Constrained and Unconstrained Optimization
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Lecture - 51
Constrained Optimization

Let us start the session. We are having 2 weeks with us; we will deal with the different methodology for solving constraint non-linear programming problem. The constraint non-linear programming problem, it can be convex, non-linear may be convex, nonconvex optimization problem as well.

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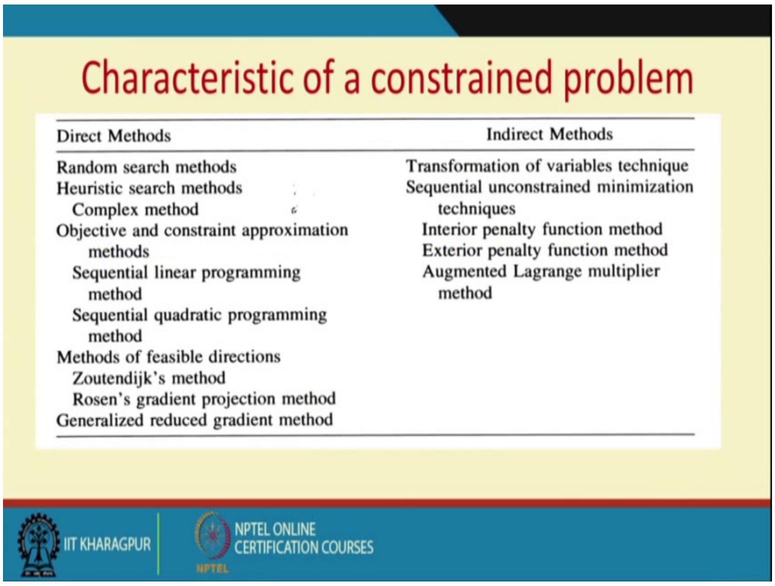

Now there are different methodology to solve the constraint non-linear programming problem, and now which one is better and which one is the best, that you will understand after few classes, but if I just start the methods one is the direct set of methods, another method set is the indirect method. In the direct method we are having the random search method, heuristic search method and sequential linear programming, sequential quadratic programming feasible direction Zoutendijks, feasible direction method that is the gradient search technique, gradient projection method. These are all the direct methods. And in the indirect method the most popular methods are the interior penalty function method, and the exterior penalty function method, and some other variation of Lagranges multiplier method is also there.

Now, I plan to have the classes on direct methods and indirect methods, and I will cover few of them, not all. Now direct random search method already I have explained little bit on that, the grid search technique, random search technique, grid walk method that I have already explained. Now, but I will start now sequential linear programming method; that is a very, that is the beauty of the problem, is that we will consider a non-linear programming problem, but at every step we will solve a linear programming problem, a sequence of linear programming problem will solve, and ultimately we will get the optimal solution, you must have been studied the, for the integer programming problem ;that is a cutting plane method. If you remember integer programming, there are two techniques branch and mound, and the cutting plane method, you must have been covered that 1 in the previous classes.

Now, one of the variation of that only, but here the cutting plane is being formed differently; that is the only difference here, that is why we will consider a non-linear programming problem. Today I will tell you how to handle the non-linear programming problem so, easily, because we are much more comfortable to solve the linear programming problem. We have several softwares to handle it. We can have a series of linear programming problem, but we will get the solution of the solution.

Let me tell you the basic philosophy of this method then I will deal with the multi variant non-linear programming problem, which is of constraint type. Let me consider a non-linear programming problem, where the, that is of single variable. Let me explain the geometry of the methodology first.

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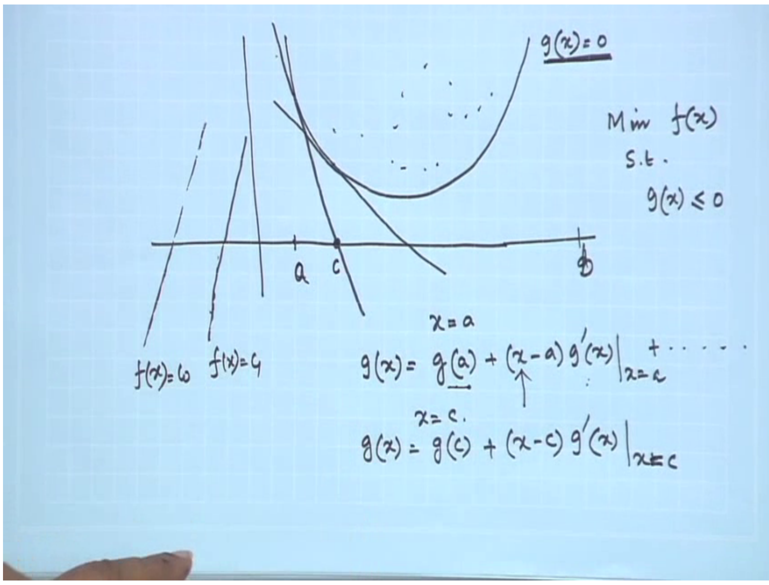

Say this is the function given to us g x is equal to 0, and we are having g x less than 0; that is why this is the feasible space for us ok.

Now, there is one objective function. This is the objective function, this is moving. These are the level curves of the objective functions; say f x. now only we are having that minimization of f x x is of, this is a linear function subject to g x less than is equal to 0. This is the only thing we have given. Now for this problem one thing we should know that, what is the range of x, we have g x less than is equal to 0, but if we can somehow from the constraint set we can find out the range of x that can be nice enough, because we have to start the process from one of the point, what we have to do. We have consider that the whole thing is a include it within c and d, say ok.

Now, say let me consider in state of c and d, a and b alright. The first 2 points, the function is in between these. What is being done. we will consider a point, then we will, this is a non-linear function g x, but how to if make the non-linear function as a linear function?We know the Taylor series, Taylors theorem, we know through the Taylor series any non-linear function can be expanded at a point. That is why if I just expand g x at a point x is equal to a, that can be written as g x is equal to g a x minus a g dot x, at x is equal to a plus, this square by 2 factorial g double dot etcetera. Let me ignore all these terms that is why if I just consider this 2 terms, then you must be understanding this is a constant value, only x is involved here that is why it will become a linear function, g a plus something alright; that is why if I consider, then I will get the value here, and these non-linear function will be approximated with the linear function ok.

Now, once this is. So, now, this function is moving, this is my objective function at different point c 0 c 1 and c 2, it is moving. Now subject to this linear constraint, then there will be the, where the solution will be at this point, there is no doubt about it. Let me consider this point as c. Once we are getting this point as at c. Again what we do at point c, we will again expand g x, then what we will get g x is equal to g c plus x minus c g dot x at x is equal to. Sorry x equal to c ignoring all other term, because we want to have the linear function.

Then at this point, again we will have another linear approximation of g x clear. This is the linear approximation of g x through the Taylors theorem, Taylor series, and the objective function is moving further and further, we will get optimum solution, in this way minimization and maximization we will handle, but where to stop, and everything I will discuss in the next, where we will deal with the optimization problem with several variable together, and we will apply Taylors series for 2 variables 3 variables etcetera.

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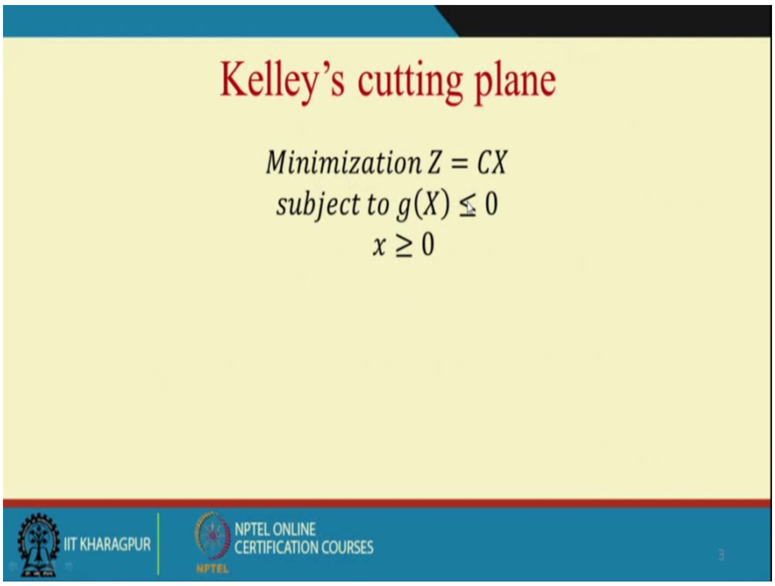

Let me consider the minimization of Z is equal to CX subject to g X less than is equal to 0. Now there is a restriction is that for this method, it is called the Kelley's cutting plane method. The restriction is that objective function must be linear in nature, it has been said.

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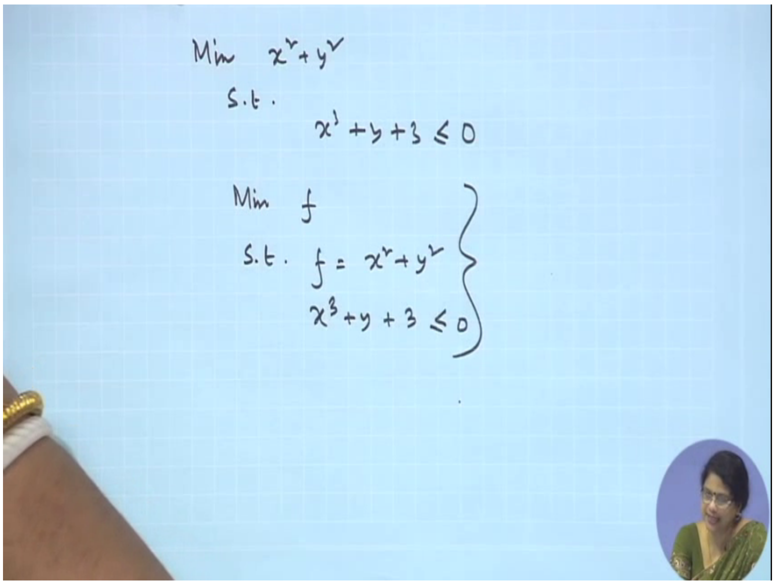

Now if we have a non-linear objective function say minimization of x square y square subject to x cube plus y plus 3 less than is equal to 0. We can have it. We can make this one as a linear function by considering this way minimization of f subject to f is equal to x square plus y square and x cube plus y plus 3 less than is equal to 0.

You see this will become a non-linear programming problem, where objective function is of linear type z is equal to c x alright; that is why is the Kelleys cutting plane method is applicable as well for the non-linear programming problem, where the objective function is non-linear in nature, but it has been advised that you apply kelleys cutting plane method, where the objective function is of the form z is equal to says; that means, the linear alright.

Now, the first step is that, if possible find out the minimum and maximum value of decision variables alright. Now for example, we are having a problem

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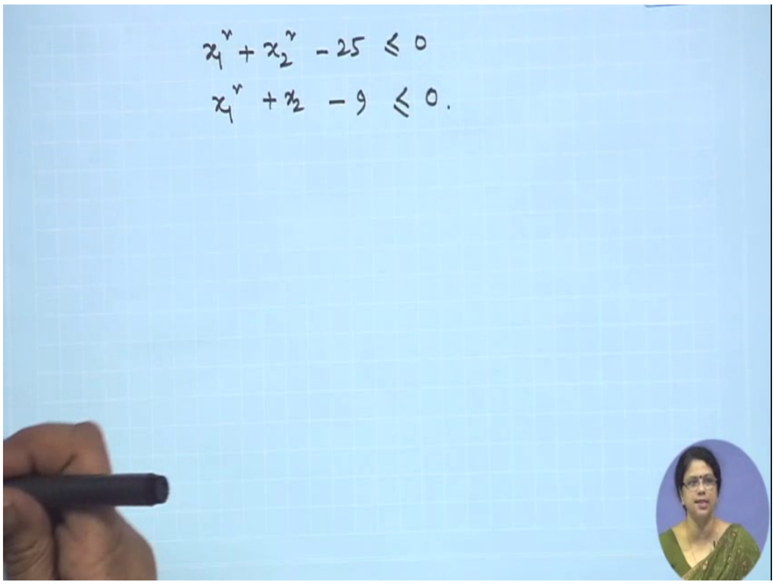

where the constraint are x 1 square plus x 2 square minus 25 less than is equal to 0 x 1 square plus x 2 minus 9 less than is equal to 0. Can you get the range of x 1 the minimum value of x 1, and the maximum value of x 1. What will be the minimum value of x 1 minimum value of x 1 would be 5 and maximum value would be 9, because from here we will get 5 and 9.

Similarly, we are more concerned about the maximize maximum value rather not the minimum value, but we will get the range of x. I will discuss all these things in the later part, let me just explain the methodology. First I will apply this 1 in the problem now here.

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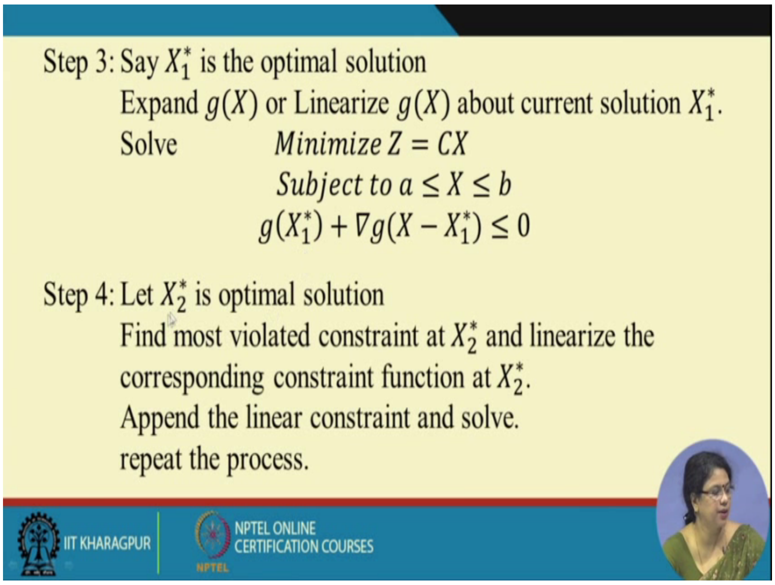

Then what we will do, we will get the optimal solution for the problem, this 1 where g X has been expanded at a point, at a given point g X 1 star now. Now if this is not there how really we call get the optimal solution, I will just explain to you in the next [FL] into problem wise [FL], this is the kelleys cutting plane method. Let me tell you the algorithm for it first, then I will just apply the algorithm for a problem of a specific. Now there is a problem where we are considering the non-linear programming problem, but objective function is of linear type, now minimization of z is equal to C X subject to G X less than is equal to 0. Now from the set, if we can find out the minimum and maximum possible value what are the minimum and maximum possible value of the decision variables, how we can find it feasible space only guide us that how the decision variables can move.

Now, feasible space how it is being constructed, this is being constructed with the constraint set; that is why I from the given constraint set. We will try to find out that what would be the maximum possible value for the individual decision variable. What is the minimum possible value of the individual decision variables. What is the minimum possible value of the individual decision variable; that is given is 0, but maximum value we will try to find out from the problem, and after that what we will do. We will find out minimization of z is equal to C X subject to X is varying within that range, we will get one optimal solution there.

Once we will get one optimal solution, let it is x 1 star, next about x 1 star we will expand the non-linear function constraint functions, whatever functions we are having around that point, then all the constraints will be converted to the non-linear linear constraint, but objective function is linear Z is equal to C X; that is why we can solve minimization of Z is equal to C X subject to X within a to b, and this is the function for us. If we solve it, we will get 1 optimal solution, as I showed you once. Well get 1 optimal solution x 2 star. Again we will expand g with respect to that point.

Then we will again find out the solution, optimal solution for this. We will add the constraint this linear constraint in the previous problem, and we will solve it again and again; that means, we are getting, we are putting different cutting planes within the feasible space in this, with the cutting planes, we will see the process will stop somewhere for this thing. Let us consider one of the problem. Now let me consider the problem the.

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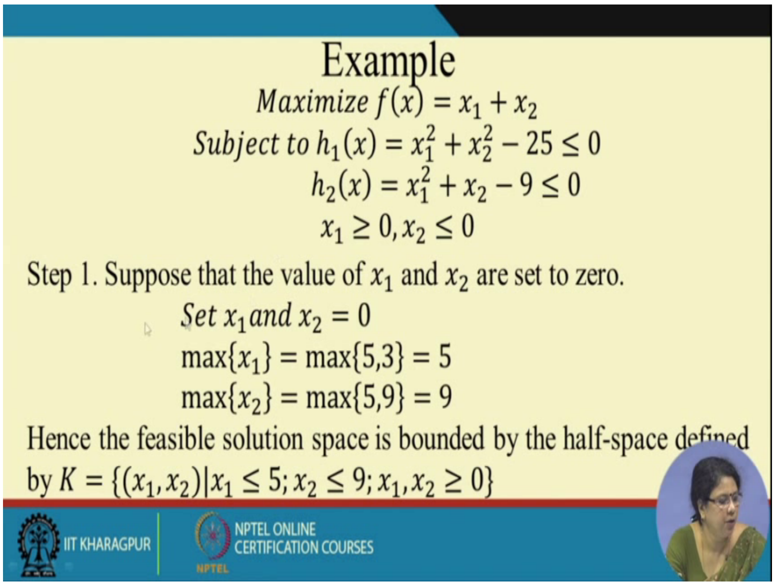

This one maximization of x 1 plus x 2 subject to x 1 square plus x 2 square minus 25 less than is equal to 0. Now from here what we can see that the value for x 1, here maximum is 5, from the next the value for x 1 is maximum three, that is why in general we can have the maximum value for x 1 is 5 minimum is 0 we have all right.

Now, for the x 2 the maximum is 5 here, but here the maximum is 9; that is why maximum of x 2 would be 9, that is why we can consider that the feasible region the we can have the space feasible region. This way instead of having the the feasible space which is bounded by the constraint space constraint our intention to make it as linear function; that is why what we will do. We will formulate the constraint space newly with the upper bound, of the upper bound of the decision variables. Then we will have a rectangle, that rectangle will be the bounded rectangle, which will contain whole feasible space. Is that clear.

Now, after that our target is to put one by one cutting planes, within that space. Now once we are getting that we will solve

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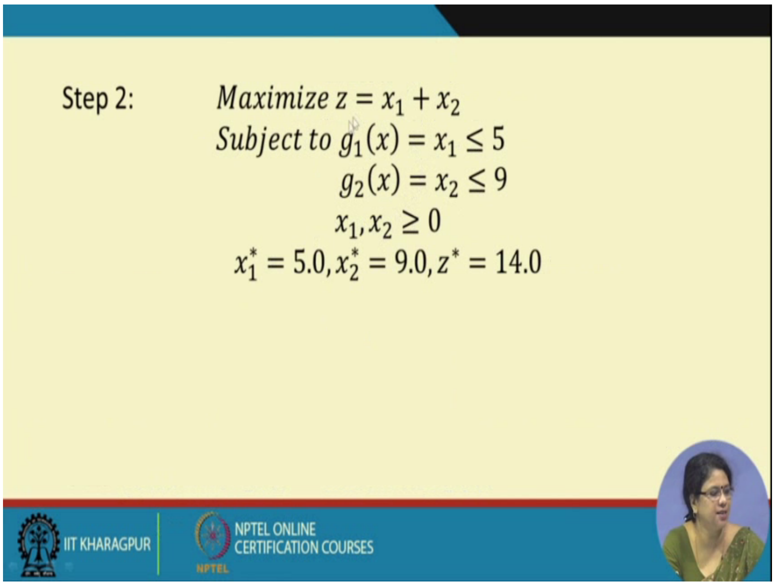

This minimization of z is equal to x 1 plus x 2 subject to x 1 less than equal to 5 x 2 less than is equal to 9. We will get 1 of the solution 5 . 0 9 . 0, and the objective functional value we will get 14. Once this is the optimal solution, what we will do. We will consider this optimal solution in the next iteration. We will consider that x 0

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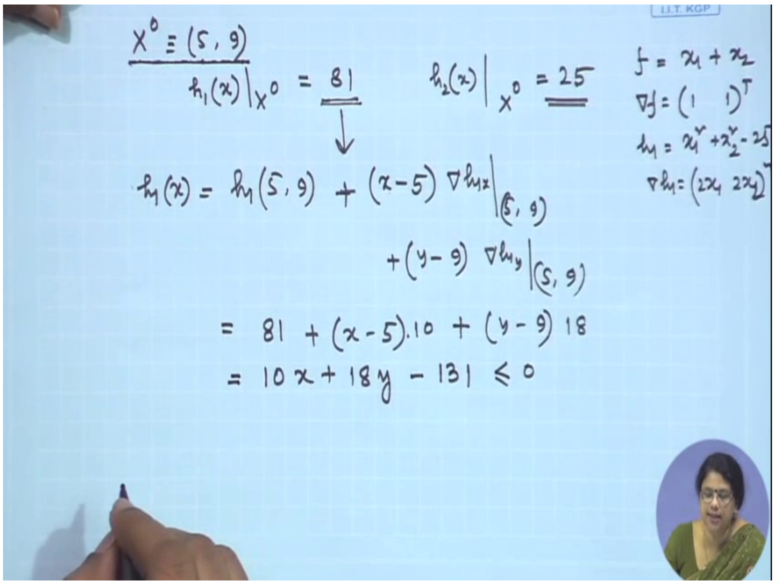

is equal to 5 9, and we will first see what is my h 1 x, at this point h 1 x at x not will be, will have the value 81, and h 2 x will have the value at x not; that is 25 ok.

Now, what is a meaning of it, it means that if I consider, this is optimal solution, than this much of violation I have to accept, and if I consider this as optimal solution, this much of violation I have to accept, because my constraint is h 1 less than is equal to 0 h 2 less then is equal to 0; that means, these value is at this point, the value is 81; that means, that value is from 0; that is the violation, that is why where is the more violation is there, certainly in h 1 is more violation is there, that is why in the cutting plane what I have to do. I have to put a cutting plane. First at one time we can put one cutting plane only.

That is why we have to find out, if I have a set of constraints we have to find out for which constraint is most violated, since h 1 is most violated, I will expand h 1 around this point, and I will put as a linear function at that constraint I will add in the problem. How I will do it, h 1 x, I am now expanding it will be h 1 5 9; that is 81 plus this is function of 2 variables, that is why x minus 5 my f function is x 1 plus x 2, that is why grad f would be 1, 1 transpose alright. Now and if I consider h 1, h 1 is x 1 square plus x 2 square minus 25 that is why grad of h 1 would be 2 x 1 2 x 1 x 2 transpose clear, that is why x minus 5 grad of h 1 at 0. 5 9 grad of h 1 the x part of that plus y minus 9 grad of h 1, the y part of it at 5 9 is that clear, that is through taylor series I am expanding.

Now, if this is. So, this is is equal to 81 plus x minus 5 2 x 1 the x part is to 2 x 1 2 x 1 means 5 into 2 that is 10 into 10 plus y minus 9 5 5 and 9 at this point. The y part is only 18 18, that is why this will become as 10 x plus 18 y minus 130 1 alright what is my constraint then this must be less than is equal to 0 my constraint, that is why what we will do. We will formulate a function of the problem as maximization

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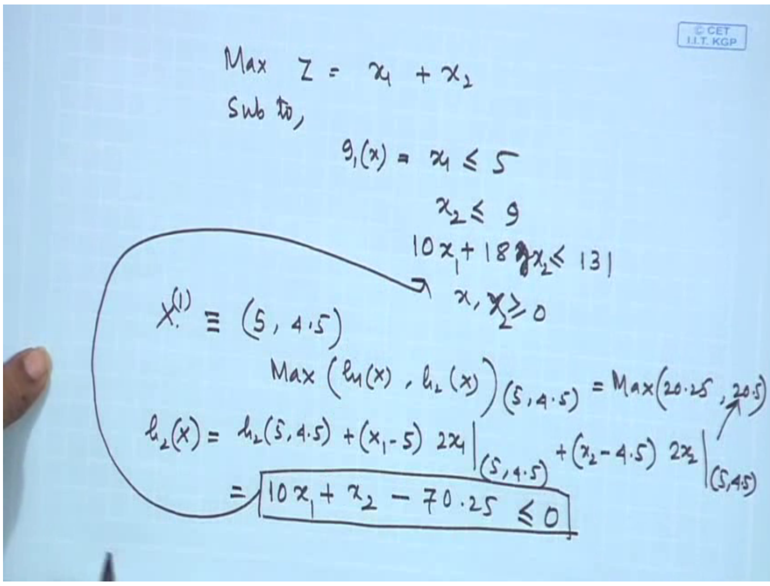

of z is equal to x 1 plus x 2 subject to g 1 x is equal to x 1 less than is equal to 5 x 2 less than is equal to 9, and newly I will put 10 x plus 18 y less than is equal to 1 30 1 x y greater than equal to 0. If we solved this problem, this is again a linear programming problem x 1 by x 2 and x 3 y

X 1 and x 2, it should not be x and y, it should be x 1 and x 2 x 1 and x 2. You are right, it cannot be y x 1 x 2 then 10 x 1 plus 18 x 2 less than is equal to 1 30 1 x 1 x 2 greater than equal to 0, if we solve this problem, this is a linear programming problem. If we solve it, we will get 1 optimal solution that is that was x 0, if I get it x 1, we will get at 0. 5 4 . 5, we will get the optimal solution.

Again we will see that in this optimal solution how much violations we have for h 1, and h 2. Now you must be understanding where the process will stop, once we will see that the constraint is not violated, we can stop it. Now, if we just proceed in this way we will find out maximum value of h 1 x on h 2 x 2 constraints, where x is a X is a vector x 1 x 2 at 0. 5 4 . 5 we will see this value will come as 20. 2 5, and 20 . 5 what we could see that h 2 is most violated constraint here, that is why again we will expand h 2 at . 5 4 . 5 h 2 x will be written as h 2 5 4 . 5 plus x 1 minus 5 2 x 1 at 5 4 . 5 plus x 2 minus 4 0. 5 2 x 2 at 5 4 0. 5. No problem if I just write it down, we will get 10 x 1 plus x 2 minus 70 . 2 5 less than is equal to 0. We will add this constraint, where here we will solve this 1, if we solve it, we will get another solution that is x 2.

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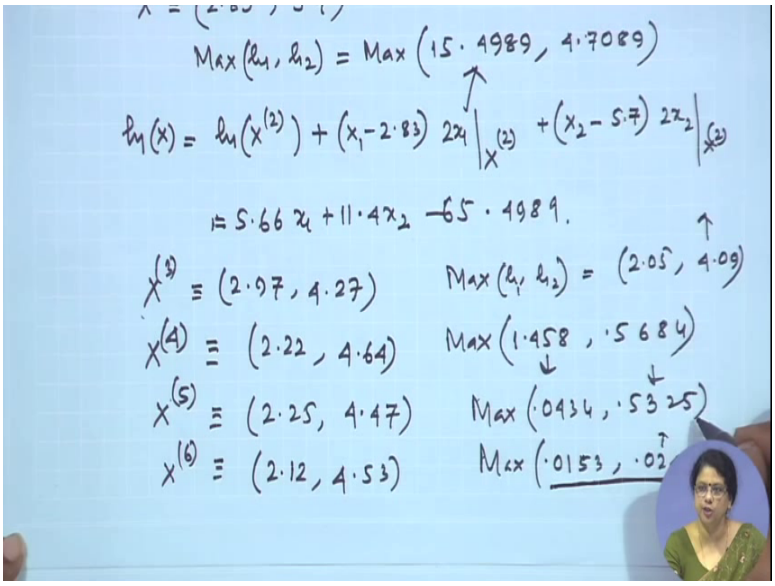

The solution will come, I will show you the series of solutions, because you will have a feel that here we need to stop x 2. Again we will find out maximum of h 1 x and h 2 x at point x 2, we will get the value will come as max of 4 15 for 4 9 8 9, what I suggest to you, you should calculate on your own all these values I, what I have done may be some calculation variations, may be there you should check it; that is why we could see h 1 is more violated. Again we will expand h 1 with respect to h 2 h 1 x 2 plus x minus 2 . 8 3 2 x 1 x 2 plus x 2 minus 5 . 7 2 x 2, this is equal to 5 . 6 6 x 1 plus 11 . 4 x 2 minus 65 . 4 9 8 9 we will get.

Again we will add this append, this constraint, in the previous problem we will get the optimal solution x three. In this way 1 by 1, let me just write down all the solutions for you 2 7, then we will get x 4 2 . 2 2 4 . 6 4, and here just to see how the h 1 values are changing h 1 and h 2 at this point h 1 value will come 2 . 0 5 4 . 0 9 the values are reducing at this point. The values are coming as 4 5 h 0. 5 6 8 4. Here I have to include here that should be, then I will get x 5 as 2 . 2 5. Hopefully you could see the values are converging, and max of point 0 4 3 4 . 5 3 2 5. Again I will add for h 2, and if I cross it just to see how, what is the value is coming 2. 1 2 4 . 5 3, and here the max values is coming 0 1 5 3 0 2 4 4 ok.

Now, you see h 1 and h 2 value, if you have satisfied that these are almost 0, you can stop here, but if you are not satisfied, then again you can include the cutting plane with h 2, you will see the values for h 1 and h 2 will reduce further and further, and 1 thing is that here I have given you the optimal solution after adding this 1. Again you have to formulate 1 cutting plane in terms of h 2 here; in terms of h 1 here in terms of h 2, all you have to append 1 after another with the original problem. Ultimately you will get the solution of the problem, as this 1. Now you could understand the advantage of this type of methodology.

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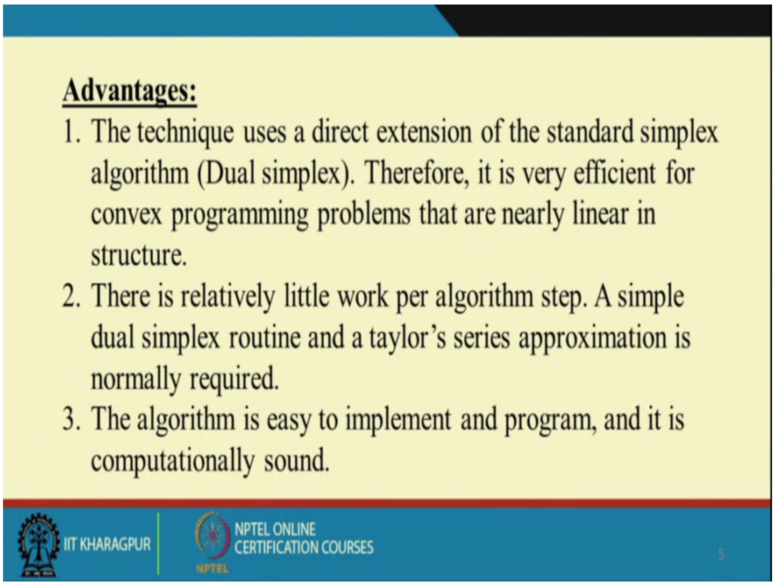

The methodology is very simple in calculation, there is no doubt about it, geometrically it is very nice to visualize. This is another part of it, because we are starting from the rectangle which contain the feasible space with that, we are starting and we are putting linear cutting planes one after another, and we are reducing the spaces after that we are converging to the solution ok.

Now, this is that is why it has been said that it is very simple to handle, and if you want to do it manually, you have learned the dual simplex method, with that after appending the solution, the dual simplex method will help you to get the optimal solution in a better way, that is why you apply the dual simplex method and easy to program even.

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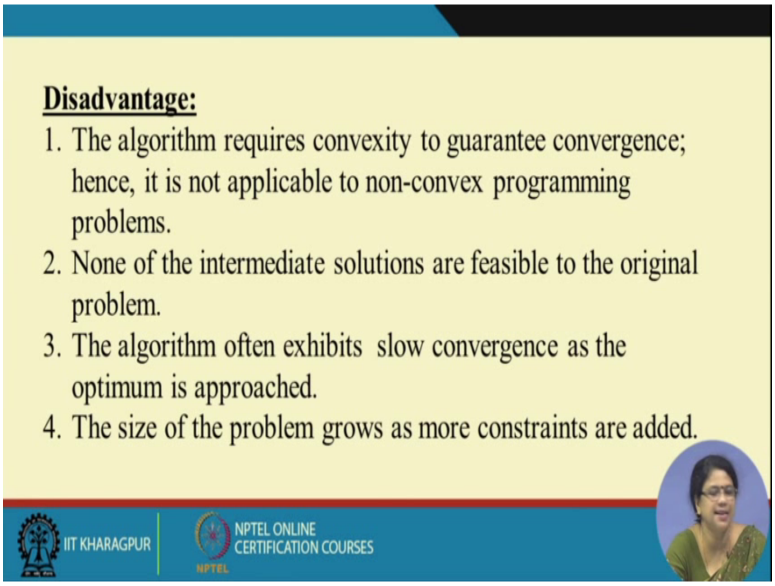

But there are certain disadvantages of this methodology as well. The first is the disadvantage is that the convexit.

If the region is convex, then certainly we will get the solution, but if the region is not convex, for example, the region looks like this. This is not a convex region, because if I add these two points, the whole line is not contained within the feasible space.

If this is my feasible space, then whatever solution we are getting through Kelleys cutting plate method, that cannot be guaranteed as a optimal solution, that is one part, and another thing none of the intermediate solutions are feasible to the original problem, because this is the feasible to the local problem, we are preparing at that stretch and algorithm, also exhibits slow convergence; that is also there. You must have been seen I had to do 7 iterations, and all these iterations always I write simple small programs to do it; otherwise I do it with x l, I do not do by hand, the calculations and I use the lingo as well with x n. You also do the same thing, and if the size of the problem is very big 1, then you must be understand the convergence will be much more slower.

That is why this process, though this is very simple easy to visualize, and that could be a starting point, for understanding the non-linear constraint, non-linear programming problem, but this methodology is not accepted in general. We that is why you must be understanding we need to learn many methodologies together, you want to. If you want to have a rough sketch of the solution, this could be the good start of it, but in the next I will tell you another method that is a feasible direction method in the next class.

Thank you very much.