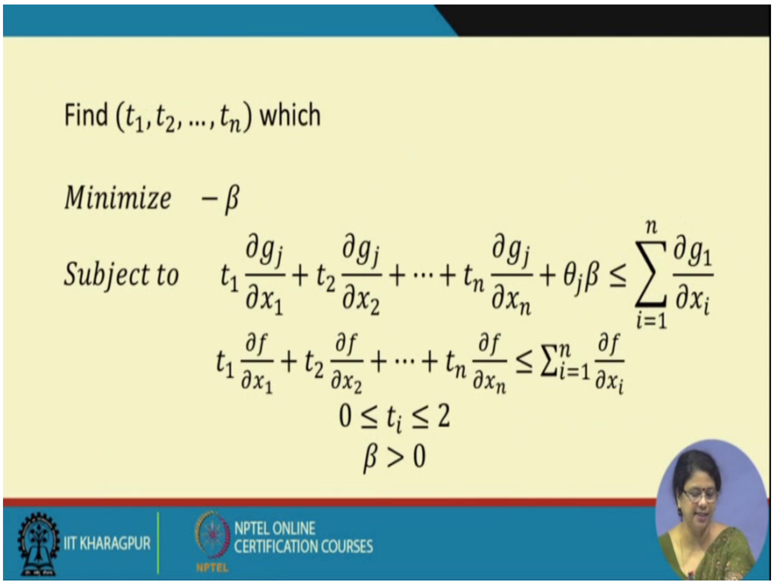

So, that s i is becoming from minus 1 to one then t, I will be from 0 to 2. That is the beauty of this method. Once we can have it then the whole problem can be rewritten as this one, that find out the direction this is the component of s such that s T delta g plus theta j that is the push factor into beta less than is equal to 0. And this is less than is equal to 0 and t I in between 0 to 2 and beta greater than 0.

What how it is being expanded that you will get it if you just a take the first order derivative of the constraint with respect to decision variable, because we have that factor with us.

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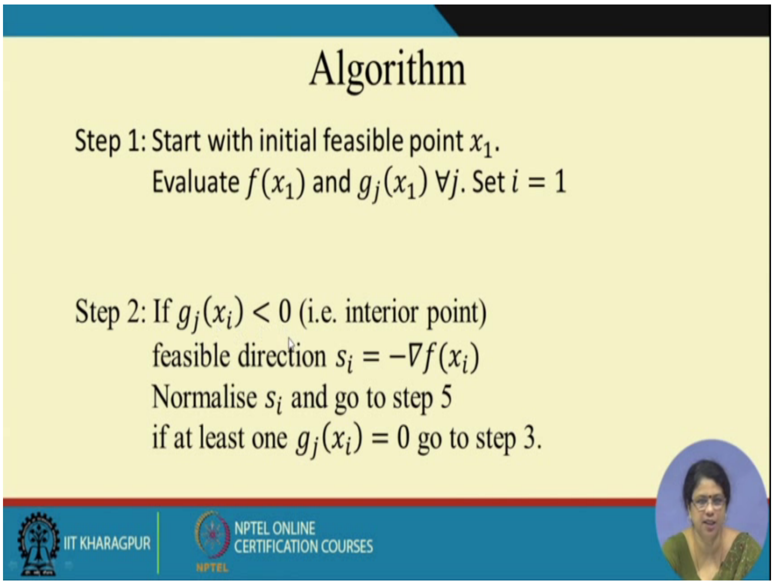

And if I just tell you I will tell you more about the zoutenijk feasible direction method, with example I show how it is being expanded in this manner in the next class, but before to that I will tell you the algorithm once.

Take the initial feasible point x 1 that is of your own choice. If your choice is good enough then your number of iterations will be less. Now find out f and g at that point then if g I is less than 0 then must be in the interior point. Now find out the feasible direction as grad f x I at that point that is the feasible direction.

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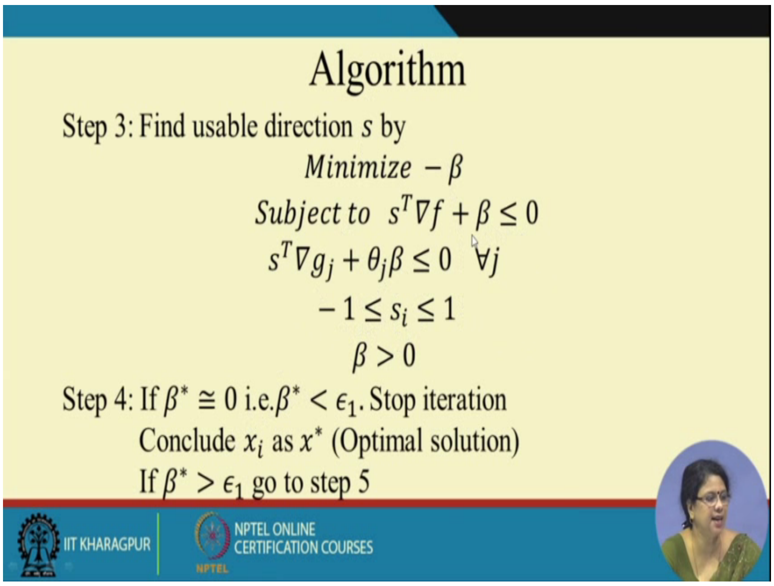

So, that s grad g must be less than 0 that is why that is a direction feasible direction and g j you have to consider, then you construct this problem considered t i in such a way that it will become from 0 to 2. Take beta corresponds to minus alpha we want to reduce alpha that is why our target to make alpha as 0 that is why in other hand we are making beta as 0.

That is why our stopping criteria would be if the beta value is very small, we can conclude that the optimal solution is that one that is the thing for you.

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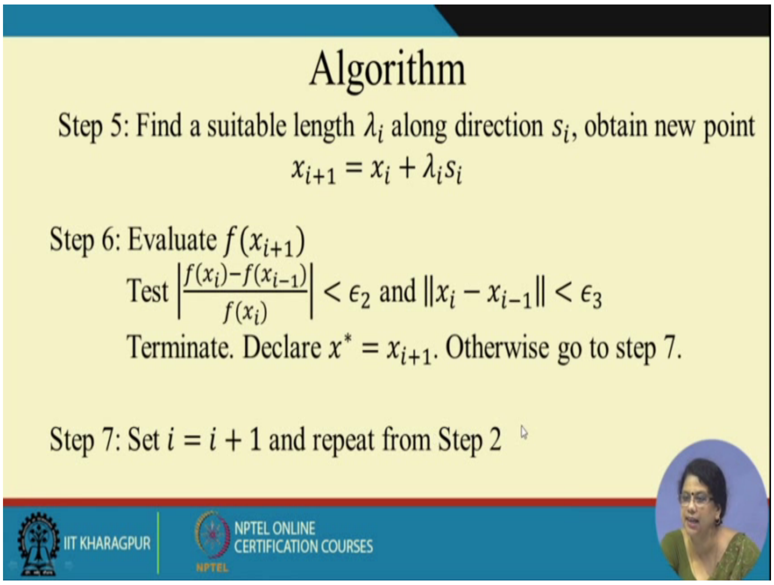

Now, this is a step 5 once we get s, I we will get the lengths suitable length we will find out. And this is the check for optimality, and we will do the solution for this. We will repeat again and again, now I will explain more about this algorithm the calculation part of the feasible direction method along with the example in the next class.

Thank you very much for today.